I. INTRODUCTION

The theoretical modeling of the multi-scale dynamics of a magnetized plasma is a top priority in laboratory and space plasma research, as well as a frontier problem in computational plasma physics. Full kinetic Lagrangian and Eulerian simulations based on the Vlasov-Maxwell system are still far from being considered to be the best tools for multi-scale plasma modeling because they are still too demanding in terms of computational resources and, in addition, because of the difficulty in constructing Vlasov equilibria capable of representing the initial large scale systems to be investigated in a realistic fashion.

An alternative approach with respect to a full kinetic description is to develop fluid models that extend the large-scale magnetohydrodynamic (MHD) (or multi-fluid) approach towards regimes where the effects related to the presence of fluctuations at scale lengths comparable to the ion Larmor radius \( \varrho_i \) (or the skin depth \( d_i \)) are included. These models can be used provided the characteristic frequencies of the system remain smaller than the ion cyclotron frequencies and phenomena related to ion cyclotron resonances remain negligible. Extended models have a wide range of applications, from laboratory and space plasma turbulence to magnetic reconnection studies. More generally, they can describe systems where the energy injected at large scales cascades efficiently towards the ion Larmor radius \( \varrho_i \) and/or the ion inertial scale \( d_i \) scale lengths, in some cases up to electron micro-scales. This is the case in magnetic reconnection events or after the development of strong fluid-like instabilities such as the Kelvin-Helmholtz instability (hereafter KHI).

The possibility of using “extended fluid models” becomes crucially important when studying systems where the large scale fields, defined as the “equilibrium fields,” vary over typical scale lengths that do not exceed the kinetic scales by too large a factor as is the case, for example, in the region at the interface between the solar wind and the Earth’s magnetosphere or the magnetotail. Such extended fluid models could also play a key role when coupling fluid and kinetic codes since they include some of the physics bridging the low frequency MHD regime and higher frequency regimes. Here, we define here as “kinetic effects” first order Finite Larmor Radius (FLR) effects such as “gyroviscosity”. We do not include the expansion of the Bessel functions, e.g., \( J_0(k_\perp \varrho_i) \simeq 1 - k_\perp^2 \varrho_i^2/2 \), which are of higher order in \( \varrho_i \).

The need to extend the standard fluid approach, which is closed in general with a polytropic equation of state, and to include terms related to the pressure tensor, so as to avoid the limitations of an isotropic pressure in the equations of motion, is specially evident in the presence of a large scale shear flow. This is due to the fact that in an almost collisionless system the shear flow and the pressure tensor are strongly coupled and interact on fast time scales. This is the case when studying the dynamics at the low latitude boundary layer between the Earth’s magnetosphere and the solar wind where large-scale MHD-type vortices are generated by the KHI driven by the velocity shear between the solar wind flow and the magnetospheric plasma at rest (see Ref. 4 and references therein). In this case the typical size of the MHD vortices is of the order of several \( d_i \) (or \( \varrho_i \)), corresponding to nearly five to ten times the typical scale length of the large scale shear flow which is of the order of a few \( d_i \) (or \( \varrho_i \)). In a theoretical description, this flow is considered as the “equilibrium” configuration that drives the KHI. Until now, in order to model such a system, MHD equilibrium configurations have been considered where the shear flow and the
density variation between the two plasmas are represented by a one dimensional profile, typically a hyperbolic tangent. The total pressure balance is ensured either by a change in the amplitude of the magnetic field (that is nearly perpendicular to the plane of the vortices) or in the temperature. An isothermal or an adiabatic polytropic equation of state is used.

However, such MHD equilibria with an isotropic pressure are unsatisfactory for fluid simulations. In fact the inclusion of, at least, first order corrections due to the pressure tensor, the so-called FLR effects, alters the shape of the equilibrium fields. As a consequence, not only is the system dynamics influenced by small-scale (non-MHD) effects but also the very initial configuration is no longer an equilibrium as it evolves on a short transient timescale. This obscures the development of the “large-scale” KHI even in its linear stage.

This emphasizes the importance of starting from a correct equilibrium configuration when dissipation processes such as collisional viscosity and resistivity are inefficient.

In the limit of massless electrons the fluid velocities are \( u_e = U - J/n \), and the model equations become

\[
\begin{align*}
\frac{\partial n}{\partial t} + \nabla \cdot (nU) &= 0, \\
\frac{\partial (nU)}{\partial t} + \nabla[nUU + \Pi_b + \Pi_e^{(0)} + \Pi_e^{(1)} + \Pi_i^{(0)} + \Pi_i^{(1)}] &= 0, \\
E &= -U \times B + \frac{J \times B}{n} - \frac{\nabla \Pi_e^{(0)}}{n}, \\
\frac{\partial B}{\partial t} &= -\nabla \times E; \quad \nabla \times B = J, \\
\frac{\partial p_{\perp}}{\partial t} + \nabla (p_{\perp} U) &= -p_{\perp} (I - bb) : \nabla U - \Pi_i^{(1)} : \nabla U, \\
\frac{\partial p_{\parallel}}{\partial t} + \nabla \left[ p_{\parallel} \left( \frac{U - J}{n} \right) \right] &= -p_{\parallel} (I - bb) : \nabla \left( U - \frac{J}{n} \right), \\
\frac{\partial p_{\perp}}{\partial t} + \nabla (p_{\parallel} U) &= -2p_{\parallel} \quad bb : \nabla U, \\
\frac{\partial p_{\parallel}}{\partial t} + \nabla \left[ p_{\parallel} \left( \frac{U - J}{n} \right) \right] &= -2p_{\parallel} \quad bb : \nabla \left( U - \frac{J}{n} \right),
\end{align*}
\]

where \( \Pi_i = (B^2/2)(I - bb) - (B^2/2)bb, \Pi_s^{(0)} = p_{\perp} (I - bb) + p_{\parallel} bb \). Here \( p_{\perp} \) and \( p_{\parallel} \) are the parallel and perpendicular components of the magnetic pressure tensor of the \( \alpha \) species, \( b = B/B \) is the unit vector along the local magnetic field, and \( I - bb \) is the projector in the perpendicular plane.
\(\Pi^{(1)}_{i} \) is the first order ion gyroviscosity tensor. In the strong magnetic guide field limit, in our case \( B \gg B_{E} \), the ion gyroviscosity tensor components \( \Pi^{(1)}_{i \perp m} \), in dimensionless form, are given by

\[
\Pi^{(1)}_{i \parallel, m} = 0,
\]

\[
\Pi^{(1)}_{i \perp, x} = \betaB \left( \partial_{x} u_{i, x} + \partial_{y} u_{i, y} \right),
\]

\[
\Pi^{(1)}_{i \perp, y} = \betaB \left( \partial_{y} u_{i, y} - \partial_{x} u_{i, x} \right),
\]

\[
\Pi^{(1)}_{i \perp, z} = \betaB \left[ (2p_{\parallel} - p_{\perp}) \partial_{x} u_{i, x} + p_{\perp} \partial_{y} u_{i, y} \right]
\]

\[
- \betaB \partial_{z} q_{i \perp},
\]

\[
\Pi^{(1)}_{i \parallel, z} = \betaB \left[ (2p_{\parallel} - p_{\perp}) \partial_{y} u_{i, x} + p_{\perp} \partial_{z} u_{i, z} \right]
\]

\[
+ \betaB \partial_{z} q_{i \perp},
\]

where \( q_{i \perp} \) and \( q_{i \parallel} \) are the heat fluxes along the magnetic field of the perpendicular and parallel thermal ion energy (hereafter neglected, as in pressure equations).

This system of equations conserves energy explicitly as can be shown by taking the scalar product of the momentum equation times \( U \), of the generalized Ohm’s law times \( J \), of the Faraday law times \( B \) and summing together the perpendicular and parallel pressure equations (the latter multiplied by 1/2) of both species which finally leads to

\[
\frac{\partial}{\partial t} (\mathcal{E}_{U} + \mathcal{E}_{B} + \mathcal{E}_{H_x} + \mathcal{E}_{H_z}) + \nabla \left\{ (\mathcal{E}_{U} + \mathcal{E}_{H_x}) I + \mathcal{E}_{I}^{(0)} \right\}
\]

\[
+ \Pi^{(1)}_{i} \cdot U + (\mathcal{E}_{H_x} I + \mathcal{E}_{I}^{(0)}) \cdot \left( \frac{U - J}{n} \right) + E \times B \right) = 0,
\]

where we have defined the following energy densities:

\[
\mathcal{E}_{U} = \frac{nU^{2}}{2}; \quad \mathcal{E}_{B} = \frac{B^{2}}{2}; \quad \mathcal{E}_{H_x} = \frac{\text{Tr}(\mathcal{E}_{I}^{(0)} I)}{2} = \frac{p_{\parallel} + 2p_{\perp}}{2}.
\]

In the following, we will define the set of Eqs. (1)–(13) as the extended two-fluid model, or the eTF model, while we define the (standard) two-fluid model, namely the TF model, as that obtained from the eTF by assuming an isotropic (scalar) pressure with \( p_{\perp} \equiv p_{\parallel} \equiv p \) and \( \Pi^{(1)}_{i} = 0 \) with a polytropic closure relation of the type \( d(p_{\parallel} \cdot n^{\gamma_{s}})/dt = 0 \) with \( \gamma_{s} \) the polytropic index: \( \gamma_{s} = 1 \) (isothermal) or \( \gamma_{s} = 5/3 \) (adiabatic).

**III. LARGE SCALE EQUILIBRIUM WITH A SHEAR FLOW**

We start by considering an initial MHD equilibrium configuration in the presence of a shear flow within the framework of the TF model. The MHD equilibrium is chosen so as to represent at least schematically the interaction of the solar wind with the Earth’s Magnetosphere at low latitude. This equilibrium and the geometry adopted in the following can be considered as a standard model for equilibrium systems with a shear flow.

We take the plasma flow along the y-direction, varying in the transverse x-direction, and a guiding magnetic field in the perpendicular z-direction. We also allow for density variations. The equilibrium reads

\[
U_{y} = U_{0} \tanh(x/L_{U}); \quad B_{eq} = B_{0}(x) \mathbf{e}_{z};
\]

\[
n = n_{0} - \frac{\Delta n}{2} \left[ 1 - \tanh(x/L_{U}) \right],
\]

where \( L_{U} \) is the equilibrium scale length. We define the smallness parameter \( \epsilon_{i} = q_{i}/L_{U} \). For a plasma with \( v_{s} \approx v_{th,i} \) (i.e., \( \beta \approx 1 \)), we obtain \( L_{U}/d_{i} = (q_{i}/d_{i}) (1/\epsilon_{i}) \approx 1/\epsilon_{i} \). In the usual MHD approach, the parameter \( \epsilon_{i} \) must be sufficiently small so that terms of order \( \epsilon_{i} \) can be neglected and the total pressure equilibrium condition requires that thermal pressure variations be compensated by a corresponding variation in the magnetic field, \( B_{0}(x) \), such that

\[
P_{\epsilon} + P_{i} + \frac{B_{0}^{2}}{2} = \text{cst},
\]

where \( P_{\epsilon} \) and \( P_{i} \) are the (isotropic) electron and ion pressures. The same equilibrium conditions would hold by considering gyrotropic pressures and by substituting \( \mathcal{E}_{I}^{(0)} = p_{\perp} \) for \( P_{\epsilon} \) and \( \mathcal{E}_{I}^{(0)} = p_{\parallel} \) for \( P_{i} \), respectively.

Here instead we consider the case of an equilibrium where the inhomogeneity scale length \( L_{U} \) is not much larger than the ion inertial skin depth, \( L_{U} \sim 2/10 d_{i} \) (or \( q_{i} \)), as typically assumed in the problem of the interaction of the Solar wind with the Earth Magnetosphere. In this case, \( \epsilon_{i} \) is not very small and ion kinetic terms such as gyroviscosity can no longer be neglected even on the equilibrium scale length. In the following, we investigate the importance of including first order terms in \( \epsilon_{i} \) in the rederivation of the standard equilibrium configuration given above. For the sake of simplicity, we shall assume \( q_{i} \sim d_{i} \).

In the presence of a shear flow, even assuming an initial uniform magnetic field (and pressure), the xx and yy-components of the ion gyro-viscosity tensor do not vanish:

\[
\Pi^{(1)}_{i \parallel, x} = - \Pi^{(1)}_{i \parallel, y} = - \frac{1}{2} \frac{p_{\parallel, 0}}{B_{0}} \frac{dU_{y}(x)}{dx}.
\]

Adopting the eTF model for the equilibrium configuration given by Eqs. (16) and (17), \( \Pi^{(1)}_{i \parallel, x} \) and \( \Pi^{(1)}_{i \parallel, y} \) are the only non-zero components of the gyroviscosity tensor \( \Pi^{(1)}_{i} \) at \( t = 0 \). As a result, the configuration spontaneously evolves because of the force imbalance in the action of equation, Eq. (2), due to the diagonal components of the ion gyroviscosity tensor in the (x, y) perpendicular plane.

We can estimate the order of magnitude of the gyroviscosity time scale \( \tau_{G} \) induced by this imbalance as follows:

\[
\frac{dU}{dt} \sim - \frac{d\Pi^{(1)}_{i \parallel, x}}{dx} \Rightarrow \frac{U_{0}}{\tau_{G}} \sim \frac{1}{2} \frac{p_{\parallel, 0}}{\Omega_{ci}} \frac{d^{2}U_{y}}{dx^{2}} \sim \frac{p_{\parallel, 0}}{\Omega_{ci}} \frac{U_{0}}{L_{U}^{2}}.
\]
which gives $\tau_G \sim L_U^2 \Omega / p_{i\perp,0}$ (we recall that in dimensionless units $\Omega_i = [B_0]$). When $L_U$ approaches unity, $\tau_G$ also approaches unity, i.e., the cyclotron time ($B_0 \approx 1$ and $p_{i\perp,0} \approx 1$ in dimensionless units), implying that the system reacts on a fast dynamical time scale. We can compare this characteristic time to the characteristic time scale of the growth of the KHI, e.g., to the inverse of the growth rate of the Fast Growing Mode (FGM) $\tau_{KH} \sim 1/\omega_{KH}(FGM) \sim 10 L_U / U_0$.\,\footnote{Assuming $U_0 \sim 1$, $B_0 \sim 1$ and $p_{i\perp} \sim 1$ as typical values for our system (see, e.g., Refs. 5, 10, and 11), we obtain $\tau_{KH} \sim 0.1 B_0 U_0 L_U / p_{i\perp} \sim 0.1 L_U \sim O(0.1), \quad (20)$}

i.e., $\tau_{KH}$ is much larger than $\tau_G$ if the typical scale of the velocity shear $L_U$ is not too large with respect to the ion scale, $q_i$ or $d_i$. Thus, if in this case we start from a MHD equilibrium, the system reacts to the absence of force balance on a very short time scale, much shorter than the ideal KHI time scale. This will influence the later development of the KHI even during its linear phase. An example of this fast reaction to the absence of a proper equilibrium condition is shown in Fig. 1 where, using the eTF model, we draw the plasma density profile at different times resulting from an initial configuration given by a uniform MHD equilibrium with a superposed shear velocity field, Eqs. (16) and (17) with $B_0 = B_0$, $\Delta n = 0$, $U_0 = 1$, $L_U = 3$ and $B_0 = 1$. The left and right frames correspond to inverse alignments of the fluid vorticity with respect to the guide field, i.e., on the sign of $\Omega \cdot B$. This is in agreement both with previous PIC simulations\,\footnote{We refer to Refs. 5 and 11 for details on the numerical scheme and references therein for a discussion of the nonlinear phase of the KHI\,\cite{111211} by influencing the transport properties and thus emphasizing the relevance of discerning between the two configurations.} and with our Vlasov-Hybrid simulations (see Sec. V). Moreover, such an asymmetry seems to emerge even in the nonlinear phase of the KHI\,\cite{111211} by influencing the transport properties and thus emphasizing the relevance of discerning between the two configurations.

In Sec. IV, we construct the equilibrium configuration analytically by solving the equations of the eTF model to first order in $\varepsilon_i$.

\section{IV. Equilibrium Configuration Including FLR Terms}

We consider an initial flow velocity field $U$ directed along the $y$-direction and varying in the perpendicular $x$-direction, $U = U_y(x) e_y$. A guide magnetic field is directed along the $z$-direction with an amplitude that may depend on $x$. A result, the time derivatives of the density and of the parallel and perpendicular pressures are identically zero. Thus, the equation of motion (2) reduces to

$$\frac{d}{dx} [\Pi^{(0)}_{i,xx} + \Pi^{(1)}_{i,xx} + \Pi^{(0)}_{e,xx} + \Pi^{(0)}_{b,xx}] = 0. \quad (21)$$

Let us assume that we have a valid equilibrium in the MHD limit, described by three functions $F(x)$, $G(x)$, and $H(x)$ such that

$$p_{i\perp}(x) = p_{i\perp,0} F(x); \quad p_{e\perp}(x) = p_{e\perp,0} G(x); \quad B^2(x) = B_0^2 H(x).$$

Let us now take three new “correction” functions $f(x)$, $g(x)$, and $h(x)$ such that
are now equilibrium profiles for the eTF model. Note that we are considering an equilibrium velocity field \( U_0(x) \) in Eq. (16) such that \( \lim_{x \to \pm \infty} dU_0(x)/dx = 0 \), i.e., the system is homogeneous outside the shear layer. Therefore, since the gyroviscosity depends on the derivative of the velocity field, the three corrective functions must approach unity as we move away from the central shear layer, thus recovering the MHD profiles. We stress that this approach, with minor modifications, remains valid for any velocity profile. For the sake of simplicity, in the following we discuss the case where FLR corrections are computed in the limit of an “asymptotic field” \( B_0 \) defined as the value of the magnetic field at the right boundary \( x_s \) of the simulation domain, i.e., \( B_0 = B(x_s) \).

We define \( p_{\perp,0} + p_{\perp,0} + B_0^2/2 \) as the asymptotic values representing the MHD equilibrium away from the shear layer where FLR corrections vanish. After some manipulations in Eq. (21), we obtain

\[
[1 - \tilde{u}'(x)] \tilde{\beta}_{\perp,0} \mathcal{F}(x)f(x) + \tilde{\beta}_{\perp,0} \mathcal{G}(x)g(x) + \frac{\mathcal{H}(x)}{1 + \tilde{\beta}_{\perp,0} + \tilde{\beta}_{\perp,0}} h(x) = 1,
\]

where \( \tilde{u}'(x) \equiv (dU_0(x)/dx)/(2B_0) = (U_0/2B_0) \cos^2(x/L_U) \), \( \tilde{\beta}_{\perp,0} = 2p_{\perp,0}/B_0^2 \) and \( \tilde{\beta}_{\perp,0} = \beta_{\perp,0}/(1 + \beta_{\perp,0} + \beta_{\perp,0}) \). We proceed by imposing the quasi-neutrality condition and by assuming the existence of a polytropic-like relation for each species. We obtain

\[
\mathcal{G}(x)g(x) = [\mathcal{F}(x)f(x)]^{1+\gamma},
\]

where \( 1 + \gamma = \gamma_{\perp,0} / \gamma_{\perp,0} \). Moreover, we assume that the perpendicular plasma beta \( \beta_{\perp,0} \) does not change with respect to the MHD case giving

\[
h(x) = f(x).
\]

Note that this implies that also \( \beta_{\perp,0} = \beta_{\perp,0} + \beta_{\perp,0} \) does not change, if and only if \( \gamma = 0 \), which is true for this model. This, in turn, implies we have the same electron and ion perpendicular temperature profiles. In fact, splitting \( \mathcal{F}(x) = \mathcal{F}_e(x) \mathcal{F}_{\perp,0}(x) \) (and the same for electrons) using the thermodynamic relation \( P = nT \), we obtain \( \mathcal{F}_e(x) = \mathcal{G}_e(x) \) from quasi-neutrality and \( \mathcal{G}_{\perp,0}(x) = [\mathcal{F}_{\perp,0}(x)]^{1+\gamma} \) from the polytropic hypothesis.

Finally, since \( \mathcal{H}(x) \) and \( \mathcal{F}(x) \) are related by the MHD equilibrium condition and by the polytropic hypothesis, i.e.

\[
\mathcal{H}(x) = 1 + \tilde{\beta}_{\perp,0} \mathcal{F}(x) - \tilde{\beta}_{\perp,0} [\mathcal{F}(x)]^{1+\gamma},
\]

the equilibrium condition including FLR corrections reads

\[
[1 - \tilde{\beta}_{\perp,0} \mathcal{F}(x) \tilde{u}'(x) + \tilde{\beta}_{\perp,0} [\mathcal{F}(x)]^{1+\gamma} [(f(x))^\gamma - 1]] f(x) = 1.
\]

In general, \( \gamma < 1 \), and we solve Eq. (23) perturbatively in this “small” parameter giving the convergence conditions

\[
a posteriori.\) The zero-order solution for \( \gamma = 0 \) is given by

\[
f_0(x) = \left[ 1 - C_0(x)a(x) \right]^{-1},
\]

\[
C_0(x) = \frac{1}{2} \frac{U_0 \tilde{\beta}_{\perp,0} + F(x)}{B_0 L_U}
\]

where \( a(x) = \cos^2(x/L_U) \). Then, solving iteratively the equation

\[
[1 - \tilde{\beta}_{\perp,0} \mathcal{F}(x) \tilde{u}'(x) + \tilde{\beta}_{\perp,0} [\mathcal{F}(x)]^{1+\gamma} [(f_{n-1}(x))^\gamma - 1]] f_n(x) = 1
\]

and Taylor expanding \( f_{n-1}(x)^\gamma = (1 - O(a(x))^{-\gamma} \)), we obtain

\[
f_n(x) = \left[ 1 - C_n(x)a(x) \right]^{-1},
\]

where

\[
C_n(x) = C_0(x) \frac{\sum_{m=0}^{n} [ - \gamma \tilde{\beta}_{\perp,0} (\mathcal{F}(x))^{1+\gamma} ]^m}{m!}
\]

This is a geometric series, which converges if and only if \( | - \gamma \tilde{\beta}_{\perp,0} (\mathcal{F}(x))^{1+\gamma} | < 1 \). Since we can always choose \( p_{\perp,0} = \max(p_{\perp,0}) \), we obtain \( 0 \leq \mathcal{F}(x) \leq 1 \) and \( \tilde{\beta}_{\perp,0} < 1 \). Therefore, the series converge for \( m \to \infty \) to

\[
f_{\infty}(x) = \left[ 1 + \gamma \tilde{\beta}_{\perp,0} (\mathcal{F}(x))^{1+\gamma} \right]^{-1} C_0(x).
\]

Note that the shape of the equilibria calculated here depends on the relative orientation of the magnetic field with respect to the fluid vorticity, i.e., on the sign of \( \Omega \cdot B \) (represented by the ratio \( U_0/B_0 \) in the \( C_0 \) coefficient). We underline that our model is based on the quasi-neutrality assumption and on the polytropic closure (from which we recover also the parallel pressure profiles, \( p_{\parallel,0}(x) = [p_{\perp,0}(x)]^{\gamma_{\perp,0}/(1+\gamma_{\perp,0})} \)). Moreover, the function \( f_0 \) in Eq. (24) is exact if \( \gamma = 0 \) irrespective of the polytropic relation if we require that ions and electrons follow the same polytropic law. The solution in Eq. (26) for \( n = \infty \) is more general and allows for a different treatment of the ion and electron pressure equations. (In our eTF model, the ion and the electron pressure equations are the same, but in general these equilibria can be used by a very broad range of kinetic models, as the hybrid ones, in which the two species are treated in different ways.) Asymptotically, as required, the solution \( f_n(x) \to 1 \) for \( x \to \pm \infty \), \( \forall n \). Finally, if \( \tilde{u}'(x) = 0 \) we obtain \( f_n(x) = 1 \forall n \), since the FLR correction vanishes. The general equilibrium taking into account the local value of the magnetic field is calculated in Appendix B.

Finally, since FLR equilibria can be relevant also for mixed Kelvin-Helmholtz and Rayleigh-Taylor unstable configurations, we add the modified equilibria for the case in which an external field \( g = -\varepsilon_0(d\phi/dx) \) is present. We will consider the \( \gamma = 0 \) case only, and thus the MHD equilibrium condition now reads
\[
\mathcal{H}(x) = \frac{1 + \beta_{\perp 0} - \beta_{\parallel 0}}{1 + \beta_{\perp 0} - \beta_{\parallel 0}} = 1 - \bar{\beta}_{\perp 0} \mathcal{F}(x) + \bar{\beta}_{\parallel 0} \phi(x),
\]
where we have defined \( \beta_{\parallel 0} = 2 \Omega_0 / B_0^2 \) and now \( \bar{\beta}_{\parallel 0} = \beta_{\parallel 0} / (1 + \beta_{\perp 0} - \beta_{\parallel 0}) \) and \( \bar{\beta}_{\perp 0} = \beta_{\perp 0} / (1 + \beta_{\perp 0} - \beta_{\parallel 0}) \).

In this case, the correction function \( f_0(x) \) in Eq. (24) is modified as follows:
\[
f_0(x) = \frac{1 + \bar{\beta}_{\parallel 0} \phi(x)}{1 + \bar{\beta}_{\perp 0} \phi(x) - C_0(x) a(x)},
\]

(27)

V. VLASOV-HYBRID CODE RESULTS

In this section, we investigate the influence of the full pressure tensor response by means of a Vlasov-Hybrid model.

A. Initialization of a velocity shear flow in kinetic plasma simulations

As already discussed in Sec. I, one of the main difficulties in the kinetic modeling of plasma dynamics driven by the presence of shear flows is the choice of the initial equilibrium conditions. Indeed, few kinetic equilibria are known in this case and may not always be useful for modeling specific “realistic” configurations, in particular with open boundary conditions. We recall in particular Ref. 12, where the magnetic field is taken to be uniform and not affected by the plasma, and Ref. 13 where the magnetic field is instead self-generated. This last case is extended in Appendix C. Note that these analytic distribution functions are “agyrotropic,” i.e., they lead to perpendicular pressure components that are not equal, as consistent with the analysis in Sec. III in terms of the contributions of the FLR corrections to the diagonal terms of the pressure tensor.

Because the geometry of the available kinetic equilibria is generally too idealized, kinetic simulations with non-trivial spatial configurations are almost always initialized with MHD equilibria (single fluid force balance). However, in the absence of dissipation processes, the system immediately reacts to the absence of kinetic equilibrium, radiating spurious waves or generating large amplitude oscillations that may strongly affect the evolution of the system.\textsuperscript{8}

In this context, it is legitimate to ask (1) whether the eTF equilibria discussed in Sec. IV, that take into account finite Larmor radius effects to first order in \( v_r \), represent a satisfactory initialization for kinetic simulations and (2) how they compare to a MHD equilibrium initialization when modeling shear flows by means of a kinetic code. To address these points, we have used a numerical code that solves the Hybrid Vlasov-Maxwell system of equations for two different shear flow initial conditions.

B. Model description and shear flow initializations

Our approach is based on the numerical solution of the hybrid Vlasov-Maxwell equations in phase space. A hybrid-Vlasov code\textsuperscript{14} is used in a 1D-3V configuration (one dimension in physical space and three dimensions in velocity space). This code has been extensively used in 1D-2V,\textsuperscript{15} 1D-3V,\textsuperscript{16-18} and 2D-3V\textsuperscript{19,20} configurations. Within this model, the kinetic dynamics of ions is investigated through the Vlasov-Maxwell set of equations
\[
\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla \mathbf{v} = 0,
\]

(28)
\[
\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}, \quad \nabla \times \mathbf{B} = \mathbf{J}
\]

(29)
while the electron response is taken into account through the generalized Ohm’s law as used in the eTF model (Eq. (3))
\[
\mathbf{E} = -\mathbf{U} \times \mathbf{B} + \frac{1}{n} (\mathbf{J} \times \mathbf{B}) - \frac{1}{n} \nabla P_e.
\]

(30)

The same ion normalizations as in the TF and eTF models are used. Here, \( f = f(x, v, t) \) is the ion distribution function and the ion density \( n \) and the mean velocity \( \mathbf{v} \) are computed from the velocity space integration of the moments of \( f \). As in the TF and the eTF models, the displacement current is neglected in Ampère equation since we consider the “low frequency” plasma dynamics. Quasi-neutrality is also assumed. An isothermal equation of state is chosen for the electrons, with the electron to ion temperature ratio \( T_e / T_i = 1 \).

Equations (28)–(30) are solved numerically in 1D-3V phase space configuration, using an Eulerian algorithm based on the coupling between the so-called splitting scheme\textsuperscript{21} and the current advance method\textsuperscript{22} for the evolution of the electromagnetic fields, generalized to the hybrid case.\textsuperscript{14} The length of the physical domain is \( L_y / d_t = 20 \pi \), using \( N_x = 512 \) uniformly distributed grid points, corresponding to a grid spatial mesh \( \Delta x / d_t \approx 0.12 \). Periodic boundary conditions are imposed in the spatial domain. The velocity domain range is limited by \( v_{\text{max},x} = v_{\text{max},z} = \pm 6 v_{th,i} \) (resp. \( v_{\text{max},y} = \pm 6 v_{th,i} \)) in the directions \( v_x \) and \( v_z \) (resp. \( v_y \)), using \( N_x = N_z = 51 \) (resp. \( N_y = 61 \)) uniformly distributed grid points, corresponding to a grid mesh \( \Delta v = 0.2 \) in each velocity direction. The time step \( \Delta t \) is constrained by the Courant-Friedrichs-Levy condition for the numerical stability of the hybrid Vlasov algorithm.

We have considered two different initializations, the first one is a MHD equilibrium, the second an eTF equilibrium. In both cases, the initial ion-electron plasma flow is characterized by a double velocity shear, imposed by setting the mean ion velocity to
\[
\mathbf{U} = U_0 \left[ \tanh \left( \frac{x - x_1}{L} \right) - \tanh \left( \frac{x - x_2}{L} \right) - 1 \right] \mathbf{e}_y
\]
with \( x_1 = 0.25 L_x \) and \( x_2 = 0.75 L_x \) the central positions of the two velocity shears and \( L = 3 \) their scale length. In the MHD initialization the ion distribution function is set to a drifting Maxwellian velocity distribution, using a uniform density. The plasma is embedded in a uniform background magnetic field \( \mathbf{B} = B_0 \mathbf{e}_z \). In the eTF initialization, the ion distribution function is set to
where the profile of the thermal velocities in the three directions $v^0_l(x), l = x, y, z$, is set according to the profiles of the diagonal elements of the pressure tensor in the eTF equilibrium, calculated in Sec. IV. The density profile $n(x)$ is ensured by the choice of $f(x)$ in Eq. (32). The initial magnetic field is set to $B = (B_0 + \delta B(x)) \mathbf{e}_z$, with $\delta B(x)$ the correction due to FLR effects as calculated in Sec. IV. The velocity shear layers at $x_1$ and $x_2$, Eq. (31), are characterized by a different orientation of the fluid vorticity $\Omega$ with respect to the mean magnetic field direction $\mathbf{B}$. In this case, the response of the ion distribution function is expected to be different on the two velocity shear layers. Finally, the ion plasma beta, defined as $\beta_i = 2e\mu_i/m_i$, is set to $\beta_i = 2$, corresponding to a ion Larmor radius $q_i = d_i$. The same simulations have been repeated with $\beta_i = 0.5$ and finally with the addition of a small component of the magnetic field in the direction of the flow $B_z = 0.05B_z$; in all these cases, our results were not qualitatively modified.

C. Kinetic evolution of the shear flow initializations

We discuss the self-consistent evolution of the ion distribution function initialized either with the MHD or with the eTF setup. In Fig. 2, we show the diagonal components, $\Pi_{xx}, \Pi_{yy}$ and $\Pi_{zz} = \Pi_{x}$ (top to bottom) of the ion pressure tensor vs. $x$ at different times as obtained by integration of the Hybrid Vlasov model, Eqs. (28)–(30). The left panels represent the results obtained using the MHD equilibrium as initial conditions while the right panels those obtained using the eTF initial equilibrium. In the figure, the dashed red-green line corresponds to the eTF profile. Note that the MHD initial profile corresponds to the black straight constant lines, $\Pi_{x} = 1$ while the eTF profile is agyrotropic (near the shear layer regions).

First, we observe that the plasma observables, initialized with the MHD or with the eTF set ups, oscillate around the same mean value corresponding to the eTF profile (red-green line). In particular, the plasma initialized with the MHD setup jumps very rapidly, on a time scale of the order of the ion cyclotron time to a “dynamical” state around the eTF equilibrium, becoming agyrotropic in the regions with velocity shear where now $\Pi_{xx}(x = x_1, x_2) \neq \Pi_{yx}(x = x_1, x_2) \neq \Pi_{yx}(x = x_1, x_2)$. This reaction of the system when initialized with the MHD equilibrium is not a relaxation towards a kinetic Vlasov equilibrium but a transition to a regime of undamped oscillations around a mean value. An additional important point is that the reaction of the pressure tensor changes the gradient of the velocity shear layers that readjust their slope in a different way depending on the alignment of the fluid vorticity $\Omega$ with respect to the mean magnetic field. Since the change of the shear gradient occurs in a few ion cyclotron times, this readjustment has important consequences on the growth rate of the KH instability that depends primarily on the gradient of the velocity field.

On the other hand, when initialized with the FLR setup, all diagonal components $\Pi_{ii}$ of the pressure tensor remain almost constant with very small oscillations in the vicinity of the eTF equilibrium profile. Again, the system does not relax toward a kinetic equilibrium but instead it oscillates without damping around a profile that is very close to the analytic solution of the eTF equilibrium. This is true for all the variables of interest in this system: magnetic field component $B_z$, parallel pressure $\Pi_{zz}$, density, etc. The absence of a relaxation of the system toward a kinetic equilibrium is due to the fact that the hybrid Vlasov algorithm is dissipation-free at the wave-lengths of interest.

In order to make it more evident that the system oscillates around the eTF equilibrium solution, in Fig. 3 we plot the time evolution of the two components of the perpendicular

\[
f(x, v) = F(x) \exp \left[ -\frac{1}{2} \left( \frac{v_x}{v^0_x(x)} \right)^2 + \left( \frac{v_y - U(x)}{v^0_y(x)} \right)^2 + \left( \frac{v_z}{v^0_z(x)} \right)^2 \right],
\]

FIG. 2. Initial evolution of the ion pressure tensor elements $\Pi_{xx}, \Pi_{yy}, \Pi_{zz}$ (top to bottom panels) as obtained by integration of the Vlasov equation for the ion distribution function $f_i$ in the presence of an initial velocity shear flow for the MHD setup (left panels) and for the FLR-corrected setup (right panels). The profiles are shown by black lines at different times during $0 < t \Omega_i < 30$, while the mean profile and the analytic solution of the FLR equilibrium are shown by green and red lines, respectively. The same vertical scale is used in the left and right panels to ease comparison.
pressure, $\Pi_{xx}$ and $\Pi_{yy}$, at the center of the two shear layers $x = x_1$, $x_2$, left and right frame, respectively. A very similar behavior is observed for the parallel pressure component $\Pi_{zz}$ (not shown here). The continuous and dashed black lines represent $\Pi_{xx}$ and $\Pi_{yy}$, respectively, in the case of the MHD initialization. We see that they separate immediately and oscillate around a mean value, the red line, corresponding to the eTF equilibrium. The typical amplitude of these oscillations is larger than 10% of the mean value. The same figure also shows that the same quantities obtained from the simulation with the eTF initialization oscillate around the eTF equilibrium with the same frequency. However, in this case the amplitude is reduced to much lower values, of the order of 1%. Finally, we note that the oscillations observed in the two layers are different both in amplitude and in frequency. Indeed, the frequency of the observed oscillations in Fig. 3 corresponds to the local ion cyclotron frequency after taking into account the value of the magnetic field at the shear layer position which is different in the eTF equilibrium from the value of the background magnetic field, $B_0 = 1$.

These results allow us to emphasize that, when using a MHD initialization for kinetic simulations of the KHI, a “see” of high amplitude oscillations would fill the simulation box thus strongly modifying both the evolution of the KHI, even at the linear stage, and the nature of the turbulence in the late non-linear stage of the KHI. This is particularly true in the case of periodic boundary conditions, as is the case in our numerical experiments, but the same conclusions are expected to be valid in the case of any boundary conditions that would not efficiently enable these artifact oscillations to escape the simulation box. Note that our results remain valid in the presence of a “small” in-plane component of the magnetic field, as for example $B_y = 1/20B_z$.

As mentioned before, gyroviscosity effects at the shear layer positions correspond to the ion velocity distribution function being agyrotropic, as consistent with the eTF equations. The distribution function evolves according to two competing processes: strain and gyration. The ion gyration around the magnetic field direction tends to rotate the perpendicular distribution function in an ion cyclotron time, while the strain due to the velocity shear tends to maintain the agyrotropy.

Finally, we stress that we have chosen a Vlasov Eulerian algorithm for this study because this kind of codes is characterized by a very low noise even during the non linear regime. This feature allows us to identify the kinetic response to the shear flow initializations with great accuracy. On the other hand, particle-in-cell algorithms, which are much less consuming in terms of computational resources, are much more noisy. When applied to a PIC code, the MHD initialization has been shown to be significantly modified and to be unsatisfactory, the kinetic relaxation being well above the noise level of the PIC simulations. On the contrary, we may expect the eTF initialization to be very useful for PIC simulations since the tiny oscillations of the Vlasov-Maxwell system around the eTF shear equilibrium would be of the order of the intrinsic thermal noise level of PIC simulations.

In conclusion, this kinetic investigation shows that taking into account the first order corrections in the ratio between the ion Larmor radius and the equilibrium scale lengths in the ion pressure tensor provides a sufficiently accurate starting point for kinetic simulations, at least in the case of shear flows with scale lengths not smaller than the ion Larmor radius. This will make it possible to study the kinetic evolution of shear flow configurations without introducing spurious effects that persist in time as they are not dissipated in a collisionless plasma.

VI. CONCLUSIONS

The eTF model developed in this article allows us to include microscopic ion dynamics effects related to the ion thermal Larmor radius (taken to be of the order of the ion inertial skin depth) in a TF plasma description. This model provides a viable alternative to computationally demanding kinetic simulations in multiscale collisionless plasma regimes where the evolution of global large spatial scales and the dynamics occurring at microscopic scales cannot be treated independently. In the present article, we have restricted ourselves to ion microscales and considered the limit of massless electrons. This limitation will be relaxed in a future paper in view of the interest in performing fully nonlinear fluid-type simulations involving microscopic electron scales as is the case, e.g., in collisionless magnetic reconnection events. In the derivation of the eTF equations, we have taken special care in ensuring energy conservation within the model equations and in stressing the role of the asymmetry of the gyroviscosity contribution to the ion pressure tensor under inversion of the magnetic field orientation.

The relative simplicity of the (eTF) model allowed us to obtain analytic expressions for a one-dimensional plasma equilibrium configuration in the presence of a shear flow that includes first order FLR effects and where the ion pressure tensor is explicitly agyrotropic.

Hybrid Vlasov simulations have then been performed using, as initial configuration with a shear flow, either a
standard MHD equilibrium or its “corrected” version within the eTF framework. Neither initialization corresponds to fully kinetic selfconsistent Vlasov equilibria which, however, are difficult to obtain for a plasma with a shear flow and are often too restricted in form to be of practical use.

However, we have shown that the eTF initialization provides a significant improvement as it strongly reduces the initial absence of force balance at the particle level that make MHD initializations unsuitable for “controlled” kinetic investigations. In fact, in a low-noise Vlasov simulation of a collisionless plasma, an initial absence of force balance at the particle level does not relax towards a new (even if uncontrolled) Vlasov equilibrium but leads to relatively large amplitude, long lasting oscillations. These oscillations affect the development of the phenomena of interest (such as the development of the Kelvin-Helmholtz instability driven by the velocity shear in the plasma) even in their linear phase. In the case of the eTF initialization, oscillations around the eTF equilibrium values are still present but their amplitude is strongly reduced.

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APPENDIX A: THE FLR EXPANSION OF THE PRESSURE TENSOR

The eTF model can be summarized as an extension of the so-called two-fluid model\(^\text{a}\) to which we add a gyrotrropic pressure for both species and FLR agyrotropic corrections. We also allow in principle for heat fluxes along the magnetic field lines, supposed to be almost straight (see the so-called 3 + 1 model\(^\text{b}\)).

Let us define the smallness parameter \(\varepsilon_x \equiv \Omega_x / L / \sim \omega / \Omega_x\), where \(\Omega_x = |e_x| B_0 / m_e c\) and \(q_x = v_{th,x} / \Omega_x\) are the cyclotron frequency and the Larmor radius of the \(x\)-species, respectively, \(L\) and \(\omega = v_{th,x} / L\) the hydrodynamic characteristic length and frequency, and \(v_{th,x}\) the thermal velocity of the \(x\)-species. Actually, the smallness parameter is defined as the ratio of the Larmor radius to the hydrodynamic length scale, \(\varepsilon_x \equiv q_x / L\). Then, we assume the ordering \(\beta_x \sim 1\) and \(u_x / v_{th,x} \sim 1\), from which we recover the frequency ordering \(\omega / \Omega_x \sim \varepsilon_x\). Note that this ordering is different from previous orderings, such as the one adopted by Chew, Goldberger and Low\(^\text{c}\) in which \(u / v_{th} \gg 1\) or the usual FLR ordering \(u / v_{th} \sim \varepsilon\) (and thus \(\omega / \Omega \sim \varepsilon^2\), see e.g. Rosenbluth and Simon\(^\text{d}\)). We call our ordering the eTF ordering. The full pressure tensor equation for the species \(x\), still in dimensionless form, reads

\begin{equation}
(\epsilon_{ilm} \Pi_{x,il} + \epsilon_{jlm} \Pi_{x,jl}) B_m = \sigma_x e_x \left[ \left( L / v_{th,x} \right) \frac{\partial \Pi_{x,jl}}{\partial t} + \frac{\partial \Pi_{x,jl}}{\partial x_k} \right] + \frac{\partial Q_{x,ijkl}}{\partial x_k},
\end{equation}

(A1)

where \(\Pi_{x,ij}\) is the \((ij)-component of the\) pressure tensor, \(B_m\) the \((m-component of the)\) magnetic field, \(u_{x,i}\) the \((k-component of the)\) fluid velocity, \(Q_{x,ijkl}\) the \((ijk)-component of the\) heat flux tensor, \(\sigma_x = \text{sign}(e_x)\) the sign of the charge species (and thus \(e_x^{-1} = \sigma_x\)).

In the limit of the maximal ordering, \(L / \tau \sim v_{th,x}\), we assume \(\omega / \Omega_x \sim \varepsilon_x \sim 1\) and we expand the pressure tensor and heat flux tensor in powers of \(\varepsilon_x\), i.e.

\[
\Pi_{ij} = \sum_{r=0}^{\infty} \varepsilon_x^r \Pi_{ij}^{(r)}; \quad Q_{ij} = \sum_{r=0}^{\infty} \varepsilon_x^r Q_{ij}^{(r)},
\]

(A2)

where (and hereafter where its suppression does not generate confusion) the species index \(x\) is omitted. The \(n\)-th order pressure tensor then reads:

\[
L_B [\Pi_{ij}^{(n)}] = R_u [\Pi_{ij}^{(n-1)}] + D [Q_{ij}^{(n-1)}],
\]

(A3)

where \(L_B [\Pi_{ij}^{(n)}]\) and \(R_u [\Pi_{ij}^{(n-1)}]\) and are the \(n\)-th order contributions to the left-hand side and to right-hand side of the pressure tensor equation involving only the pressure tensor and \(D [Q_{ij}^{(n-1)}]\) is the contribution to its right-hand side involving the heat flux tensor. These three operators depend only on \(B, u,\) and \(Q,\) respectively. At zero order, \(n = 0, (R_u [\Pi_{ij}^{(1)}] \equiv 0\) and \(D [Q_{ij}^{(1)}] \equiv 0\), and at first order, \(n = 1, \) we obtain

\[
(\epsilon_{ilm} \Pi_{ij}^{(0)} + \epsilon_{jlm} \Pi_{ij}^{(0)}) B_m = 0,
\]

(A4)

\[
(\epsilon_{ilm} \Pi_{ij}^{(1)} + \epsilon_{jlm} \Pi_{ij}^{(0)}) B_m = \sigma_x \left[ \frac{d \Pi_{ij}^{(0)}}{dt} + \Pi_{ij}^{(0)} \frac{\partial u_{x,k}}{\partial x_k} \right] + \left( \Pi_{ij}^{(0)} \frac{\partial u_{x,i}}{\partial x_k} + \Pi_{ij}^{(0)} \frac{\partial u_{x,j}}{\partial x_k} \right) + \frac{\partial Q_{ij}^{(0)}}{\partial x_k},
\]

(A5)

where \(\partial / \partial t + u_{x,i} \partial / \partial x_i\) has been replaced by the total time derivative \(d / dt\). Note that the pressure tensor equation, Eq. (A5), is not invariant under magnetic field inversion, \(B \rightarrow -B\).

At zero order, we recover the gyrotrropic pressure tensor
\[ \Pi^{(0)} = p_\perp (I - bb^\perp) + p_\parallel bb^\parallel \] 
\[ \Pi_{ij}^{(0)} = \Pi_{ij}^{(0)} \equiv p_\perp; \quad \Pi_j^{(0)} \equiv p_\parallel. \]  
\[ (A6) \]

Here \( b = B/B \) is the unit vector along the magnetic field and, without loss of generality, we have assumed the z-axis aligned with the magnetic field. Note that the zero-order solution for the pressure tensor is invariant under magnetic field inversion.

Let us now consider the first order equation
\[ \mathcal{L}_B[\Pi_{ij}^{(1)}] = \mathcal{R}_u[\Pi_{ij}^{(0)}] + D(Q_{(ijk)}^{(0)}), \]
\[ (A7) \]

If we assume, following the standard approach (see, e.g., Refs. 26 and 27), invariance with respect to magnetic field inversion \( B \rightarrow -B \) of \( \Pi^{(1)} \), we obtain
\[ \mathcal{L}_B[\Pi_{ij}^{(1)}] = -\mathcal{L}_B[\Pi_{ij}^{(1)}]; \quad \mathcal{R}_u[\Pi_{ij}^{(0)}] \rightarrow \mathcal{R}_u[\Pi_{ij}^{(0)}]; \]
\[ D(Q_{(ijk)}^{(0)}) \rightarrow -D(Q_{(ijk)}^{(0)}), \]
so that \( \Pi^{(1)} \) is no more solution and thus the invariance assumption is wrong. As a result, \( \Pi^{(1)} \) is not invariant with respect to the inversion \( B \rightarrow -B \). For these reasons, we introduce a coefficient that takes into account the relative orientation of the magnetic field with respect to the coordinate axes \( s_m \equiv \text{sign}(b \cdot e_m) = \text{sign}(b_{nm}) \) (such that \( s_m^{-1} = s_m \)), where \( e_m \) is the unit vector along the m-axis of the reference system. Then, by defining \( \mathcal{L}_B[\Pi_{ij}^{(1)}] \equiv s_m \mathcal{L}_B[\Pi_{ij}^{(1)}] \) and \( D(Q_{(ijk)}^{(0)}) \equiv s_m D(Q_{(ijk)}^{(0)}) \), Eq. \( (A7) \) can be now cast in a invariant form, at first-order, that correctly takes into account the gyromotion of the particles with respect to the coordinate axes, \( \mathcal{L}_B[\Pi_{ij}^{(1)}] = \mathcal{R}_u[\Pi_{ij}^{(0)}] + D(Q_{(ijk)}^{(0)}) \), which, using \( s_m^{-1} = s_m \), explicitly reads
\[ (\epsilon_{ilm} \Pi_{ij}^{(1)} + \epsilon_{ilm} \Pi_{ij}^{(1)} )B_m = s_m \sigma_s \left[ \frac{d\Pi_{ij}^{(0)}}{dt} + \Pi_{ij}^{(0)} \right] \frac{\partial u_{x_k}}{\partial x_k} + \left( \Pi_{ik}^{(0)} \frac{\partial u_{x_j}}{\partial x_k} + \Pi_{jk}^{(0)} \frac{\partial u_{x_i}}{\partial x_k} \right) \]
\[ + \sigma_s \frac{\partial Q_{(ijk)}^{(0)}}{\partial x_k}. \]
\[ (A8) \]

In order to evaluate the right-hand side of Eq. \( (A8) \), we calculate the total time derivative of the zero-order pressure tensor \( \Pi_{ij}^{(0)} \)
\[ \frac{d\Pi_{ij}^{(0)}}{dt} = \frac{d}{dt}[p_\perp \delta_{ij} + (p_\parallel - p_\perp) b_i b_j] \]
\[ = \left( \frac{dp_\perp}{dt} \right) \delta_{ij} \left( \frac{dp_\parallel}{dt} \right) b_i b_j + \left( \frac{p_\parallel - p_\perp}{|B|^2} \right) b_i b_j \]
\[ \times \left[ \left( \frac{dB_i}{dt} \right) B_j + \left( \frac{dB_j}{dt} \right) B_i - \frac{2B_i B_j}{|B|^2} \left( \frac{dB}{dt} \right) \right], \]
\[ (A9) \]

where we have introduced \( \delta_{ij} \equiv \delta_{ij} - b_i b_j \) for shortness. Then, we take the curl of the generalized Ohm’s law, Eq. \( (3) \), and retaining only terms that are consistent with the adopted ordering. In fact, remember that in this limit \( u_e = U \) and \( u_i, = U - J/n \), thus the generalized Ohm’s law reads either \( E + u_e \times B = \frac{1}{n} \nabla \cdot \Pi^{(0)} \) or \( E + u_i \times B = \frac{1}{n} (J \times B - \nabla \cdot \Pi^{(0)}). \) In both cases, after taking the curl, the righthand side is of order \( v_e \sim d_e \) and thus zero, for consistency with the massless electron limit that we adopt in the present paper. Thus, we evaluate the total time derivative of the i-component of the magnetic field:
\[ \frac{dB_i}{dt} = B_i \frac{\partial u_j}{\partial x_k} - \left( \frac{\partial u_{x_k}}{\partial x_i} \right) B_j, \]
\[ (A10) \]

where we used Eq. \( (4) \). Then, we finally rewrite \( \mathcal{R}_u[\Pi_{ij}^{(0)}] \) as
\[ \mathcal{R}_u[\Pi_{ij}^{(0)}] = \sigma_s \left\{ \left( \frac{dp_\perp}{dt} \right) b_{ij} + \left( \frac{dp_\parallel}{dt} \right) b_{ij} + \Pi_{ij}^{(0)} \left( \frac{\partial u_{x_k}}{\partial x_l} \right) \right\} + \left( \frac{p_\parallel - p_\perp}{|B|^2} \right) b_{ij} \frac{\partial u_{x_k}}{\partial x_k} \]
\[ \times \left[ b_i b_{jk} \frac{\partial u_{x_j}}{\partial x_k} + b_i b_{kj} \frac{\partial u_{x_j}}{\partial x_k} - b_j b_{ik} \frac{\partial u_{x_k}}{\partial x_i} \right], \]
\[ (A11) \]

where we used the fact that \( d|B|/dt = b_i (dB_i/dt) \). To evaluate the divergence of the zero-order heat flux tensor, we use the gyrotropic expression:\[28\]
\[ Q_{(ij)}^{(0)} = q_\perp [rb]_{ij} + q_\parallel b b b_{ij}; \quad [rb]_{ij} \equiv \tau_{ij} b_k + \tau_{jk} b_i + \tau_{ki} b_j. \]

We now project the first order equation on the two orthogonal spaces using the projectors \( b b \) and \( r \), which are the kernel of the \( \mathcal{L}_B \) operator. In this way we obtain two equations for the \( \Pi^{(0)} \) only, which determine the “secular” evolution of the zero order quantities, \( p_\parallel \) and \( p_\perp \)
\[ \frac{dp_{\parallel}}{dt} + p_{\parallel} (V \cdot u_{\parallel}) + 2p_{\parallel} (b b : V u_{\parallel}) \]
\[ + V (q_{\parallel} b) - 2q_{\perp} (V : b) = 0, \]
\[ (A12) \]
\[ \frac{dp_{\perp}}{dt} + p_{\perp} (V \cdot u_{\perp}) + p_{\perp} (r : V u_{\perp}) \]
\[ + V (q_{\perp} b) + q_{\perp} (V : b) = 0, \]
\[ (A13) \]

where we the orthogonality relationships: \( r : b b = r : b = 0 \) together with \( b b : b b = 1, \tau : r = 2 \). In deriving the heat flux terms, we used the property \( b b : V b = b_{ij} \partial_{x_j} b_i = 0 \). Note that, when written in terms of the \( U \) and \( J/n \) variables and the heat fluxes are neglected, Eqs. \( (A12) \) and \( (A13) \) coincide with Eqs. \( (5) \)–\( (8) \), except for the gyroviscosity term in the ion perpendicular pressure equation. Indeed, in order to conserve energy, we must add a second order term to the perpendicular pressure equation, Eq. \( (A13) \), when \( z = i \). Formally, such a term comes from projecting the equation for \( \Pi^{(0)} + \Pi^{(1)} \) on the two eigenspaces \( b b \) and \( r \). Neglecting the second order changes in the magnetic field direction, we obtain the following equation for the \( p_{\parallel} \) and \( p_{\perp} \) evolution with FLR corrections:
\[ \frac{dp_{\parallel}}{dt} + p_{\parallel} (V \cdot u_{\parallel}) + 2p_{\parallel} (b b : V u_{\parallel}) \]
\[ + V (q_{\parallel} b) - 2q_{\perp} (V : b) = 0 \]
\[ (A14) \]
\[ \frac{dp_{\perp}}{dt} + p_{\perp} (V \cdot u_{\perp}) + p_{\perp} (r : V u_{\perp}) \]
\[ + V (q_{\perp} b) + q_{\perp} (V : b) + \Pi^{(1)} : V u_{\perp} = 0, \]
\[ (A15) \]
where $\boldsymbol{\Pi}^{(1)}$, $\nabla \mathbf{u}_e$ term will be retained for the ions only since we are considering the massless electron limit. Retaining this term is necessary for energy conservation in the model, which is a fundamental consistency property of a model.\(^\text{30}\) We can now solve Eq. (A8) for $\Pi^{(1)}$. Without loss of generality, we assume again that the z-axis is aligned to the magnetic field, so $\mathbf{b} = \hat{z}$. Since the parallel component $\Pi^{(1)}_{z,z} = 0$. Then, using the traceless property of the first-order tensor $\mathrm{Tr}[\Pi^{(1)}] = \Pi^{(1)}_{xx} + \Pi^{(1)}_{yy} = 0$, we solve Eq. (A8) for the $\Pi^{(1)}$ components

$$\Pi^{(1)}_{x,x} = -\Pi^{(1)}_{x,y} = -\frac{s_3 s_2 p_{z\perp}}{B^2} \left( \frac{\partial u_{x,y}}{\partial y} + \frac{\partial u_{x,x}}{\partial x} \right), \quad (A16)$$

$$\Pi^{(1)}_{x,y} = \Pi^{(1)}_{y,x} = -\frac{s_3 s_2 p_{z\perp}}{B^2} \left( \frac{\partial u_{x,y}}{\partial y} - \frac{\partial u_{x,x}}{\partial x} \right), \quad (A17)$$

$$\Pi^{(1)}_{x,z} = \Pi^{(1)}_{z,x} = -\frac{s_3 s_2 p_{z\perp}}{B} \left[ \left( 2p_{z\parallel} - p_{z\perp} \right) \frac{\partial u_{x,z}}{\partial \epsilon} + p_{z\perp} \frac{\partial u_{x,z}}{\partial \epsilon} \right] - \frac{s_2 \partial u_{\perp}}{B} \frac{\partial u_{\perp}}{\partial \epsilon}, \quad (A18)$$

$$\Pi^{(1)}_{z,y} = \Pi^{(1)}_{y,z} = \frac{s_3 s_2 p_{z\perp}}{B} \left( 2p_{z\parallel} - p_{z\perp} \right) \frac{\partial u_{x,z}}{\partial \epsilon} + p_{z\perp} \frac{\partial u_{x,z}}{\partial \epsilon}, \quad (A19)$$

$$\Pi^{(1)}_{z,z} = 0. \quad (A20)$$

The above expressions for the FLR corrections take into account both the orientation of the magnetic field with respect to the z-axis and heat flux contributions. We underline that, contrary to previous works, the derivation of Eqs. (A16)–(A20) presented here accounts for the orientation of the magnetic field with respect to the z-axis via the $s_3$ coefficient, making the physical asymmetry under the inversion $\mathbf{B} \to -\mathbf{B}$ explicit. In fact, if for the sake of simplicity one neglects heat fluxes this inversion causes a change of sign in the $\Pi^{(1)}$ tensor, i.e., $\Pi^{(1)} \to -\Pi^{(1)}$. This asymmetry has a direct consequence on the dynamical evolution of the system and on its equilibrium configuration, as shown in this article and pointed out by recent kinetic simulations.\(^\text{8}\)

Finally, we remark that the expressions for the $\Pi^{(1)}$ components obtained here are derived assuming $\mathbf{B}$ along the z-axis and thus they can be considered valid as long as the magnetic field component in the xy-plane remains negligible with respect to the $z$-component, i.e., $(B_x^2 + B_y^2)^{1/2} \ll |B_z|$.

**APPENDIX B: EQUILIBRIA INCLUDING FLR CORRECTIONS COMPUTED WITH THE LOCAL FIELD**

In general, a formal treatment of the FLR correction should take into account the local value of the magnetic field in their expressions, if it is not spatially homogeneous. This corresponds to allow for a spatial dependence of the cyclotron frequency (and thus of the Larmor radius proportional to the ratio $\sqrt{T}/B$). In this case, it is easy to show that Eq. (21) should be rewritten as

$$\left( 1 - \frac{\tilde{u}'(x)}{\sqrt{H(x)} h(x)} \right) \tilde{p}_{z,0} F(x) f(x) + \tilde{p}_{z,0} G(x) g(x)$$

$$+ \frac{H(x)}{1 + \tilde{p}_{z,0} + \tilde{p}_{y,0}} h(x) = 1. \quad (B1)$$

Then, by using the same relations between $f(x)$, $g(x)$ and $h(x)$, as in Sec. IV, we obtain

$$\left( 1 - \frac{\tilde{u}'(x)}{\sqrt{H(x)}} \right) \tilde{p}_{z,0} F(x) \sqrt{f(x)} + \tilde{p}_{z,0} [F(x)f(x)]^{1/2}$$

$$+ \left[ 1 + \tilde{p}_{z,0} F(x) + \tilde{p}_{y,0} (F(x))^{1/2} \right] f(x) = 1. \quad (B2)$$

We now consider only the case $\tilde{u}' = 0$, since it is easier and supported by the simulations results. In this case, we obtain a quadratic equation for $w(x) = \sqrt{f(x)}$ with non-constant coefficients:

$$A(x)w^2(x) + B(x)w(x) - 1 = 0, \quad (B3)$$

where the coefficients are given by

$$A(x) = 1 - \tilde{p}_{z,0} F(x)$$

$$B(x) = \left( 1 - \frac{\tilde{u}'(x)}{\sqrt{H(x)}} \right) \tilde{p}_{z,0} F(x).$$

Since we are considering MHD functions such that $0 \leq F(x) \leq 1$ and $\tilde{p}_{z,0} < 1$, $A(x) > 0$ for all $x$ and we can solve for $w_{\pm}(x)$, retaining only the meaningful $w_{+}(x)$ solution

$$w_{+}(x) = \frac{1}{2A(x)} \left( \sqrt{B^2(x) + 4A(x) - B(x)} \right). \quad (B4)$$

One can easily verify that $w_{-}(x) \leq 0 \forall x$, while $w_{+}(x) > 0 \forall x$. This is based on the fact that, for all $x$, $A(x) > 0$ and $|B(x)| \leq \sqrt{B^2(x) + 4A(x)}$. We finally remark that the solution $w_{+}(x) \to 1$ for $x \to \pm \infty$ and if $\tilde{u}' = 0$ we obtain $w_{+}(x) = 1$, as to be expected.

**APPENDIX C: EXACT SELF-CONSISTENT KINETIC EQUILIBRIA**

The distribution function discussed in Ref. 13 can be rewritten as a function of the integrals of the motion $\epsilon = m(v_x^2 + v_y^2 + v_z^2)/2 + q\phi$, and $P_y = mv_y + qA_y/c$ in general terms in the form

$$f(\epsilon, P_y) = C \exp \left[-\frac{\epsilon - c_1 P_y^2}{m} - \frac{v_y P_y}{\theta} \right], \quad c_1 > -1/2,$$

where $\epsilon$ is the particle energy, $P_y$ is the y-component of the canonical momentum of a particle of mass $m$ and charge $q$; $v_y$ and $\theta$ are parameters; and $\phi(x)$ and $A_y(x)$ are the electrostatic and the y component of the vector potential, respectively. It can be rewritten as
Note that this distribution function is agyrotropic, i.e., the two perpendicular pressure components are not equal for \( c_1 \neq 0 \), consistent with the analysis in Sec. III in terms of the contributions of the diagonal terms of the gyroviscosity. Reintroducing the species index \( x \) we obtain the charge and current densities \( q \) and \( j_x \) and the \( xx \) component of the total pressure tensor \( \Pi_{xx} \) (which plays the role of the Sagdeev potential) as

\[
f(\epsilon, P_y) = C^* \exp \left[ -\frac{\epsilon}{\theta} - \frac{c_1 (P_y + m v^*/(2c_1))^2}{m\theta} \right].
\]

The well known Harris distribution \(^{11} \) (with in general a non-vanishing electrostatic potential) is obtained by setting \( c_1 = 0 \). In terms of the particle velocities and density, the distribution function can be rewritten as

\[
f(v_x, v_y, v_z, x) = \frac{n_0 (1 + 2c_1)^{1/2}}{(2\theta/m)^{3/2}} \exp \left\{ -\frac{q\phi + [1/(1 + 2c_1)] [c_1 (qA_x)^2/(mc^2) + qA_v v^*/c]}{\theta} \right. \right. \times \exp \left. \left[ \frac{m (v_x^2 + v_y^2)}{2\theta} \right] \right. \exp \left. \left[ -\frac{m (1 + 2c_1)(v_z - V_{z0})^2}{2\theta} \right] \right. \right. , \text{ with } V_{z0} = -\frac{2c_1 qA_y}{mc(1 + 2c_1)} - \frac{v^*}{1 + 2c_1}.
\]

More general distributions can be constructed following the procedure devised in Ref. 32 by using a proper (positive) superposition of distribution functions with different values of the parameters \( c_1 \) and \( v^* \). This can lead to a Sagdeev potential \( \Pi_{xx}(\phi, A_x) \) that has the required dependence on \( \phi \) and \( A_x \) in order to obtain through Poisson and Ampère equations a physically relevant equilibrium configuration.

An interesting case is obtained by considering a neutral plasma with no electric field, \( \phi = 0 \), and \( v^* = 0 \). This implies that \( n_0 \exp[-c_1^2 (qA_x)^2/(mc^2 \theta_x)] \) must be independent of \( x \) for all \( x \), i.e., \( n_0 \epsilon = n_0 \) and \( c_1^2 A_x^2/(mc^2 \theta_x) = c_1^2 A_y^2/(mc^2 \theta_y) \). If we take equal temperatures, \( \theta_x = \theta_y \), we obtain equal and opposite \( x \)-dependent velocities for the two species that lead to a fluid velocity \( U_x(x) \) and to a current density \( j_x(x) \). If we require the magnetic field to be even in \( x \), \( B_x^2 \) must be sufficiently large compared to \( \Pi_{xx} \). In this case, the vector potential \( A_x(x) \) is odd in \( x \) and so is the fluid velocity \( U_x(x) \). Note that in this equilibrium the plasma flow is not due to an \( E \times B \) drift and is fully related to the agyrotropicity of the distribution function.


\(^18\) L. Marradi, F. Valentini, and F. Califano, EPL 92, 49002 (2010).


