Continuum Mechanics

Lecture 6

Waves in Fluids

Prof. Romain Teyssier

http://www.itp.uzh.ch/~teyssier

Universität Zürich

ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Outline

- Sound waves
- Jeans instability
- Shallow water equations
- Gravity waves
- Rayleigh-Taylor and Kelvin-Helmholtz instabilities
- Quasi-linear waves and shock formation
- Shock waves and Rankine-Hugoniot relations

Sound waves

We use the fluid equation in one dimension without gravity source term.

In conservative form, they write

$$\partial_t \rho + \partial_x (\rho v) = 0$$

$$\partial_t (\rho v) + \partial_x \left(\rho v^2 + P\right) = 0$$

We assume a barotropic EoS $P = P(\rho)$

Far from any discontinuities, we can also use the quasi-linear form:

$$\partial_t \rho + v \partial_x \rho + \rho \partial_x v = 0$$

$$\partial_t v + v \partial_x v + \frac{1}{\rho} \partial_x P = 0$$

We consider the reference equilibrium state $\rho = \rho_0 \text{ and } v = v_0$ *everywhere in space*.

Waves are small disturbances of this equilibrium state

$$\rho(x,t) = \rho_0 + \delta\rho(x,t) \text{ with } \delta\rho \ll \rho_0$$

$$v(x,t) = v_0 + \delta v(x,t) \text{ with } \delta v \ll v_0 \text{ and } c_0$$

Sound waves

We linearize the quasi-linear form, dropping high-order terms.

$$\partial_t(\delta\rho) + v_0\partial_x(\delta\rho) + \rho_0\partial_x(\delta v) = 0$$

$$\partial_t(\delta v) + v_0\partial_x(\delta v) + \frac{c_0^2}{\rho_0}\partial_x(\delta\rho) = 0$$

where we have used the definition of the sound speed $c^2 = P'(\rho)$

We are looking for monochromatic planar wave solutions:

$$\delta\rho = \Delta\rho \exp^{i(kx-\omega t)} \qquad \delta v = \Delta v \exp^{i(kx-\omega t)}$$

We obtain the following linear system for the amplitudes:

$$(-i\omega + ikv_0)\Delta\rho + ik\rho_0\Delta v = 0$$

$$ik\frac{c_0^2}{\rho_0}\Delta\rho + (-i\omega + ikv_0)\Delta v = 0$$

In order to have a non vanishing solution, the determinant must be zero.

We obtain the dispersion relation for sound waves:

$$\boxed{(\omega - kv_0)^2 - k^2c_0^2 = 0}$$

The velocities of sound waves are $v = \frac{\omega}{k} = v_0 \pm c_0$

Riemann invariants and characteristics

The previous linear system of partial differential equations can be written as

$\partial_t W + A\partial_x W = 0$ where the vector of unknowns is $W = (\delta\rho, \delta v)^T$

The matrix A is given by $A = \begin{pmatrix} v_0 & \rho_0 \\ \frac{c_0^2}{\rho_0} & v_0 \end{pmatrix}$. The eigenvalues are $\lambda^\pm = v_0 \pm c_0$

and the eigenvectors components are given by

$$\delta\alpha^+ = \frac{1}{2}\left(\delta\rho + \frac{\rho_0}{c_0}\delta v\right)$$

$$\delta\alpha^- = \frac{1}{2}\left(\delta\rho - \frac{\rho_0}{c_0}\delta v\right)$$

Since the matrix is diagonal in the eigenvector basis, we have :

$$\partial_t(\delta\alpha^+) + (v_0 + c_0)\partial_x(\alpha^+) = 0 \qquad \partial_t(\delta\alpha^-) + (v_0 - c_0)\partial_x(\alpha^-) = 0$$

We define the characteristic curves (different from the trajectories) as:

$$\frac{dx}{dt}^{\pm} = (v_0 \pm c_0)$$

$\delta\alpha^\pm$ are conserved quantities along their corresponding characteristic curves: they are called Riemann invariants.

Given the initial conditions at t=0, we can reconstruct the final solution by combining Riemann invariants along each crossing characteristic (in this case straight lines).

Self-gravitating fluids

The fluids equation in conservative form in presence of gravity write:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0$$

$$\frac{\partial \rho \vec{v}}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v} \otimes \vec{v}) + \vec{\nabla} P = \rho \vec{F}$$

In a self-gravitating fluid, the gravitational potential follow the Poisson equation

$$\Delta \Phi = 4\pi G \rho \text{ with } \vec{F} = -\vec{\nabla}\Phi$$

We drop the constant 4πG from now on: $\rho = \Delta\Phi = -\vec{\nabla} \cdot \vec{F}$

For each component, we have $\rho F_x = -\left(\vec{\nabla} \cdot \vec{F}\right) F_x = -\vec{\nabla} \cdot \left(F_x \vec{F}\right) + \left(\vec{F} \cdot \vec{\nabla}\right) F_x$

We then use the relations

$$\partial_x F_x = -\partial^2_{xx}\Phi = \partial_x F_x$$

$$\partial_y F_x = -\partial^2_{xy}\Phi = \partial_x F_y$$

$$\partial_z F_x = -\partial^2_{xz}\Phi = \partial_x F_z$$

The tidal tensor $\partial_j F_i = -\partial_{ij}\Phi$ is symmetric

Finally, we have $\rho F_x = -\vec{\nabla} \cdot \left(F_x \vec{F}\right) + \vec{F}\partial_x \vec{F} = -\vec{\nabla} \cdot \left(F_x \vec{F}\right) + \partial_x \frac{\left|\vec{F}\right|^2}{2}$

Momentum conservation:

$$\boxed{\frac{\partial \rho \vec{v}}{\partial t} + \vec{\nabla} \cdot \left(\rho \vec{v} \otimes \vec{v} + P\overline{\overline{1}} + \vec{F} \otimes \vec{F} - \frac{F^2}{2}\overline{\overline{1}}\right) = 0}$$

Jeans instability

We consider an equilibrium state with $\rho = \rho_0, \ \Phi = 0 \text{ and } v = 0$

In this infinite medium, the Poisson equation has to be modified $\Delta\Phi = 4\pi G\left(\rho - \rho_0\right)$

The perturbed state satisfies $\Delta(\delta\Phi) = 4\pi G(\delta\rho)$

The linearized continuity equation is $\partial_t(\delta\rho) + \rho_0 \vec{\nabla} \cdot (\delta\vec{v}) = 0$

The Euler equation becomes $\partial_t(\delta\vec{v}) + \frac{c_0^2}{\rho_0}\vec{\nabla}(\delta\rho) + \vec{\nabla}(\delta\Phi) = 0$

Taking the partial time derivative of the continuity equation leads to

$$\partial_t^2(\delta\rho) = -\rho_0 \vec{\nabla} \cdot (\partial_t(\delta\vec{v})) = c_0^2\Delta(\delta\rho) + \rho_0\Delta(\delta\Phi) = c_0^2\Delta(\delta\rho) + 4\pi G\rho_0(\delta\rho)$$

We are looking for plane wave solution $\delta\rho = \Delta\rho \exp^{i(kx-\omega t)}$

We get the following dispersion relation $\boxed{\omega^2 = c_0^2 k^2 - 4\pi G\rho_0 = c_0^2(k^2 - k_J^2)}$

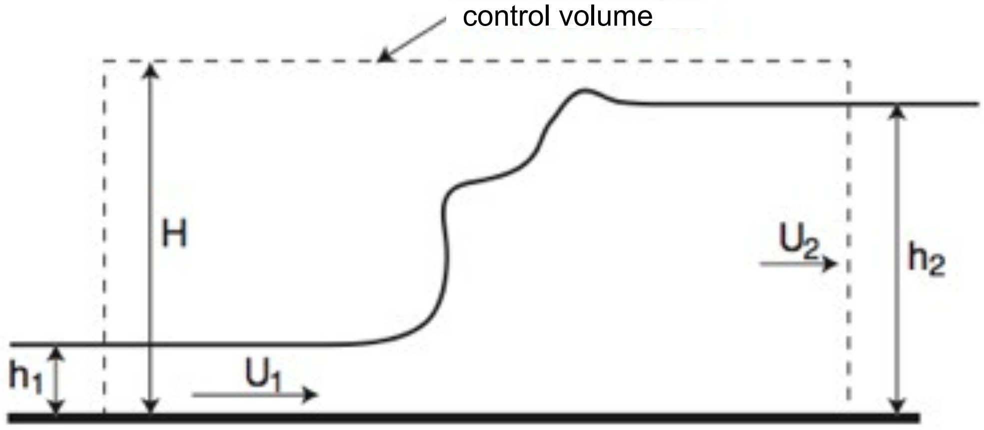
where we have introduce the Jeans length $k_J = \frac{2\pi}{\lambda_J} = \sqrt{\frac{4\pi G\rho_0}{c_0^2}}$

For short wavelength, $k > k_J$ we have propagating waves with $v \leq c_0$

For long wavelength, $k < k_J$ small perturbations grow exponentially fast.

$\omega^2 < 0$ so ω is purely imaginary and $\delta\rho \propto \exp^{\pm|\omega|t}$ → Instability !

Shallow water equations



In the water layer of varying thickness h(x,t) and constant density ρ, the pressure is given by hydrostatic equilibrium $\boxed{p(x,z) = p_0 + \rho g \left(h(x,t) - z\right)}$

In air, we have $p=p_0$ and $\rho=0$.

We write mass conservation in integral form in the control volume with

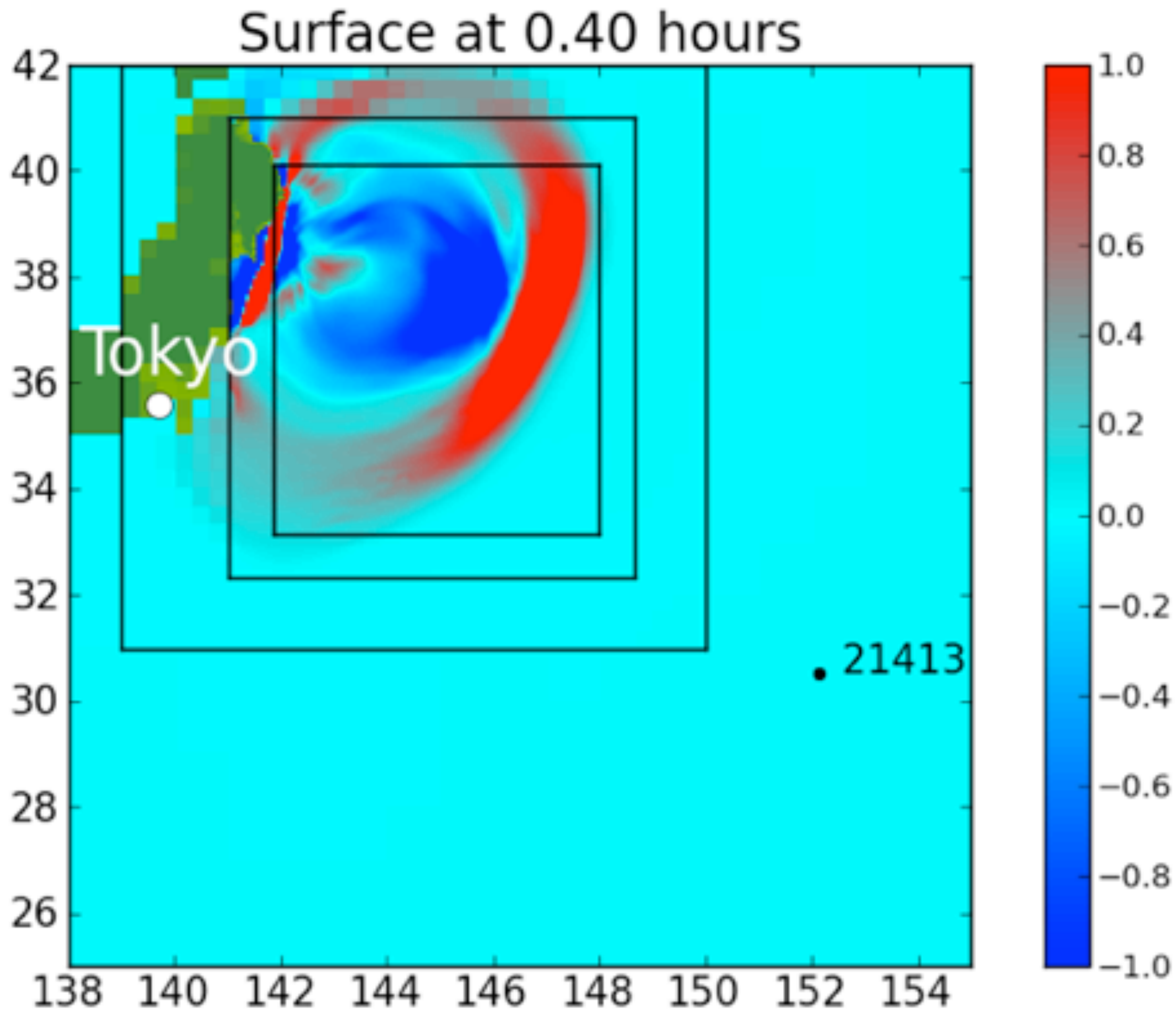
$\Delta x = x_2 - x_1$ and $h_1 \simeq h_2 \simeq h(x,t)$, assuming $v_z = 0$ and $v_x(x,z,t) = v(x,t)$

$$\frac{\partial}{\partial t}\int_{x_1}^{x_2}\int_0^h \rho \mathrm{d}x\mathrm{d}z = \frac{\partial}{\partial t}\left(\rho h \Delta x\right) = -\rho\left(v_2 h_2 - v_1 h_1\right) \qquad \boxed{\frac{\partial}{\partial t}(h) + \frac{\partial}{\partial x}(vh) = 0}$$

Using momentum conservation on the same control volume, we have

$$\boxed{\frac{\partial}{\partial t}(hv) + \frac{\partial}{\partial x}\left(v^2 h + g\frac{h^2}{2}\right) = 0}$$

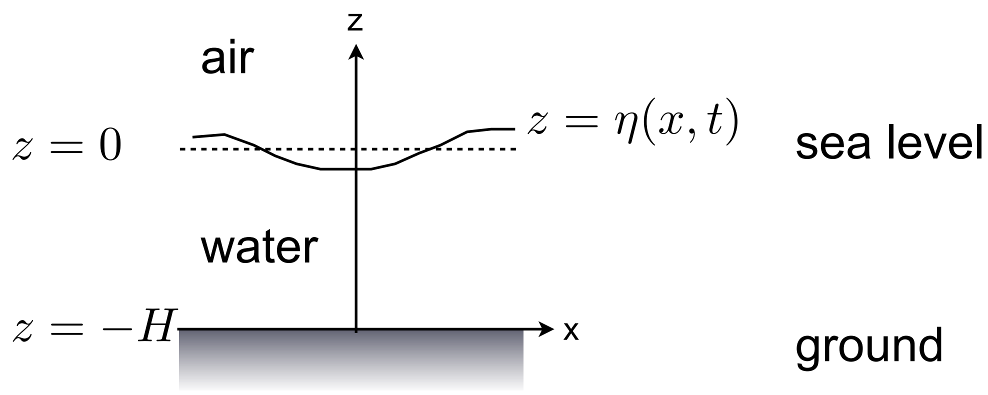
Tsunami modeling using shallow water eq.



http://kingkong.amath.washington.edu/clawpack/

Gravity waves

Incompressible fluid dynamics in deep water under constant gravity.



We consider the equilibrium state $\eta(x,t) = 0$ and $\vec{v}(x,z,t) = 0$

In air, we have $p = p_0$ and in water, $p = p_0 - \rho_0 g z$.

Using the second Bernoulli theorem, using $\vec{v}(x,z,t) = \vec{\nabla}\phi$ we have in the volume:

$$\partial_t \phi(x,z,t) + \frac{v^2}{2} + gz + \frac{p(x,z,t)}{\rho_0} = C(t) \quad \text{and} \quad \vec{\nabla}\cdot\vec{v} = 0 \longrightarrow \Delta\phi = 0$$

We add to this the boundary condition at the bottom $v_z(z = -H) = 0$

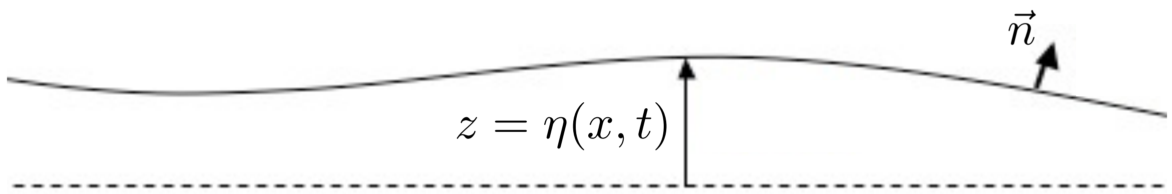
and the kinematic boundary condition at the top $\partial_t \eta(x,t) + v_x \partial_x \eta(x,t) = v_z$

Kinematic condition on a free surface

For an incompressible inviscid fluid, we have to solve a Poisson equation with a boundary condition on the outer surface $\vec{v} \cdot \vec{n} = 0$.

When the outer surface is fixed, both the location and the normal vector are function of space only. This results in a Neumann BC for the potential.

For a free surface that moves, this is more complicated.



A point on the free surface has position and velocity given by

$$\vec{r}(t) = (x(t), \eta(x(t), t)) \qquad \vec{v}(t) = (x', \partial_t \eta + x' \partial_x \eta)$$

The BC writes (no vacuum between the fluid and the free surface):

$\vec{v}(t) \cdot \vec{n} = (v_x, v_z) \cdot \vec{n}$ where the fluid velocity is (v_x, v_z) and $\vec{n} = \dfrac{(-\partial_x \eta, 1)}{\sqrt{1 + \partial_x \eta^2}}$

The final kinematic boundary condition writes

$$\boxed{\partial_t \eta + v_x(x, \eta, t) \partial_x \eta = v_z(x, \eta, t)}$$

Gravity waves

We linearize the previous set of equations:

At the upper surface, we have $\partial_t \phi(x,0,t) + g\eta(x,t) + \frac{p_0}{\rho_0} = C(t)$

and $\partial_t \eta(x,t) = v_z(x,0,t) = \partial_z \phi(x,0,t)$

At the bottom, we have $\partial_z \phi(x,-H,t) = 0$

We are looking for propagating waves in the x direction $\begin{cases} \eta(x,t) = A\exp^{i(kx-\omega t)} \\ \phi(x,z,t) = \phi(z)\exp^{i(kx-\omega t)} \end{cases}$

The Poisson equation in the volume writes $\phi''(z) = k^2\phi(z)$

for which the general solution is $\phi(z) = \phi^+ \exp^{+kz} + \phi^- \exp^{-kz}$

The lower BC gives us the first relation $\phi^+ \exp^{-kH} - \phi^- \exp^{+kH} = 0$

The upper BC gives us the second relation $k\left(\phi^+ - \phi^-\right) = -i\omega A$

Bernoulli relation at the upper BC gives us $-i\omega\left(\phi^+ + \phi^-\right) = -gA$

where we absorbed the constants in the velocity potential.

The dispersion relation writes $\boxed{\omega^2 = gk\frac{\exp^{+kH} - \exp^{-kH}}{\exp^{+kH} + \exp^{-kH}} = gk\tanh\left(kH\right)}$

Gravity waves

Two interesting limiting cases: $kH \ll 1$ and $kH \gg 1$

1- Deep water: $H \gg 1/k$ $\omega = \sqrt{gk}$ $v_g = \frac{d\omega}{dk} = \frac{1}{2}\frac{\omega}{k}$

2- Shallow water: $H \ll 1/k$ $\omega = k\sqrt{gH}$ $v = \sqrt{gH}$

We use the shallow water equations to derive directly the second result.

We linearize the quasi-linear form:

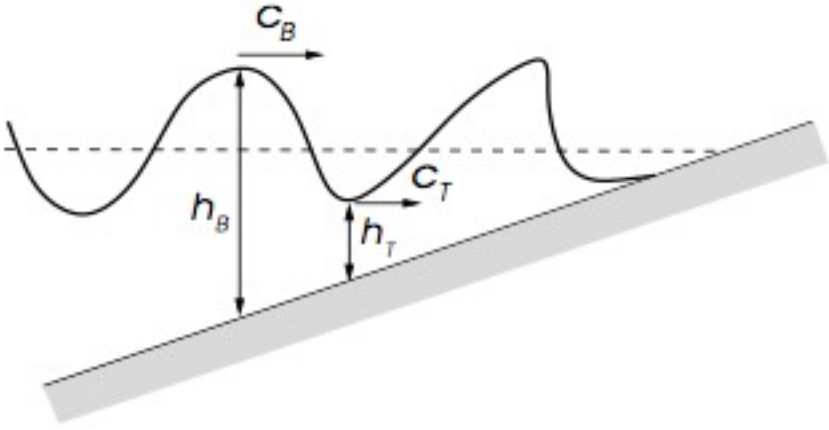
$$\begin{cases} \partial_t(\delta h) + H\partial_x(\delta v) = 0 \\ \partial_t(\delta v) + g\partial_x(\delta h) = 0 \end{cases}$$

$$\boxed{\omega^2 = k^2(gH)}$$

In shallow waters, the speed increases as the square root of the depth.

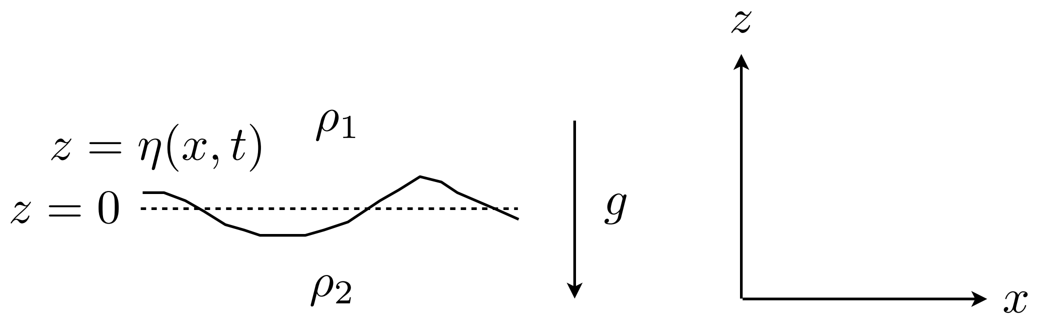
Close to the shore, waves tend to decelerate. Peaks decelerate slower than the troughs. They tend to catch up. At the shore, the trough stops, while the next peak still travels fast: the wave is breaking.

See «formation of a shock wave».



Rayleigh-Taylor instability

2 semi-infinite incompressible fluids separated by an horizontal interface.



In the 2 separate volume, we have $\vec{v}_1 = \vec{\nabla}\phi_1 \qquad \vec{\nabla}\cdot\vec{v}_1 = 0 = \Delta\phi_1$

$$\vec{v}_2 = \vec{\nabla}\phi_2 \qquad \vec{\nabla}\cdot\vec{v}_2 = 0 = \Delta\phi_2$$

The boundary conditions for the velocity field are $\vec{v} \rightarrow 0 \text{ when } z \rightarrow \pm\infty$

and at z=0 we have $\partial_t\eta + v_{x,1}\partial_x\eta = v_{z,1}$ and $\partial_t\eta + v_{x,2}\partial_x\eta = v_{z,2}$.

We also impose pressure continuity at the interface $p_1 = p_2$ and from Bernoulli:

$$\partial_t\phi_1 + \frac{v_1^2}{2} + g\eta + \frac{p_1}{\rho_1} = C_1 \qquad \partial_t\phi_2 + \frac{v_2^2}{2} + g\eta + \frac{p_2}{\rho_2} = C_2$$

As usual, we linearized these equations and look for planar wave solutions:

$$\phi_1 = \phi_1(z)\exp^{i(kx-\omega t)} \qquad \phi_2 = \phi_2(z)\exp^{i(kx-\omega t)} \qquad \eta = A\exp^{i(kx-\omega t)}$$

Rayleigh-Taylor instability

The Poisson equation in each domain is $\phi_1'' = k^2\phi_1$ and $\phi_2'' = k^2\phi_2$

The unique solutions that satisfy the velocity BC at infinity are

$$\phi_1(z) = \phi_1 \exp^{-kz} \qquad \phi_2(z) = \phi_2 \exp^{+kz}$$

Linearizing the Bernoulli equations and imposing equal pressures give:

$$\rho_1 \left(\partial_t \phi_1 + g\eta\right) = \rho_2 \left(\partial_t \phi_2 + g\eta\right)$$

where we have absorbed the 2 constants C_1 and C_2 in the velocity potentials.

Linearizing the 2 interface kinematic conditions gives: $\partial_t \eta = \partial_z \phi_1 = \partial_z \phi_2$

We use the planar wave solutions in the previous equations to get the system:

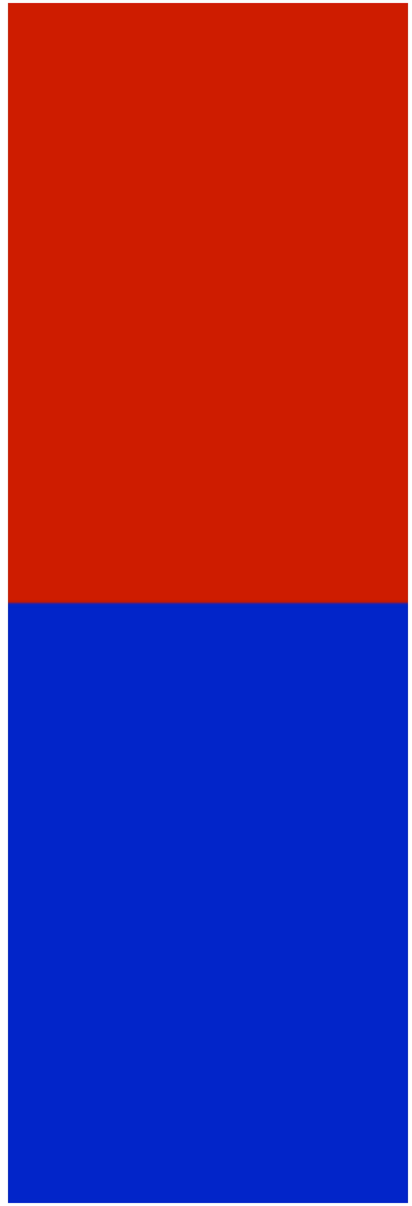
$$-i\omega A = -k\phi_1 = +k\phi_2 \qquad \rho_1(-i\omega\phi_1 + gA) = \rho_2(-i\omega\phi_2 + gA)$$

The dispersion relation follows: $\boxed{\omega^2 = \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1} gk}$

If $\rho_2 > \rho_1$, we obtain stable gravity waves in deep water (especially if $\rho_1 \ll \rho_2$)

If $\rho_2 < \rho_1$, the perturbation is unstable.

Rayleigh-Taylor instability



Kelvin-Helmholtz instability

We consider exactly the same set-up as for the RT instability, except that gravity is absent and the boundary conditions at infinity are different (shearing flow).

$$v_{x,1} \rightarrow U_1 \text{ when } z \rightarrow +\infty \qquad v_{x,2} \rightarrow U_2 \text{ when } z \rightarrow -\infty$$

The planar wave solutions are now

$$\phi_1(x,z,t) = \phi_1 \exp^{-kz} \exp^{i(kx-\omega t)}$$

$$\vec{v}_1 = \vec{\nabla}\phi_1 + U_1\vec{e}_x \qquad \phi_2(x,z,t) = \phi_2 \exp^{+kz} \exp^{i(kx-\omega t)}$$

$$\vec{v}_2 = \vec{\nabla}\phi_2 + U_2\vec{e}_x \qquad \eta(x,t) = A\exp^{i(kx-\omega t)}$$

The boundary conditions at the interface are now more complicated.

The kinematic conditions are linearized as $\partial_t\eta + U_1\partial_x\eta = \partial_z\phi_1$

and $\partial_t\eta + U_2\partial_x\eta = \partial_z\phi_2$

Pressure equilibrium and Bernoulli relations (absorbing the constants) give

$$\rho_1\partial_t\phi_1 + \rho_1 U_1\partial_x\phi_1 = \rho_2\partial_t\phi_2 + \rho_2 U_2\partial_x\phi_2$$

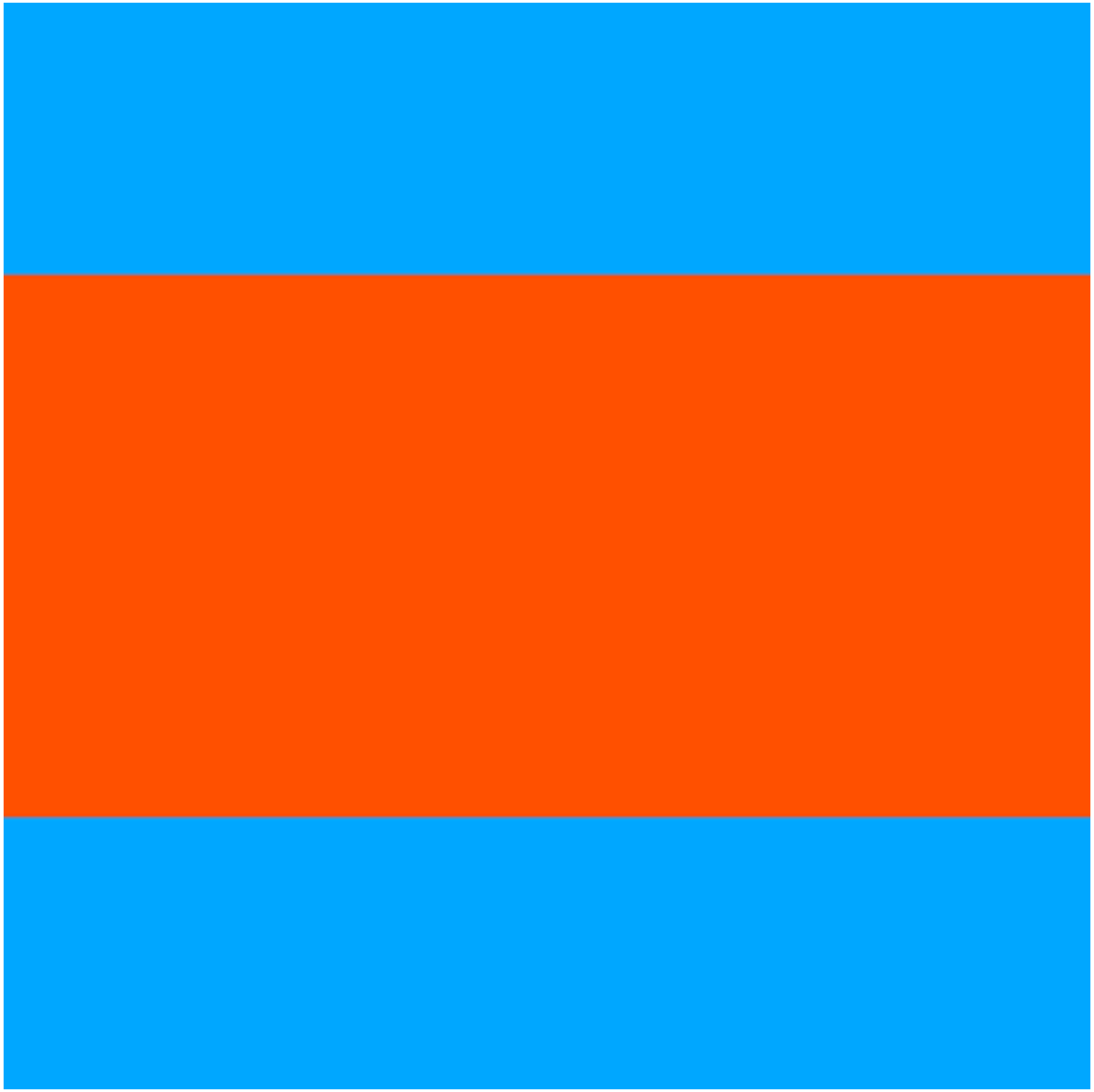
Using the plane wave solutions, we find the system $(-i\omega + ikU_1)A = -k\phi_1$

$$\rho_1(-i\omega + ikU_1)\phi_1 = \rho_2(-i\omega + ikU_2)\phi_2 \qquad (-i\omega + ikU_2)A = +k\phi_2$$

The dispersion relation is:

$$\boxed{\omega = k\left(\frac{\rho_1 U_1 + \rho_2 U_2}{\rho_1 + \rho_2} \pm i\frac{\sqrt{\rho_1\rho_2}}{\rho_1+\rho_2}|U_1 - U_2|\right)}$$

Kelvin-Helmholtz instability



Quasi-linear waves and Riemann invariants

The 1D isothermal fluid equation in quasi-linear form write:

$$\partial_t \rho + v\partial_x \rho + \rho\partial_x v = 0 \qquad \partial_t v + v\partial_x v + \frac{1}{\rho}\partial_x P = 0 \qquad \text{with} \qquad P = \rho c_0^2$$

The 2 quantities $\alpha^+(x,t) = v(x,t) + c_0 \ln \rho(x,t)$ and $\alpha^-(x,t) = v(x,t) - c_0 \ln \rho(x,t)$

satisfy $\partial_t \alpha^+ + (v + c_0)\partial_x \alpha^+ = 0$ and $\partial_t \alpha^- + (v - c_0)\partial_x \alpha^- = 0$.

They are Riemann invariants along the characteristic curves:

$$\frac{dx^+}{dt} = v(x^+(t),t) + c_0 \qquad \frac{dx^-}{dt} = v(x^-(t),t) - c_0$$

Characteristic curves are different than the fluid trajectories: $\frac{dx^0}{dt} = v(x^0(t),t)$

At any point in space-time (x,t), we can compute the fluid velocity as:

$$v(x,t) = \frac{\alpha^+(x_1) + \alpha^-(x_2)}{2}$$

where x_1 and x_2 are the starting points in the initial conditions of the 2 characteristics.

What happens if two «right-going» characteristics cross at the same point ?

→ formation of a shock.

A simple non-linear example: Burger's equation

Burger's equation writes in quasi-linear form as $\partial_t v + v\partial_x v = 0$

and in conservative form as $\partial_t v + \partial_x \frac{v^2}{2} = 0$

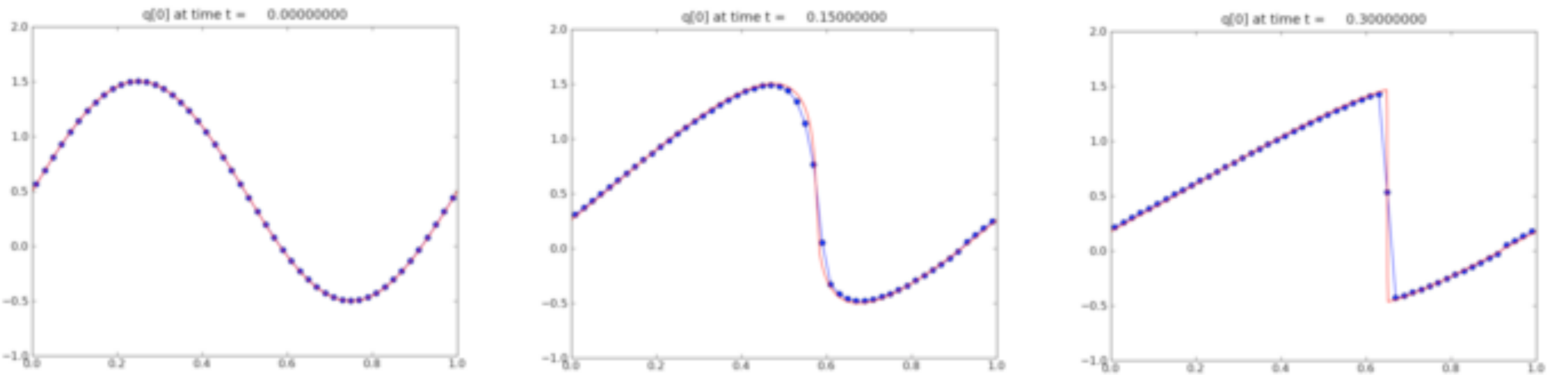
In this case, characteristic curves are equal to the trajectories and the velocity is the Riemann invariant. It follows that characteristic curves are straight lines.

We consider the general initial condition $v_0(x)$.

The solution is given by the implicit equation $v(x,t) = v_0(x - v(x,t)t)$

Taking the time derivative leads to $\partial_t v = (-v + t\partial_t v)\, v_0'(x - vt)$ and $\boxed{\partial_t v = -v\frac{v_0'}{1 + t v_0'}}$

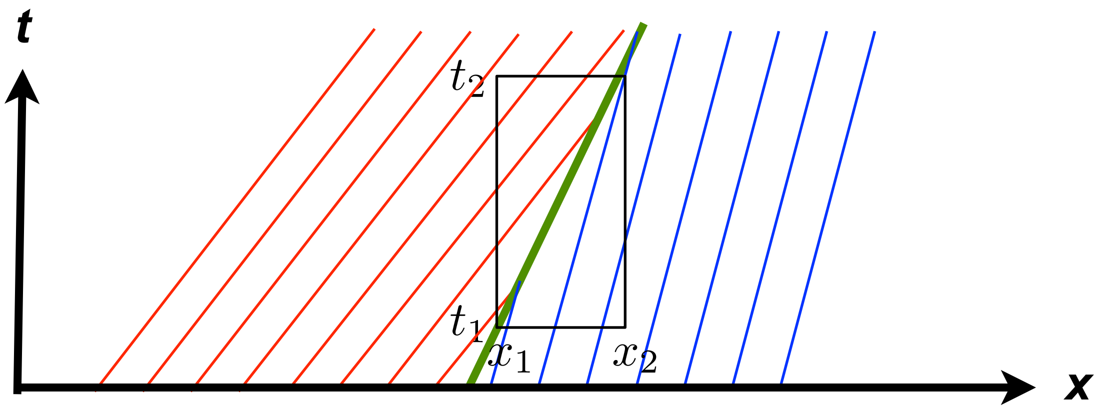
The solution blows out at the finite time $t = -\frac{1}{v_0'}$ $\rightarrow$ formation of the shock.



Shock waves and the Rankine-Hugoniot relations

Shock waves are discontinuities propagating in the flow that arise naturally from characteristic crossings and non-linear waves steepening.

We consider here the 1D case (perpendicular to the shock surface).



The fluid equations write in conservative form: $\partial_t U + \partial_x F(U) = 0$

We use a small enough control volume around the moving discontinuity (speed S) so that the flow quantities can be considered as homogeneous ($x_1=St_1$ and $x_2=St_2$).

$$\int_{t_1}^{t_2}\int_{x_1}^{x_2} (\partial_t U + \partial_x F)\,\mathrm{d}x\mathrm{d}t = U_2(x_2-x_1) - U_1(x_2-x_1) + F_2(t_2-t_1) - F_1(t_2-t_1) = 0$$

$$\boxed{F_2 - F_1 = S(U_2 - U_1)}$$

These are conservation laws connecting the upstream and downstream regions.

Rankine-Hugoniot relations

Burger's equation: $U = v$ and $F = \frac{v^2}{2}$ the shock speed is $S = \frac{v_1 + v_2}{2}$

Isothermal shocks: $U = (\rho, \rho v)$ and $F = (\rho v, \rho v^2 + \rho c_0^2)$

Mass conservation: $\rho_2 v_2 - \rho_1 v_1 = S(\rho_2 - \rho_1)$

Momentum conservation: $\rho_2 v_2^2 - \rho_1 v_1^2 + \rho_2 c_0^2 - \rho_1 c_0^2 = S(\rho_2 v_2 - \rho_1 v_1)$

Change of variables : $w_1 = v_1 - S \qquad w_2 = v_2 - S$

We have now in the frame of the shock: $\rho_2 w_2 - \rho_1 w_1 = 0$

$$\rho_2 w_2^2 - \rho_1 w_1^2 + \rho_2 c_0^2 - \rho_1 c_0^2 = 0$$

We define the compression ratio $r = \frac{\rho_2}{\rho_1}$ and the *Mach number* $\mathcal{M} = \frac{w_1}{c_0}$

The shock relations lead to $r^2 - r(\mathcal{M}^2 + 1) + \mathcal{M}^2 = 0$ or $r = 1$ and $r = \mathcal{M}^2$

$$\boxed{\begin{aligned} \rho_2 &= \mathcal{M}^2 \rho_1 \\ w_2 &= \frac{1}{\mathcal{M}^2} w_1 \end{aligned}}$$

Rankine-Hugoniot relations are a one-parameter (S) family of solutions.

The shock speed is usually determined using boundary conditions downstream.

Examples of shock waves solutions

Shock wave on a wall for an isothermal ideal fluid.

We assume that we have a wall boundary condition on the left $v_2 = 0$ and $v_1 < 0$.

We don't know the shock speed yet. We have $\rho_2 = \mathcal{M}^2 \rho_1 \quad w_2 = \frac{1}{\mathcal{M}^2} w_1$ with $\mathcal{M} = \frac{w_1}{c_0}$

Since $v_2 = 0$, we have $S = -w_2$ which leads to $S^2 - v_1 S - c_0^2 = 0$.

Among the 2 solutions, only one is physically admissible : compressive wave.

$$\boxed{S = \frac{1}{2}\left(v_1 + \sqrt{v_1^2 + 4c_0^2}\right)}$$

For a strong shock $(|v_1| \gg c_0)$, we have $S \simeq \frac{c_0^2}{|v_1|} \quad \rho_2 = \frac{|v_1|^2}{c_0^2}\rho_1$

Hydraulic jump: shallow water. We have at the wall of the sink $v_2 = 0$.

RH relations are: $-h_1 v_1 = S(h_2 - h_1)$

$$-h_1 v_1^2 + \frac{1}{2} g h_2^2 - \frac{1}{2} g h_1^2 = S(-h_1 v_1)$$

We define the height ratio $r = \frac{h_2}{h_1}$.

We have $r^3 - r^2 - r(1 + 2F^2) + 1 = 0$

For $F \gg 1, \quad r \simeq \sqrt{2}F$

Froude number

$$F = \frac{v_1}{\sqrt{g h_1}}$$

$$\boxed{S \simeq \sqrt{\frac{g h_1}{2}}}$$