Continuum Mechanics

Lecture 5

Ideal fluids

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Outline

- Helmholtz decomposition
- Divergence and curl theorem
- Kelvin's circulation theorem
- The vorticity equation
- Vortex dynamics and vortex flow
- Bernoulli theorem and applications

Helmholtz decomposition of the velocity field

For a continuous and differentiable velocity field, we have the following unique decomposition:

$$\vec{v} = \vec{\nabla}\phi + \vec{\nabla} \times \vec{A} \qquad \text{with Gauge condition} \quad \vec{\nabla} \cdot \vec{A} = 0$$

The scalar and vector potential are solutions of $\Delta\phi = \vec{\nabla} \cdot \vec{v}$ and $\Delta\vec{A} = -\vec{\nabla} \times \vec{v}$

with appropriate boundary conditions.

The source terms for these 2 Poisson equations are respectively

$\vec{\nabla} \cdot \vec{v}$: the divergence of the velocity field

$\vec{\nabla} \times \vec{v}$: the curl of the velocity field

Limiting cases:

1- $\vec{\nabla} \cdot \vec{v} = 0$ for an *incompressible flow*. The velocity field is solenoidal or divergence free.

2- $\vec{\nabla} \times \vec{v} = 0$ for a *potential flow*, because in this case the velocity field derives from a scalar potential. The velocity is said to be curl free.

Physical interpretation of the divergence

We have seen in the previous lecture that the variation of a Lagrangian volume is given by

$$\frac{dV_t}{dt} = \int_{V_t} \mathrm{d}x^3 \vec{\nabla} \cdot \vec{v}$$

The rate of change of the specific volume $V = 1/\rho$ is $\frac{1}{V}\frac{DV}{Dt} = \vec{\nabla} \cdot \vec{v}$

Using the divergence theorem, we can express the total volume variation as the net flux of volume across the outer surface as:

$$\frac{dV_t}{dt} = \int_{S_t} \vec{v} \cdot \vec{n} \mathrm{d}S$$

Let's consider the case of a point source (or sink) of divergence at r=0.

$$\vec{\nabla} \cdot \vec{v} = Q \ \delta(\vec{x} = 0)$$

We have a spherically symmetric velocity field around the source. Using the divergence theorem, we have: $Q = 4\pi r^2 v_r$

A source (or sink) velocity field is thus $\boxed{\vec{v} = \frac{Q}{4\pi r^2}\vec{e_r}}$

Physical interpretation of the curl

We define the vorticity as the following vector field $\vec{\omega} = \vec{\nabla} \times \vec{v}$

In components, we have $\vec{\omega} = \begin{vmatrix} \partial_y v_z - \partial_z v_y \\ \partial_z v_x - \partial_x v_z \\ \partial_x v_y - \partial_y v_x \end{vmatrix}$

For a rigid body motion $\vec{v} = \vec{v}_0 + \vec{\Omega} \times \vec{r}$, we have $\vec{\omega} = \vec{\nabla} \times \vec{\Omega} \times \vec{r} = 2\vec{\Omega}$

The vorticity is thus twice the local rotation rate in the fluid.

A vortex is a vorticity line along the axis $\vec{t} = \vec{\omega}/|\vec{\omega}|$

We introduce Stoke's theorem or curl's theorem.

We define the circulation Γ as the integral of the parallel velocity along a closed contour.

$$\Gamma = \int_L \vec{v} \cdot \mathrm{d}\vec{l} = \int_L v_{\parallel} \mathrm{d}l$$

We have the following identity $\boxed{\Gamma = \int_L \vec{v} \cdot \mathrm{d}\vec{l} = \int_S (\vec{\nabla} \times \vec{v}) \cdot \vec{n} \mathrm{d}S}$

Let us consider a vortex line at r=0. $\vec{\nabla} \times \vec{v} = \Omega_z \vec{e}_z \; \delta(\vec{r} = 0)$

Using Stoke's theorem, we have $\Gamma = 2\pi r v_\theta = \Omega_z$

A vortex velocity field is thus $\boxed{\vec{v} = \frac{\Omega_z}{2\pi r} \vec{e}_\theta}$

Velocity field induced by a vortex distribution

This is the *Biot-Savart law for vortices.*

We consider an incompressible fluid for which $\vec{\nabla} \cdot \vec{v} = 0$

We know from the Helmholtz decomposition that $\vec{v} = \vec{\nabla} \times \vec{A} + \vec{\nabla}\phi$

together with the Gauge condition $\vec{\nabla} \cdot \vec{A} = 0$

The potential vector satisfies the Poisson equation $\Delta \vec{A} = -\vec{\omega}$

The solution is

$$\boxed{\vec{v}(\vec{x}) = -\frac{1}{4\pi}\int_V \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3} \times \vec{\omega}(\vec{x}')\mathrm{d}x'^3}$$

We need to add the scalar potential contribution, solving $\Delta\phi = 0$

with the appropriate boundary conditions (see lecture on potential flows).

We consider a filament of vorticity $\vec{\omega}(\vec{x}') = \Omega\delta(r = 0)$ using the curvilinear coordinate *s*.

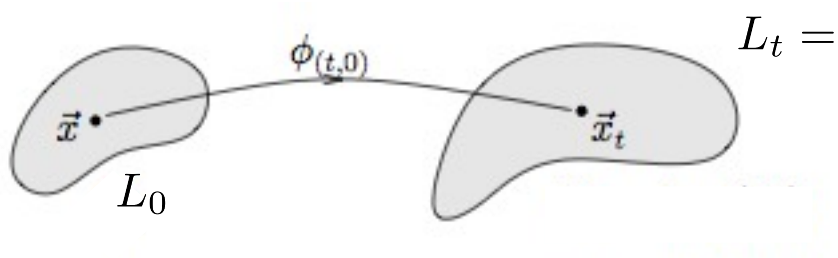
$$\vec{v}(\vec{x}) = -\frac{\Omega}{4\pi}\int_L \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3} \times \vec{t}(\vec{x}')\mathrm{d}s$$

For a vertical filament, we have $\vec{x} - \vec{x}' = r\vec{e}_r + (z - z')\vec{e}_z$ with $\vec{t} = \vec{e}_z$

and $\vec{e}_r \times \vec{e}_z = -\vec{e}_\theta$

$$\boxed{\vec{v}(\vec{x}) = \frac{\Omega}{4\pi}\int_{-\infty}^{+\infty} \frac{r\mathrm{d}z}{(r^2 + z^2)^{3/2}}\vec{e}_\theta = \frac{\Omega}{2\pi r}\vec{e}_\theta}$$

Kelvin's circulation theorem



$$L_t = \phi_{(t,0)}(L_0)$$

We consider a closed contour evolving with the flow. We use the inverse Lagrangian mapping to compute the time derivative of the circulation.

$$\frac{d}{dt}\Gamma(t) = \frac{d}{dt}\int_{L_t} \vec{v}\cdot \mathrm{d}\vec{x} = \frac{d}{dt}\int_{L_0} \vec{v}(\vec{x}(\vec{y},t),t)\cdot(\frac{\partial \vec{x}}{\partial \vec{y}}\mathrm{d}\vec{y})$$

$$\frac{d}{dt}\Gamma(t) = \int_{L_0} \frac{D\vec{v}}{Dt}\cdot(\frac{\partial \vec{x}}{\partial \vec{y}}\mathrm{d}\vec{y}) + \int_{L_0} \vec{v}\cdot(\frac{\partial \vec{v}}{\partial \vec{y}}\mathrm{d}\vec{y}) = \int_{L_t} \frac{D\vec{v}}{Dt}\cdot \mathrm{d}\vec{x} \qquad \int_{L_0} \vec{v}\cdot \mathrm{d}\vec{v} = \left[\frac{v^2}{2}\right] = 0$$

We now inject the Euler equation for an ideal fluid $\frac{D\vec{v}}{Dt} = \vec{F} - \frac{1}{\rho}\vec{\nabla}P$

$$\frac{d}{dt}\Gamma(t) = \int_{L_t} \vec{F}\cdot \mathrm{d}\vec{x} + \int_{S_t} \frac{1}{\rho^2}\vec{\nabla}\rho \times \vec{\nabla}P \cdot \vec{n}\mathrm{d}S$$

If the external force derives from a potential $\vec{F} = -\vec{\nabla}\Phi$

and if the fluid is barotropic $\frac{1}{\rho}\vec{\nabla}P = \vec{\nabla}\Pi$ then $\frac{d}{dt}\Gamma = 0$

Lagrange theorem: if initially the vorticity is zero, then it remains zero everywhere.

The vorticity equation

We start with the Euler equation for ideal fluids $$\frac{D\vec{v}}{Dt} = \vec{F} - \frac{1}{\rho}\vec{\nabla}P$$

Taking the curl leads to $$\frac{\partial\vec{\omega}}{\partial t} + \vec{\nabla} \times (\vec{v}\cdot\vec{\nabla}\vec{v}) = \vec{\nabla} \times \vec{F} - \vec{\nabla} \times (\frac{1}{\rho}\vec{\nabla}P)$$

Using the identity $\vec{v}\cdot\vec{\nabla}\vec{v} = \vec{\nabla}(\frac{v^2}{2}) + \vec{\omega}\times\vec{v}$ we have

$$\boxed{\frac{\partial\vec{\omega}}{\partial t} + \vec{\nabla} \times (\vec{\omega}\times\vec{v}) = \vec{\nabla} \times \vec{F} + \frac{1}{\rho^2}\vec{\nabla}\rho \times \vec{\nabla}P}$$

Using the identity $\vec{\nabla} \times (\vec{\omega}\times\vec{v}) = (\vec{\nabla}\cdot\vec{v})\vec{\omega} + (\vec{v}\cdot\vec{\nabla})\vec{\omega} - (\vec{\omega}\cdot\vec{\nabla})\vec{v}$

we find the vorticity equation:

$$\frac{D\vec{\omega}}{Dt} = (\vec{\omega}\cdot\vec{\nabla})\vec{v} - (\vec{\nabla}\cdot\vec{v})\vec{\omega} + \vec{\nabla} \times \vec{F} + \frac{1}{\rho^2}\vec{\nabla}\rho \times \vec{\nabla}P$$

For a barotropic fluid under gravity, we have $$\boxed{\frac{D}{Dt}\left(\frac{\vec{\omega}}{\rho}\right) = \left(\frac{\vec{\omega}}{\rho}\cdot\vec{\nabla}\right)\vec{v}}$$

Helmholtz theorem: vortex lines move with the fluid.

Proof: a line element that moves with the fluid satisfies $$\frac{D}{Dt}\left(\delta\vec{\ell}\right) = (\delta\vec{\ell}\cdot\vec{\nabla})\vec{v}$$

Vortex dynamics

For a barotropic fluid, the vorticity equation writes in component form:

$$\frac{D\omega_i}{Dt} = \omega_x \partial_x v_i + \omega_y \partial_y v_i + \omega_z \partial_z v_i - (\vec{\nabla} \cdot \vec{v})\omega_i$$

Let's consider a vertical vortex line $\vec{\omega} = \omega_z \vec{e}_z$

$$\underbrace{\frac{D\omega_x}{Dt} = \omega_z \partial_z v_x \qquad \frac{D\omega_y}{Dt} = \omega_z \partial_z v_y}_{\text{vortex tilting due to shear}} \qquad \underbrace{\frac{D\omega_z}{Dt} = -\omega_z \left(\partial_x v_x + \partial_y v_y\right)}_{\text{vortex stretching due to 2D divergence}}$$

vortex tilting due to shear   vortex stretching due to 2D divergence

The 2D divergence is the rate of change of the section of the vortex tube

$$\frac{1}{S}\frac{DS}{Dt} = \left(\partial_x v_x + \partial_y v_y\right)$$

For a 2D velocity field, the total vorticity in the vortex tube is conserved.

$$\omega_z S = \text{constant}$$

First Bernoulli Theorem

We start with the Euler equations in Lagrangian form $\frac{D\vec{v}}{Dt} = -\vec{\nabla}\Phi - \frac{1}{\rho}\vec{\nabla}P$

with equations for the thermodynamical variables

$$\frac{D\rho}{Dt} = -\rho\vec{\nabla}\cdot\vec{v} \qquad \frac{D\epsilon}{Dt} = -\frac{P}{\rho}\vec{\nabla}\cdot\vec{v}$$

Multiplying by velocity and defining the specific enthalpy as $h = \epsilon + \frac{P}{\rho}$, we have

$$\frac{D}{Dt}\left(\frac{v^2}{2}\right) = -\vec{v}\cdot\vec{\nabla}\Phi - \frac{\vec{v}}{\rho}\cdot\vec{\nabla}P$$

and $\frac{D}{Dt}(\Phi) = \frac{\partial\Phi}{\partial t} + \vec{v}\cdot\vec{\nabla}\Phi \qquad \frac{D}{Dt}(h) = \frac{1}{\rho}\frac{\partial P}{\partial t} + \frac{\vec{v}}{\rho}\cdot\vec{\nabla}P$

Collecting everything, we have the following relation:

$$\boxed{\frac{D}{Dt}\left(\frac{v^2}{2} + \Phi + h\right) = \frac{1}{\rho}\frac{\partial P}{\partial t} + \frac{\partial\Phi}{\partial t}}$$

Theorem follows trivially: *in a stationary flow, the total enthalpy* $H = \frac{v^2}{2} + \Phi + h$ *is conserved along streamlines.*

Validity: no viscosity, no dissipation (reversible isentropic flow)

Second Bernoulli Theorem

We consider a *curl free* flow $\vec{v} = \vec{\nabla}\phi$ in a *barotropic* fluid $\frac{1}{\rho}\vec{\nabla}P = \vec{\nabla}\Pi$

Using the now well known vector relation $\vec{v}\cdot\vec{\nabla}\vec{v} = \vec{\nabla}(\frac{v^2}{2}) + \vec{\omega}\times\vec{v}$

the Euler equation becomes $\vec{\nabla}\left(\frac{\partial\phi}{\partial t} + \frac{v^2}{2} + \Phi + \Pi\right) = 0$

The theorem follows:

For a potential flow, we have everywhere in the flow (not only along streamlines):

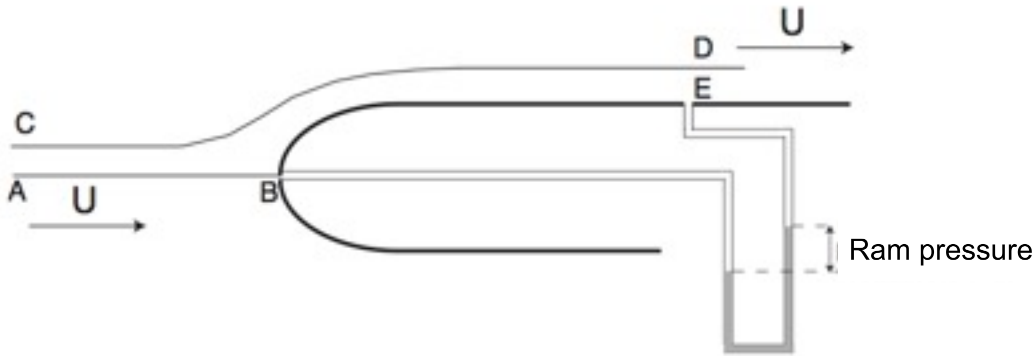
$$\boxed{\frac{\partial\phi}{\partial t} + \frac{v^2}{2} + \Phi + \Pi = \mathcal{C}(t)}$$

The constant depends only on time. The flow doesn't have to be stationary.

For a stationary flow, the quantity $H = \frac{v^2}{2} + \Phi + \Pi$ is uniform everywhere.

For a curl free incompressible fluid, we have $\boxed{H = \frac{v^2}{2} + \Phi + \frac{P}{\rho}}$

Application of the Bernoulli Theorem: Pitot tube



We would like to measure the velocity of the fluid at infinity.

We consider a probe with section AC equal to section ED.

The flow is stationary and incompressible: $\rho\frac{v^2}{2} + P = \mathrm{constant}$

Mass conservation implies $v_A S_A = v_D S_D$ so that $v_A = v_D = v_\infty$

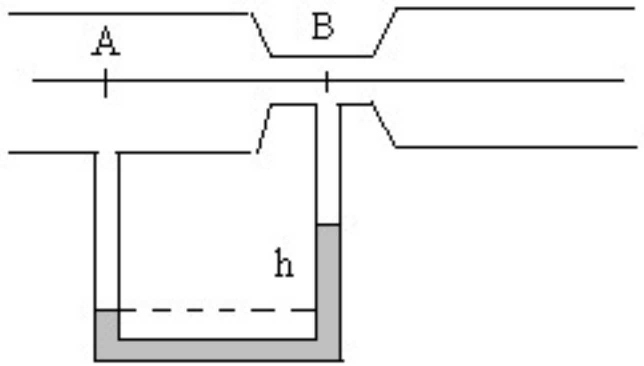
Point B, however, is a stagnation point with $v_B = 0$.

We conclude that $P_B = \rho\frac{v_\infty^2}{2} + P_\infty$. Using the probe, we measure $\Delta P = P_B - P_\infty$

The velocity is just $\boxed{v_\infty = \sqrt{\frac{2\Delta P}{\rho}}}$ and ΔP is called the *ram pressure*.

These probes (also called *Pitot tube*) are used in planes to measure the velocity.

Application of the Bernoulli Theorem: Venturi tube



We would like to measure the incoming velocity in a pipe.

We modify slightly the section of the pipe around point B.

Mass conservation implies $v_A S_A = v_B S_B$.

Bernoulli theorem implies $\rho \frac{v_B^2}{2} + P_B = \rho \frac{v_A^2}{2} + P_A$

Assuming that $S_B = S_A(1-\epsilon)$, if we measure ΔP, we have:

$$v_A = \sqrt{\frac{\Delta P}{\rho \epsilon}}$$

This probe is called a *Venturi tube*.

Hugoniot theorem

For an stationary incompressible fluid, mass conservation implies $vS = \text{constant}$.

If the section decreases, the velocity increases $\frac{dv}{v} = -\frac{dS}{S}$.

For a compressible fluid, we now have $\rho v S = \text{constant}$.

$$\frac{dv}{v} + \frac{d\rho}{\rho} = -\frac{dS}{S}$$

The stationary Euler equation gives us $v dv = -\frac{1}{\rho} dP$.

Introducing the sound speed $c^2 = \frac{dP}{d\rho}$,

combining the 2 equations results in

$$\boxed{\frac{dv}{v}\left(1 - \frac{v^2}{c^2}\right) = -\frac{dS}{S}}$$

The dimensionless number $\mathcal{M} = \frac{v}{c}$ is called the *Mach number of the flow*.

If $\mathcal{M} < 1$, the fluid behaves qualitatively like an incompressible fluid.

If $\mathcal{M} > 1$, it is reversed: the velocity will increase if the section increases.