Continuum Mechanics

Lecture 9

Magnetized flows

Prof. Romain Teyssier

http://www.itp.uzh.ch/~teyssier

Universität Zürich

Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Outline

- Maxwell equations
- The equations of Magneto-Hydrodynamique (MHD)
- Generation of magnetic fields
- Ideal MHD equations
- Dynamics of magnetic flux tube
- Conservation of magnetic flux
- Magnetic stress
- The MHD equations in conservative form
- Alfvén waves
- Effect of magnetic resistivity

Maxwell equations in a plasma

Maxwell equations for the evolution of the magnetic field B and the electric field E.

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \cdot \vec{E} = 4\pi q \qquad q = \sum_\alpha n_\alpha q_\alpha$$

$$\frac{1}{c}\partial_t \vec{B} = -\vec{\nabla} \times \vec{E} \qquad \frac{1}{c}\partial_t \vec{E} + \frac{4\pi}{c}\vec{J} = \vec{\nabla} \times \vec{B} \qquad \vec{J} = \sum_\alpha n_\alpha q_\alpha \vec{v}_\alpha$$

$$\partial_t q + \vec{\nabla} \cdot \vec{J} = 0$$

q is the charge density, J is the current density.

The electromagnetic field is tightly coupled to the charge density and the current density.

They are present in a plasma: a ionized fluid, with mostly charged particles (electrons and ions). The most common plasma in the universe is the Hydrogen plasma, with protons of mass m_p and charge +e and electrons of mass m_e and charge -e.

We decompose the plasma into 2 fluids (electrons and protons).

Each particle is accelerated by the Lorentz force: $m_\alpha \frac{D\vec{v}_\alpha}{Dt} = q_\alpha \left(\vec{E} + \frac{1}{c}\vec{v}_\alpha \times \vec{B} \right)$

Plasma fluid equations

For the electron and proton fluids, we have the following Euler equations:

$$\rho_e \frac{D\vec{v}_e}{Dt} = -\vec{\nabla} p_e - n_e e \left(\vec{E} + \frac{1}{c} \vec{v}_e \times \vec{B} \right) - \frac{n_e m_e}{\tau_{ei}} \left(\vec{v}_e - \vec{v}_i \right)$$

$$\rho_i \frac{D\vec{v}_i}{Dt} = -\vec{\nabla} p_i + n_i e \left(\vec{E} + \frac{1}{c} \vec{v}_i \times \vec{B} \right) + \frac{n_e m_e}{\tau_{ei}} \left(\vec{v}_e - \vec{v}_i \right)$$

The last term on the RHS is the friction between electrons and ions due to electron-ion microscopic collisions.

Because $m_e \ll m_i$, the inertial term in the electron fluid equation is removed (electrons dynamical equilibrium).

$$\vec{E} + \frac{1}{c} \vec{v}_e \times \vec{B} = -\frac{1}{n_e e} \vec{\nabla} p_e + \eta \vec{J}$$

where we define the resistivity coefficient $\eta = \frac{m_e}{n_e e^2 \tau_{ei}}$

The proton velocity becomes the reference velocity: it carries most of the mass.

$$\boxed{\vec{E} + \frac{1}{c} \vec{v} \times \vec{B} = -\frac{1}{n_e e} \vec{\nabla} p_e + \frac{1}{n_e e c} \vec{J} \times \vec{B} + \eta \vec{J}}$$

Non-relativistic limit and charge neutrality

We consider *non-relativistic flows* $v \ll c$

We perform an order of magnitude analysis on the Maxwell equations.

From the previous equation $\vec{E} \simeq -\frac{1}{c}\vec{v} \times \vec{B}$, we get $E \simeq \frac{v}{c}B \simeq \frac{L}{cT}B$

In the Maxwell equation $\frac{1}{c}\partial_t \vec{E} + \frac{4\pi}{c}\vec{J} = \vec{\nabla} \times \vec{B}$, we have:

$\frac{1}{c}\partial_t \vec{E} \simeq \frac{E}{cT} \simeq \frac{L^2}{c^2T^2}\frac{B}{L}$ while $\vec{\nabla} \times \vec{B} \simeq \frac{B}{L}$.

$$\boxed{\frac{4\pi}{c}\vec{J} = \vec{\nabla} \times \vec{B}}$$

From this, we get $\vec{\nabla} \cdot \vec{J} = 0$ and $\partial_t q = 0$.

This is consistent with the additional approximation of *charge neutrality*,

which is valid for length scales larger than the Debye length.

$$L \gg \lambda_D = \sqrt{\frac{k_B T_e}{n_e e^2}}$$

We get $q \ll n_e e$ and $\boxed{n_e \simeq n_i}$

The non-ideal MHD equations

Injecting the electric field equation into the ion Euler equation gives:

$$\boxed{\rho \frac{D\vec{v}}{Dt} = -\overrightarrow{\nabla}(p_e + p_i) + \frac{1}{c}\overrightarrow{J} \times \overrightarrow{B}}$$

together with the magnetic field and current density equations

$$\boxed{\partial_t \overrightarrow{B} = -c\overrightarrow{\nabla} \times \overrightarrow{E}} \qquad \boxed{\frac{4\pi}{c}\overrightarrow{J} = \overrightarrow{\nabla} \times \overrightarrow{B}}$$

where the electric field in the general case is given by

$$\boxed{\overrightarrow{E} = -\frac{1}{c}\vec{v} \times \overrightarrow{B} + \frac{1}{n_e e}\overrightarrow{\nabla} p_e + \frac{1}{n_e e c}\overrightarrow{J} \times \overrightarrow{B} + \eta \overrightarrow{J}}$$

induction term

thermoelectric effect

Hall effect

Ohmic dissipation

The is no general rule to remove those non-ideal terms.

We must check a posteriori.

Generation of magnetic fields

For very weak magnetic fields, like in the early universe, the dominant non-ideal term in the electric field equation is the electron pressure gradient.

$$\vec{E} = -\frac{1}{c}\vec{v} \times \vec{B} + \frac{1}{n_e e}\vec{\nabla} p_e$$

The induction equation $\partial_t \vec{B} = -c\vec{\nabla} \times \vec{E}$

becomes

$$\partial_t \vec{B} + \vec{\nabla} \times (\vec{B} \times \vec{v}) = \frac{c}{n_e^2 e}\vec{\nabla} n_e \times \vec{\nabla} p_e$$

The vorticity equation we have derived for an ideal fluid reads

$$\frac{\partial \vec{\omega}}{\partial t} + \vec{\nabla} \times (\vec{\omega} \times \vec{v}) = \frac{1}{\rho^2}\vec{\nabla} \rho \times \vec{\nabla} P$$

In both cases, the field is generated when pressure and density gradients are misaligned. For a barotropic evolution, no field is generated.

What is striking here is that both fields are evolving in a tightly coupled manner. If they are both initially zero, they subsequently evolve proportionally to each other.

$$\boxed{\vec{B} = \frac{c m_p}{e}\vec{\omega}}$$

Ideal MHD equations

In the ideal MHD limit, the electric field is given by $\vec{E} = -\frac{1}{c}\vec{v} \times \vec{B}$

and the magnetic field evolves according to the induction equation:

$$\boxed{\partial_t \vec{B} = \vec{\nabla} \times (\vec{v} \times \vec{B})}$$

with the divergence-free constraint: $\boxed{\vec{\nabla} \cdot \vec{B} = 0}$

The mass conservation is the same as for the fluid equations

$$\boxed{\frac{D\rho}{Dt} + \rho\vec{\nabla} \cdot \vec{v} = 0}$$

The Euler equation differ from the fluid's one by the Lorentz force

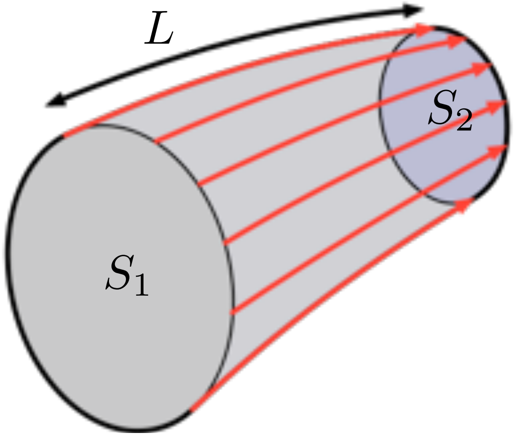
$$\boxed{\rho\frac{D\vec{v}}{Dt} = -\vec{\nabla}p + \frac{1}{4\pi}(\vec{\nabla} \times \vec{B}) \times \vec{B}}$$

In case we have an isothermal or barotropic fluid, the system is closed by the equation of state p(ρ). Otherwise, we have to add the energy equation

$$\boxed{\rho\frac{D\vec{\epsilon}}{Dt} = -p\vec{\nabla} \cdot \vec{v}}$$

with the fluid Equation of State: $\boxed{p = p(\rho, \epsilon)}$

Magnetic flux conservation



A flux tube is defined as a cylindrical surface tangent to the field lines.

Divergence theorem : $$\int_{S_1} \vec{B} \cdot \vec{n} \mathrm{d}S + \int_{S_2} \vec{B} \cdot \vec{n} \mathrm{d}S = \int_V \vec{\nabla} \cdot \vec{B} \mathrm{d}V = 0$$

We have along the flux tube: $$\boxed{\int_S \vec{B} \cdot \vec{n} \mathrm{d}S = \phi = \mathrm{constant}}$$

In the Sun, magnetic tubes are rising buoyantly above the surface (magnetic arches) and sometimes are ejected (Coronal Mass Ejection), causing intense solar eruptions.

The «frozen-in» theorem

We now follow the surface element as it evolves in time and space from $S_1 = S(t)$ to $S_2 = S(t+dt)$

We use the cylindrical volume swept by the surface element. The cylinder vertical surface is $S_3 = L(t)|\vec{v}|dt$

Since $\vec{\nabla} \cdot \vec{B} = 0$ at all times, we have

$$-\phi_1(t+\Delta t) + \phi_2(t+\Delta t) + \phi_3 = 0$$

The flux variation is $\displaystyle \frac{d\phi}{dt} = \lim_{\Delta t \to 0} \frac{\phi_2(t+\Delta t) - \phi_1(t)}{\Delta t}$

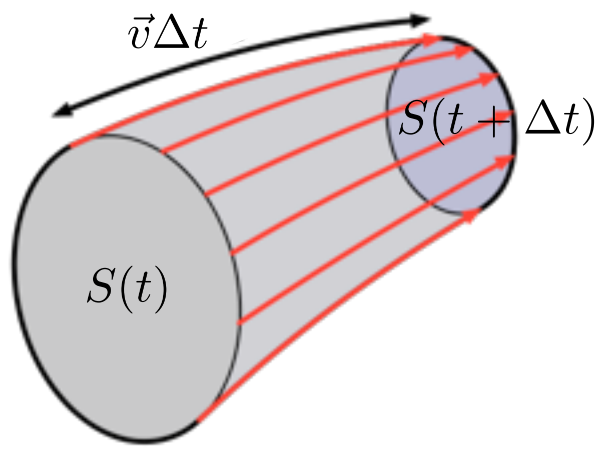
so we get $\displaystyle \frac{d\phi}{dt} = \lim_{\Delta t \to 0} \frac{\phi_1(t+\Delta t) - \phi_1(t) - \phi_3}{\Delta t}$

The first term on the RHS is just $\displaystyle \int_{S_1} \partial_t \vec{B} \cdot \vec{n} \mathrm{d}S$

The second term is $\displaystyle \phi_3 = \int_{S_3} \vec{B} \cdot \vec{n} \mathrm{d}S \simeq \int_{L(t)} \vec{B} \cdot (\vec{t} \times \vec{v}) \mathrm{d}l \Delta t$

Geometrically, $\vec{B} \cdot (\vec{t} \times \vec{v}) = (\vec{v} \times \vec{B}) \cdot \vec{t}$ so that we get

$$\phi_3 = \int_{L(t)} (\vec{v} \times \vec{B}) \cdot \vec{t} \mathrm{d}l \Delta t = \int_{S_1} \vec{\nabla} \times (\vec{v} \times \vec{B}) \mathrm{d}S \Delta t$$



$$\boxed{\frac{d\phi}{dt} = \int_{S_1} \left[\partial_t \vec{B} - \vec{\nabla} \times (\vec{v} \times \vec{B})\right] \mathrm{d}S = 0}$$

Magnetic field lines are frozen in the plasma flow. This property is similar to the dynamics of vortex tubes.

Magnetic pressure and magnetic tension

The Lorentz force writes $\vec{F} = \frac{1}{c}\vec{J} \times \vec{B}$ where the current is $\vec{J} = \frac{c}{4\pi}\vec{\nabla} \times \vec{B}$

The force can be written as $4\pi\vec{F} = -\vec{\nabla}\left(\frac{B^2}{2}\right) + \left(\vec{B}\cdot\vec{\nabla}\right)\vec{B}$

We use curvilinear coordinates along the field lines so that $\vec{B} = B\vec{t}$

We have $\left(\vec{B}\cdot\vec{\nabla}\right)\vec{B} = B\frac{\partial}{\partial s}(B\vec{t}) = \partial_s\left(\frac{B^2}{2}\right)\vec{t} + \frac{B^2}{R}\vec{n}$

Finally, the total force acting on the fluid element is given by

$$\boxed{\rho\frac{D\vec{v}}{Dt} = -\vec{\nabla}(p+p_B) + (\partial_s p_B)\,\vec{t} + \left(\frac{2p_B}{R}\right)\vec{n}}$$

where $p_B = \frac{B^2}{8\pi}$ is the magnetic pressure.

The first term on the RHS acts on the fluid as an isotropic pressure, similar to the gas pressure, but whose relative magnitude is controlled by the β parameter

$$\boxed{\beta = \frac{p}{p_B} = \frac{8\pi p}{B^2}}$$

The third term on the RHS is analogous to a tension force $T = 2p_B$ perpendicular to the field lines and works against flow motions to restore small curvature.

It is called *magnetic tension*.

The Maxwell stress tensor

The Lorentz force writes $\vec{F} = \frac{1}{c}\vec{J} \times \vec{B}$ where the current is $\vec{J} = \frac{c}{4\pi}\vec{\nabla} \times \vec{B}$

The force can be written as $4\pi\vec{F} = -\vec{\nabla}\left(\frac{B^2}{2}\right) + \left(\vec{B}\cdot\vec{\nabla}\right)\vec{B}$

Using the relation $\vec{\nabla}\left(B_x\vec{B}\right) = B_x\left(\vec{\nabla}\cdot\vec{B}\right) + \left(\vec{B}\cdot\vec{\nabla}\right)B_x,$

we can write the Lorentz force as $\vec{F} = \vec{\nabla}\cdot\overline{\overline{\mathcal{M}}}$

where we introduce the Maxwell tensor defined as

$$\boxed{\mathcal{M}_{ij} = \frac{1}{4\pi}\left(B_iB_j - \frac{B^2}{2}\delta_{ij}\right)}$$

Note that the Maxwell tensor is symmetric.

The Euler equation for ideal MHD can thus be written in conservative form as

$$\boxed{\partial_t(\rho\vec{v}) + \vec{\nabla}\cdot\left(\rho\vec{v}\vec{v} + P\overline{\overline{1}}\right) - \vec{\nabla}\cdot\overline{\overline{\mathcal{M}}} = 0}$$

$\vec{\nabla}\cdot\overline{\overline{\mathcal{M}}}$ is the Maxwell stress field, the magnetic force per unit area acting on the fluid.

Ideal MHD equation in 1D

In one dimension, the equations write in conservative form (we use 4π=1 here):

$$\partial_t \rho + \partial_x(\rho v_x) = 0$$

$$\partial_t(\rho v_x) + \partial_x(\rho v_x^2 + P + \frac{B^2}{2} - B_x^2) = 0$$

$$\partial_t(\rho v_y) + \partial_x(\rho v_x v_y - B_x B_y) = 0$$

$$\partial_t(\rho v_z) + \partial_x(\rho v_x v_z - B_x B_z) = 0$$

$$B_x = \text{constant}$$

$$\partial_t B_y + \partial_x(v_x B_y - v_y B_x) = 0$$

$$\partial_t B_z + \partial_x(v_x B_z - v_z B_x) = 0$$

Ideal MHD equation in 1D

We now write the *isothermal* MHD equations in quasi-linear form $\partial_t W + \overline{\overline{A}}\partial_x W = 0$

$$\partial_t \rho + v_x \partial_x \rho + \rho \partial_x v_x = 0$$

$$\partial_t v_x + v_x \partial_x v_x + \frac{a^2}{\rho}\partial_x \rho + \frac{B_y}{\rho}\partial_x B_y + \frac{B_z}{\rho}\partial_x B_z = 0$$

$$\partial_t v_y + v_x \partial_x v_y - \frac{B_x}{\rho}\partial_x B_y = 0$$

$$\partial_t v_z + v_x \partial_x v_z - \frac{B_x}{\rho}\partial_x B_z = 0$$

$$\partial_t B_y + v_x \partial_x B_y + B_y \partial_x v_x - B_x \partial_x v_y = 0$$

$$\partial_t B_z + v_x \partial_x B_z + B_z \partial_x v_x - B_x \partial_x v_z = 0$$

We have $$\overline{\overline{A}} = \begin{bmatrix} v_x & \rho & 0 & 0 & 0 & 0 \\ \frac{a^2}{\rho} & v_x & 0 & 0 & \frac{B_y}{\rho} & \frac{B_z}{\rho} \\ 0 & 0 & v_x & 0 & -\frac{B_x}{\rho} & 0 \\ 0 & 0 & 0 & v_x & 0 & -\frac{B_x}{\rho} \\ 0 & B_y & -B_x & 0 & v_x & 0 \\ 0 & B_z & -B_x & 0 & 0 & v_x \end{bmatrix}$$

MHD waves

We consider now a reference equilibrium state $W_0 = (\rho^0, v_x^0, v_y^0, v_z^0, B_y^0, B_z^0)$

that we perturb slightly with $W = W_0 + \delta W$ and $\delta W = (\delta\rho, \delta v_x, \delta v_y, \delta v_z, \delta B_y, \delta B_z)$

The waves equation is given by $\boxed{\partial_t(\delta W) + \overline{\overline{A}}_0 \partial_x(\delta W) = 0}$

The wave speeds are given by the eigenvalue decomposition $\det\left(\overline{\overline{A}}_0 - \lambda \overline{\overline{1}}\right) = 0$

We have 7 eigenvalues ordered from left-going waves to right-going waves

$$u - c_f \le u - c_a \le u - c_s \le u \le u + c_s \le u + c_a \le u + c_f$$

The Alfvén waves have waves speed $\boxed{c_a^2 = \frac{B^2}{\rho}}$. They are transverse waves with no pressure and density variations.

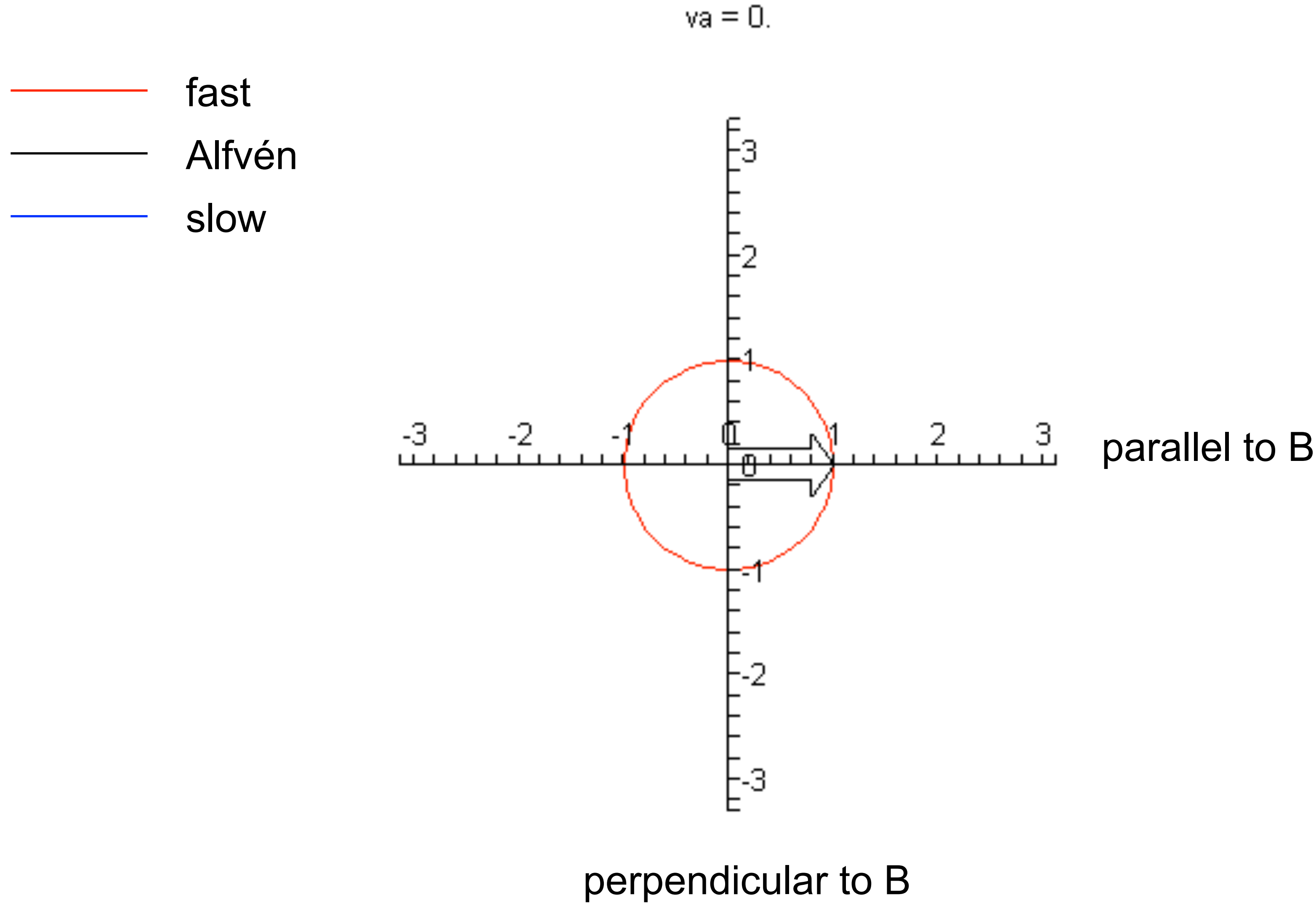
The fast magnetosonic waves are longitudinal waves with pressure and density variations correlated with magnetic fields variations.

$$\boxed{c_f^2 = \frac{1}{2}(a^2 + c_a^2) + \frac{1}{2}\sqrt{(a^2 + c_a^2)^2 - 4a^2 c_{a,x}^2}} \qquad c_{a,x}^2 = \frac{B_x^2}{\rho}$$

The slow magnetosonic waves are also longitudinal waves, but pressure and density variations are anti-correlated with magnetic field variations.

$$\boxed{c_s^2 = \frac{1}{2}(a^2 + c_a^2) - \frac{1}{2}\sqrt{(a^2 + c_a^2)^2 - 4a^2 c_{a,x}^2}}$$

The Friedrichs diagram



Alfvén waves

We now consider a reference state at rest $v_x^0 = 0$ and we restrict ourselves to incompressible perturbations $\delta\rho = \delta P = \delta v_x = 0$.

We work in the frame where the x-axis is along the field lines $B_y^0 = B_z^0 = 0$

The previous system simplifies into

$$\partial_t(\delta\vec{v}_\perp) - \frac{B_x^0}{\rho^0}\partial_x(\delta\overrightarrow{B}_\perp) = 0$$

$$\partial_t(\delta\overrightarrow{B}_\perp) - B_x^0\partial_x(\delta\vec{v}_\perp) = 0$$

where $\delta\vec{v}_\perp = (\delta v_y, \delta v_z)$ and $\delta\overrightarrow{B}_\perp = (\delta B_y, \delta B_z)$

Looking for plane waves solution $\delta\vec{v}_\perp = \vec{v}_\perp^0 \exp^{i(k_x x - \omega t)}$ $\delta\overrightarrow{B}_\perp = \overrightarrow{B}_\perp^0 \exp^{i(k_x x - \omega t)}$

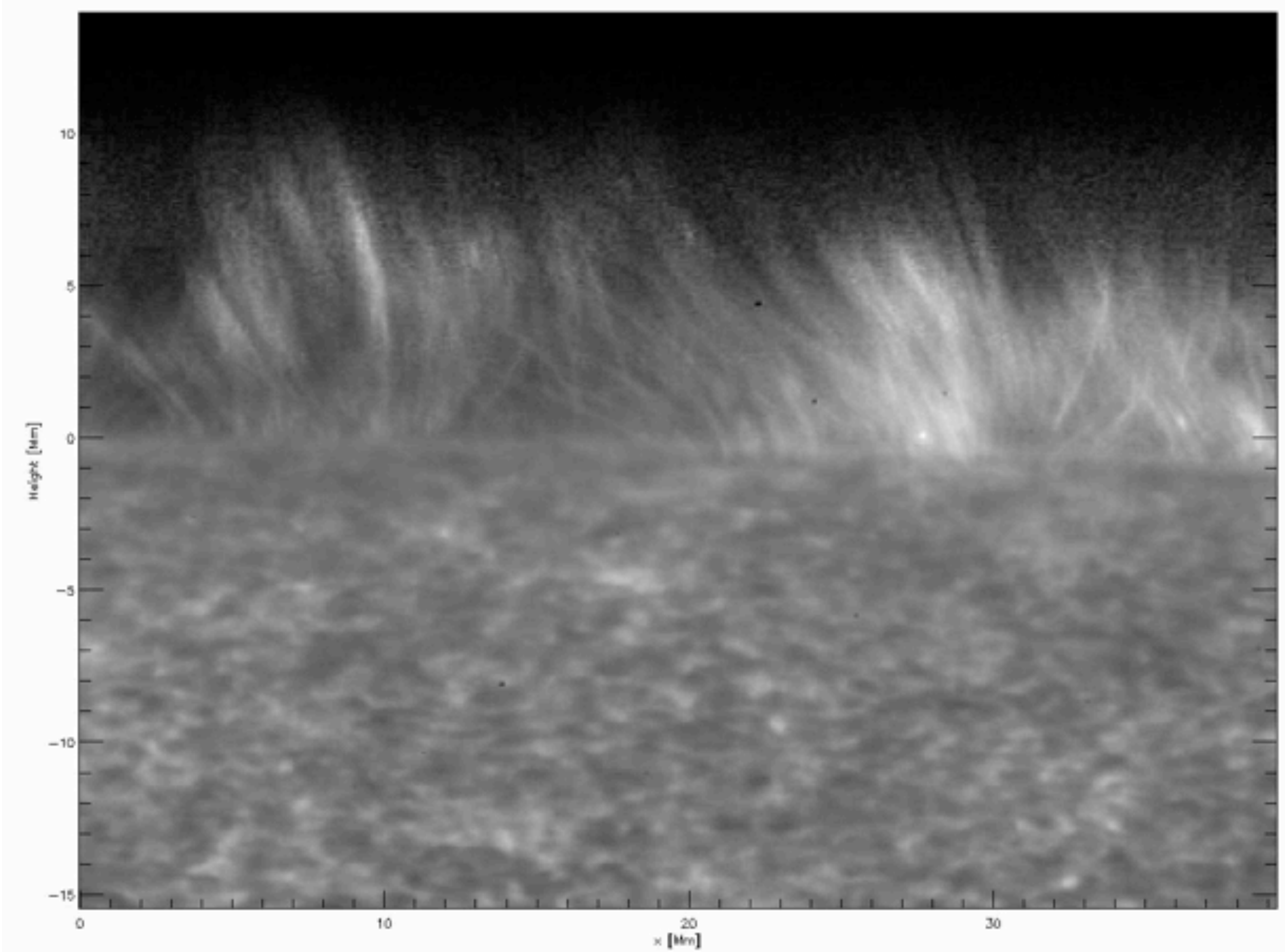
we obtain the dispersion relation for Alfvén waves $\boxed{\omega^2 = c_a^2 k_\|^2}$

We have $\dfrac{\delta\overrightarrow{B}_\perp}{B_x^0} = \pm\dfrac{\delta\vec{v}_\perp}{c_a}$ so that transverse magnetic perturbations are correlated or anti-correlated with transverse velocity perturbations,

depending on the direction of propagation $\omega = \pm c_a k_\|$

Alfvén waves are analogous to waves along a string with tension.

Alfvén waves observed in the Sun corona



Non-ideal MHD equations

Including non-ideal effects in the previous equations boils down to 2 main physical ingredients (we restrict ourselves to incompressible fluids).

1- including fluid viscosity $\sigma_{ij} = -p\delta_{ij} + \eta(\partial_{x_i} v_j + \partial_{x_j} v_i)$

2- including Ohmic dissipation $\vec{E} = -\frac{1}{c}\vec{v} \times \vec{B} + \eta_B \vec{J}$

The Navier-Stokes equation becomes $\frac{D\vec{v}}{Dt} = -\frac{1}{\rho}\vec{\nabla} p + \nu\Delta\vec{v} + \frac{1}{\rho c}\vec{J} \times \vec{B}$

The induction equation becomes $\partial_t \vec{B} = \vec{\nabla} \times (\vec{v} \times \vec{B}) + \nu_B \Delta \vec{B}$

The Reynolds number $Re = U_\infty L/\nu$ was introduced to estimate the importance of the inertial term $(\vec{v} \cdot \vec{\nabla})\vec{v} \simeq \frac{U_\infty^2}{L}$ compared to the viscous stress $\nu\Delta\vec{v} \simeq \nu\frac{U_\infty}{L^2}$

The magnetic Reynolds number $Re_B = U_\infty L/\nu_B$ can be defined to compare the induction term $\vec{\nabla} \times (\vec{v} \times \vec{B}) \simeq \frac{U_\infty B}{L}$ to magnetic resistivity $\nu_B \Delta \vec{B} \simeq \nu_B \frac{B}{L^2}$

To weight the relative importance of the 2 dissipative processes, we introduce the magnetic Prandtl number

$$\boxed{Pr_B = \frac{Re_B}{Re} = \frac{\nu}{\nu_B} = \frac{\text{viscous diffusion rate}}{\text{magnetic diffusion rate}}}$$

Usually, $Pr_B \ll 1$

Non-ideal Alfvén waves

We reconsider the incompressible case. The linearized equations become:

$$\partial_t(\delta\vec{v}_\perp) - \frac{B_x^0}{\rho^0}\partial_x(\delta\vec{B}_\perp) = \nu\Delta(\delta\vec{v}_\perp)$$

$$\partial_t(\delta\vec{B}_\perp) - B_x^0\partial_x(\delta\vec{v}_\perp) = \nu_B\Delta(\delta\vec{B}_\perp)$$

and the wave dispersion relation is now more complicated

$$(\omega + ik^2\nu)(\omega + ik^2\nu_B) = c_a^2 k^2$$

Solving for this second order polynomial gives

$$\boxed{\omega = \pm c_a k\sqrt{1 - \frac{k^2(\nu-\nu_B)^2}{4c_a^2}} - ik^2\frac{\nu+\nu_B}{2}}$$

The second term on the RHS is a damping term, as expected.

The propagation speed is also reduced by non-ideal effects.

For strong magnetic resistivity, the right-going Alfvén wave becomes unstable above a critical wave number $k_c = 2c_a/\nu_B$

or below a critical scale given by $\boxed{\frac{\lambda_c}{L} \sim \frac{U_\infty}{c_a}\frac{1}{Re_B}}$

On small scales, the fluid approximation breaks down: collisionless plasma regime.