Continuum Mechanics

Lecture 8

Incompressible viscous flows

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Outline

- general equations and boundary conditions
- planar viscous flows
- Couette and Poiseuille stationary flows
- viscous non-stationary interface
- viscous drag on a sphere
- boundary layer theory
- pressure gradients
- turbulence

Incompressible viscous flow equations

The Navier-Stokes equations for a viscous incompressible fluid write

$$\rho \frac{D\vec{v}}{Dt} = \rho \overrightarrow{F} + \overrightarrow{\nabla} \cdot \overline{\overline{\sigma}} = \rho \overrightarrow{F} - \overrightarrow{\nabla} P + \eta \Delta \vec{v} \qquad \overrightarrow{\nabla} \cdot \vec{v} = 0$$

We also have boundary conditions at the (moving) domain boundary.

We start from the Rankine-Hugoniot relations (normal to the interface).

$$(\rho_1 \vec{v}_1 - \rho_2 \vec{v}_2) \cdot \vec{n} = (\rho_1 - \rho_2) \overrightarrow{S} \cdot \vec{n}$$

$$\rho_1 \vec{v}_1 (\vec{v}_1 \cdot \vec{n}) - \overline{\overline{\sigma_1}} \vec{n} - \rho_2 \vec{v}_2 (\vec{v}_2 \cdot \vec{n}) + \overline{\overline{\sigma_2}} \vec{n} = (\rho_1 \vec{v}_1 - \rho_2 \vec{v}_2) \overrightarrow{S} \cdot \vec{n}$$

Rearranging terms, we have: $\boxed{\rho_1 \left(\vec{v}_1 - \overrightarrow{S}\right) \cdot \vec{n} = \rho_2 \left(\vec{v}_2 - \overrightarrow{S}\right) \cdot \vec{n} = \dot{m}}$

$$\boxed{\dot{m} (\vec{v}_1 - \vec{v}_2) = \left(\overline{\overline{\sigma_1}} - \overline{\overline{\sigma_2}}\right) \vec{n}}$$

The divergence free condition imposes $\vec{v}_1 \cdot \vec{n} = \vec{v}_2 \cdot \vec{n} = \overrightarrow{S} \cdot \vec{n}$

and the continuity of the stress field follows $\overline{\overline{\sigma}}_1 \vec{n} = \overline{\overline{\sigma}}_2 \vec{n}$

Since the stress tensor is proportional to the rate of strain tensor, the transverse velocity has to be differentiable at the boundary, and therefore at least continuous.

We get the «no-slip» boundary condition: $\vec{v} = \overrightarrow{S}$ at the boundary surface S.

Viscous versus inviscid flows

The two set of equations can be compared in more details.

We consider a rigid body (wall) at rest $\overrightarrow{S} = 0$ on a surface S bounding volume V.

Viscous flow

$$\rho \frac{D\vec{v}}{Dt} = \rho \overrightarrow{F} - \overrightarrow{\nabla} P + \eta \Delta \vec{v}$$

$$\overrightarrow{\nabla} \cdot \vec{v} = 0$$

On S: $\vec{v} = 0$

Inviscid flow

$$\rho \frac{D\vec{v}}{Dt} = \rho \overrightarrow{F} - \overrightarrow{\nabla} P$$

$$\overrightarrow{\nabla} \cdot \vec{v} = 0$$

On S: $\vec{v} \cdot \vec{n} = 0$

Besides the additional stress field, the main difference comes with the boundary condition.

The transverse velocity can be discontinuous at the wall for an ideal fluid. It is called the slip condition.

The transverse velocity has to be zero at the wall for a viscous fluid. It is called the no-slip condition. At the molecular level, it means the particles have to stick to the wall when they come too close.

The Reynolds number

We write the Navier-Stokes equation $\partial_t \vec{v} + \left(\vec{v} \cdot \vec{\nabla}\right) \vec{v} = \nu \Delta \vec{v} - \frac{1}{\rho} \vec{\nabla} p$

where we define the viscosity coefficient $\nu = \frac{\eta}{\rho}$

We define the dimensionless variables $\vec{v} = \tilde{v} U_\infty$ $\quad t = \tilde{t} \frac{L}{U_\infty}$ $\quad \vec{x} = \tilde{x} L$ $\quad p = \tilde{p} \rho U_\infty^2$

We obtain the new dimensionless equation $\partial_{\tilde{t}} \tilde{v} + \left(\tilde{v} \cdot \tilde{\nabla}\right) \tilde{v} = \frac{\nu}{U_\infty L} \tilde{\Delta} \tilde{v} - \frac{1}{\rho} \tilde{\nabla} \tilde{p}$

with only one parameter, the Reynolds number $\boxed{Re = \frac{U_\infty L}{\nu}}$

For small Reynolds numbers, we can neglect the inertial term $\left(\vec{v} \cdot \vec{\nabla}\right) \vec{v}$

and the Navier-Stokes equation becomes a linear equation.

Stationary flows are characterized by $\eta \Delta \vec{v} = \vec{\nabla} p$

Using $\vec{\nabla} \cdot \vec{v} = 0$, we have $\Delta p = 0$ + boundary conditions.

Using $\vec{\omega} = \vec{\nabla} \times \vec{v}$, we have $\Delta \vec{\omega} = 0$ + boundary conditions.

The velocity field does not depend on the viscosity coefficient !

For large Reynolds number and far away from boundaries, we can neglect the viscous term and we *almost* converge back to the ideal fluid limit.

Viscous planar flows

We consider 2D flows with planar symetry $\vec{v} = v_x(x, y, t)\vec{e}_x$

The divergence free condition writes $\vec{\nabla} \cdot \vec{v} = \partial_x v_x = 0$ or $v_x = v(y, t)$

The Navier-Stokes equation is just $\partial_t v = \nu \partial_y^2 v - \frac{1}{\rho}\partial_x p$ $\quad \partial_y p = 0 \rightarrow p(x, t)$

For planar flows, the inertial term vanishes because of the divergence free constraint.

Stationary case: $\boxed{\nu \partial_y^2 v = \frac{1}{\rho}\partial_x p}$

Equilibrium between shear stresses and pressure gradient.

Since v depends only on y and p only on x, the pressure gradient is a constant.

It is the free parameter of the flow, together with velocity boundary conditions.

We have therefore to solve for the PDE $\partial_y^2 v = \frac{1}{\eta}\partial_x p$

with proper boundary conditions.

We can use the superposition principle since it is a linear BVP.

Planar Couette flow

We want to solve the previous equation between two moving rigid walls.

The boundary conditions are (for example) $v(y=0)=0$ and $v(y=L)=U_\infty$

The general solution is $v(y)=\frac{1}{2\eta}\partial_x p y^2 + Ay + B$ and if we insert the BC,

$$\boxed{v(y)=U_\infty\left[\frac{y}{L}-\left(\frac{L^2\partial_x p}{2\eta U_\infty}\right)\frac{y}{L}\left(1-\frac{y}{L}\right)\right]}$$

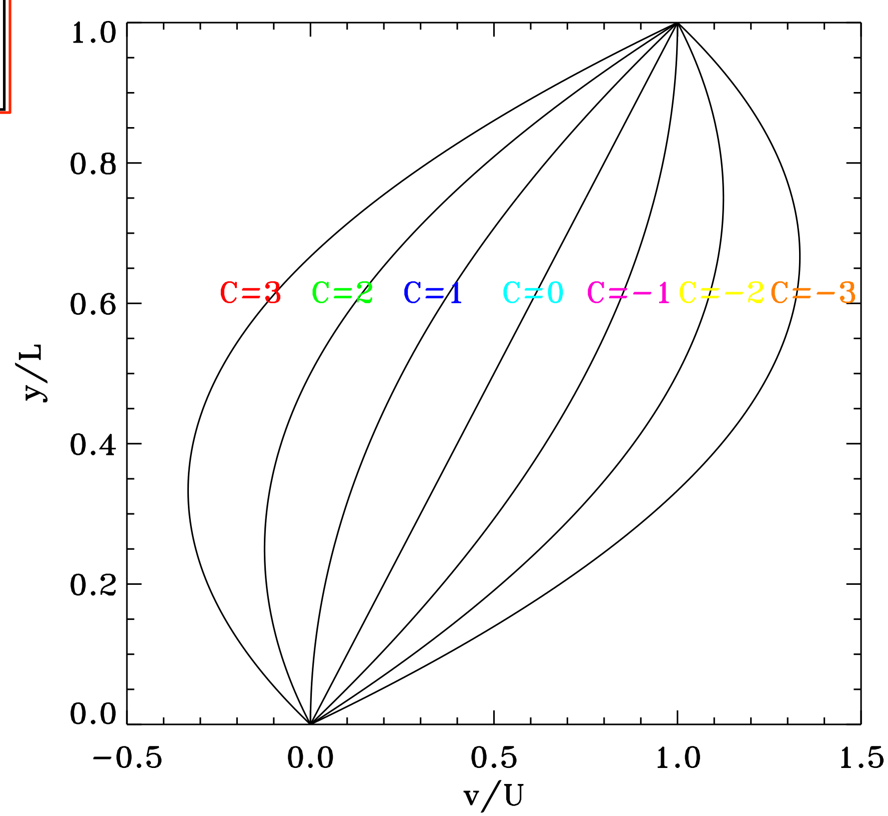
The family of curves we have obtained

depends on one parameter

$$\boxed{C=\frac{L^2\partial_x p}{2\eta U_\infty}}$$

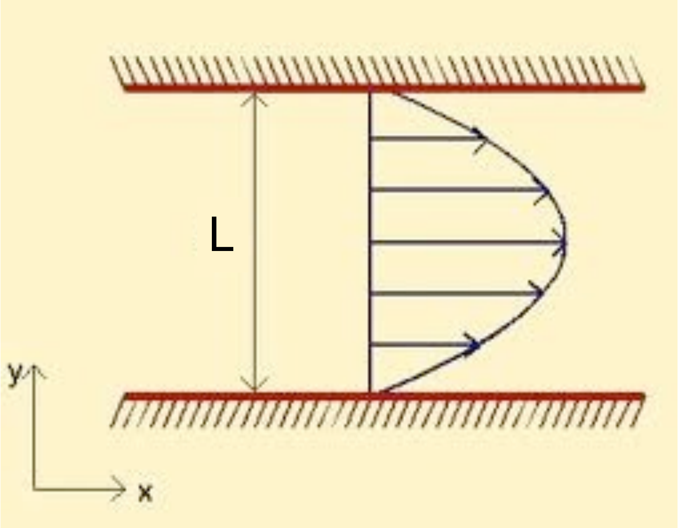
For $U_\infty = 0$, we get the Poisseuille flow

$$\boxed{v(y)=-\left(\frac{L^2\partial_x p}{2\eta}\right)\frac{y}{L}\left(1-\frac{y}{L}\right)}$$



Properties of the Poiseuille flow

$$\boxed{v(y) = -\left(\frac{L^2\partial_x p}{2\eta}\right)\frac{y}{L}\left(1-\frac{y}{L}\right)}$$



If the pressure gradient is negative, the flow is traveling to the right.

The mass flux is $\dot{m} = \rho\int_0^L v(y)\mathrm{d}y = -\frac{\rho L^3}{12\eta}\partial_x p$

The vorticity is $\omega_z = -\partial_y v = \frac{\partial_x p}{\eta}\left(y-\frac{L}{2}\right)$

The stress field (force per unit surface) acting on any horizontal layer is

$$\vec{T} = \overline{\overline{\sigma}}\vec{e}_y = \eta\partial_y v\vec{e}_x - p\vec{e}_y = \partial_x p\left(y-\frac{L}{2}\right)\vec{e}_x - p\vec{e}_y$$

The Poiseuille and Couette flows are shear flows.

For small Reynolds number ($Re < Re_c = 5722$) the planar Poiseuille flow is stable.

For high Reynolds number, it becomes unstable (see Kelvin-Helmholtz instability) and turbulence develops.

Non-stationary planar boundary layer.

We consider the case for which the pressure gradient vanishes.

The Navier-Stokes equation reads $\partial_t v = \nu \partial_y^2 v$

This equation is equivalent to the «heat equation» (v becomes the temperature).

We consider now the «first Stokes problem»:

the wall is impulsively put in motion at velocity U_∞ at time t=0

The boundary conditions are $v(y=0,t) = U_\infty$ and $v(y,t) \rightarrow 0 \text{ when } y \rightarrow +\infty$

Scaling properties of the Navier-Stokes equation:

We scale the variables by $t = \tilde{t}T$ $y = \tilde{y}L$ $v = \tilde{v}\frac{L}{T}$ and get $\partial_{\tilde{t}}\tilde{v} = \frac{\nu T}{L^2}\partial_{\tilde{y}}^2\tilde{v}$

If we choose $L = \sqrt{\nu T}$ the problem becomes scale-invariant.

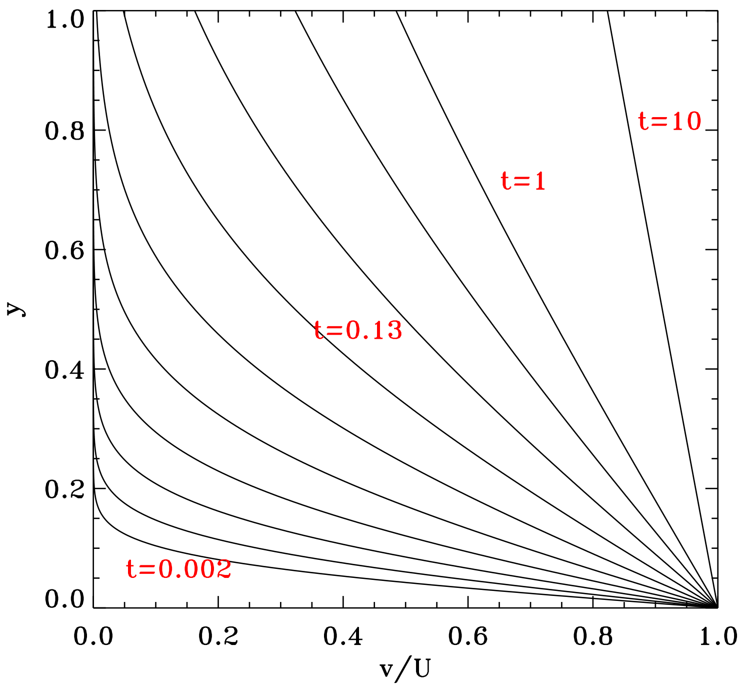
This suggests we choose a solution of the form $f(\xi)$ where $\xi = \frac{y}{2\sqrt{\nu t}}$

We have $d\xi = \frac{\xi}{y}dy - \frac{\xi}{2t}dt$ so that $\partial_t v = -f'\frac{\xi}{2t}$ and $\partial_x^2 v = f''\frac{\xi^2}{y^2}$

The equation becomes $\boxed{f'' + 2\xi f' = 0}$ and lead to $f' = A\exp^{-\xi^2}$

The solution is $v(y,t) = U_\infty\left(1 - \mathrm{erf}(\xi)\right)$ where $\mathrm{erf}(\xi) = \frac{2}{\sqrt{\pi}}\int_0^\xi \exp^{-u^2}\mathrm{d}u$

Properties of the first Stokes problem



The transverse velocity is diffusing away from the wall due to viscous stresses.

The thickness of the diffusive layer increases with time.

We can define the thickness of the layer by

$v = 1\% U_\infty$ which corresponds to $\xi = 2$.

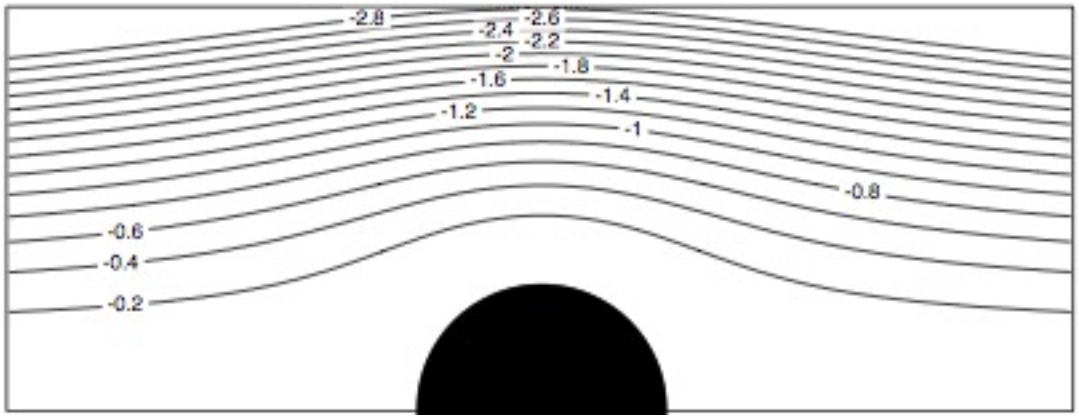
The thickness of the boundary layer is $\boxed{\delta = 4\sqrt{\nu t}}$

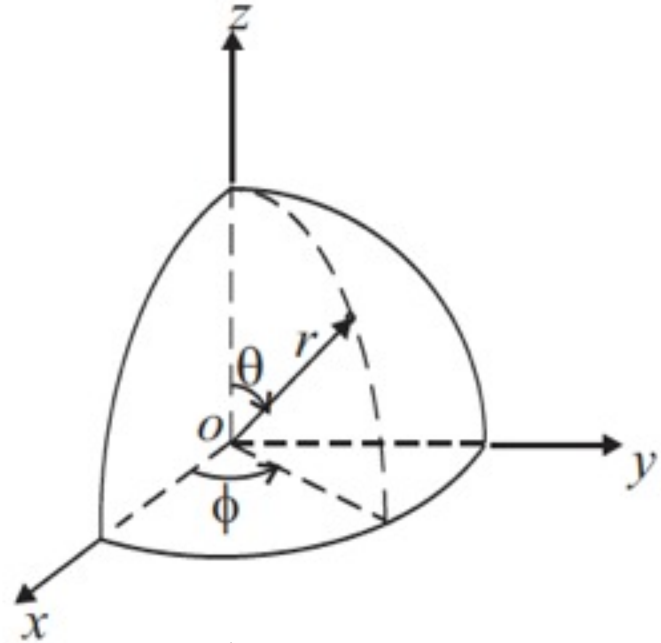
The vorticity profile is $\omega_z = -\partial_y v = -\frac{U_\infty}{\sqrt{\pi \nu t}} \exp^{-y^2/(4\nu t)}$

Initially, the vorticity is a Delta function at the wall. It then diffuses away from the wall, always confined within the growing boundary layer.

At late time, the vorticity vanishes while the flow follows uniformly the wall velocity.

Viscous flow past a sphere





This is also called the «second Stokes problem».

For small Reynolds number stationary flows, we have $\eta\Delta\vec{v} = \vec{\nabla}p$ and $\vec{\nabla}\cdot\vec{v} = 0$

We use spherical coordinates.

For symmetry reasons, we have $\vec{v} = v_r\vec{e}_r + v_\theta\vec{e}_\theta$ and $\vec{\omega} = \vec{\nabla}\times\vec{v} = \omega_\phi\vec{e}_\phi$

The boundary conditions are for $r \to +\infty$ $\vec{v} \to U_\infty\vec{e}_z = U_\infty\left(\cos\theta\vec{e}_r - \sin\theta\vec{e}_\theta\right)$

and the no-slip condition on the sphere ($r = a$) is $\vec{v} = 0$

We use a lot $\boxed{\vec{\nabla}\times\vec{B} = \frac{1}{r\sin\theta}\partial_\theta(\sin\theta B_\phi)\vec{e}_r - \frac{1}{r}\partial_r(rB_\phi)\vec{e}_\theta + \frac{1}{r}\left(\partial_r(rB_\theta) - \partial_\theta B_r\right)\vec{e}_\phi}$

Viscous flow past a sphere

We use the vector potential approach (Helmholtz decomposition) to satisfy automatically the divergence free condition.

We have $\vec{v} = \overrightarrow{\nabla} \times \overrightarrow{A}$ and $\vec{\omega} = \overrightarrow{\nabla} \times \vec{v}$

The symmetry of the flow requires $\overrightarrow{A} = A_\phi \vec{e}_\phi$

The velocity field writes $\vec{v} = \frac{1}{r\sin\theta}\partial_\theta(\sin\theta A_\phi)\vec{e}_r - \frac{1}{r}\partial_r(rA_\phi)\vec{e}_\theta$

and the vorticity $\overrightarrow{\omega} = -\frac{1}{r}\left(\partial_r^2(rA_\phi) + \frac{1}{r}\partial_\theta(\frac{1}{\sin\theta}\partial_\theta(\sin\theta A_\phi))\right)\vec{e}_\phi$

We also know from the Navier-Stokes equation that $\Delta\overrightarrow{\omega} = -\overrightarrow{\nabla} \times \overrightarrow{\nabla} \times \overrightarrow{\omega} = 0$

We are looking for solutions of the form $\boxed{rA_\phi(r,\theta) = f(r)\sin\theta}$

The vorticity is $-r\omega_\phi = \left(f'' - \frac{2f}{r^2}\right)\sin\theta = -g(r)\sin\theta$ while NS gives $g'' - \frac{2g}{r^2} = 0$

The final equation we need to solve is $f'''' - \frac{4f''}{r^2} + \frac{8f'}{r^3} - \frac{8f}{r^4} = 0$

With power law functions $f(r) = r^\alpha$ we have $(\alpha+1)(\alpha-1)(\alpha-2)(\alpha-4) = 0$

The general solution is $\boxed{\vec{v} = \frac{2f}{r^2}\cos\theta\vec{e}_r - \frac{f'}{r}\sin\theta\vec{e}_\theta}$ with $\boxed{f(r) = \frac{A}{r} + Br + Cr^2 + Dr^4}$

Viscous flow past a sphere

We now impose the boundary conditions.

For $r \to +\infty$, we have $\vec{v}_\infty = 2C\cos\theta\vec{e}_r - 2C\sin\theta\vec{e}_\theta$ $\quad D = 0 \quad C = \frac{U_\infty}{2}$

For $r = a$, we have $f(a) = 0$ and $f'(a) = 0$ $\quad A = \frac{1}{4}U_\infty a^3 \quad B = -\frac{3}{4}U_\infty a$

$$\boxed{v_r = U_\infty \cos\theta \left(1 + \frac{a^3}{2r^3} - \frac{3a}{2r}\right)}$$

$$\boxed{v_\theta = -U_\infty \sin\theta \left(1 - \frac{a^3}{4r^3} - \frac{3a}{4r}\right)}$$

The vorticity is $\omega_\phi = -U_\infty \frac{3a}{2r^2}\sin\theta$

Using the Navier-Stokes equation, we have $\partial_r p = \eta U_\infty \frac{3a}{r^3}\cos\theta$

Integrating from 0 to ∞, we have $p = p_\infty - \eta U_\infty \frac{3a}{2r^2}\cos\theta$

We now compute the stress field on the sphere $\bar{\bar{\sigma}}\vec{e}_r = (\sigma_{rr}, \sigma_{r\theta}, 0)$

$$\sigma_{rr} = -p + \eta\partial_r v_r = -p = -p_\infty + \eta U_\infty \frac{3}{2a}\cos\theta$$

$$\sigma_{r\theta} = \eta\left(\frac{1}{r}\partial_\theta v_r + r\partial_r\left(\frac{v_\theta}{r}\right)\right) = -\eta U_\infty \frac{3}{2a}\sin\theta$$

$$F_z = \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \mathrm{d}\theta \sin\theta \eta U_\infty \frac{3a}{2}$$

The Stokes formula for the drag force is

$$\boxed{F_z = 6\pi\eta a U_\infty}$$

The boundary layer problem

We saw that the Navier-Stokes equation in dimensionless form writes

$$\frac{D\vec{v}}{Dt} = -\frac{1}{\rho}\vec{\nabla}p + \frac{1}{Re}\Delta\vec{v} \quad \text{with} \quad \vec{\nabla}\cdot\vec{v} = 0 \quad \text{and} \quad \vec{v} = 0 \quad \text{at the boundary.}$$

For large Reynolds number, the equation converges towards the ideal fluid Euler equation, but not uniformly.

The main reason is that the Euler equations use a slip condition: $\vec{v}\cdot\vec{n} = 0$

A simple analogy:

We want to solve $\partial_x f = 0$ with $\lim_{x\to+\infty} f(x) = 1$.

The solution is trivially $f(x) = 1$ and is analogous to a inviscid flow in a half-plane.

We now solve $\epsilon\partial_x^2 f + \partial_x f = 0$ with the additional constraint $f(0) = 0$

The solution is now $f(x) = 1 - \exp^{-x/\epsilon}$

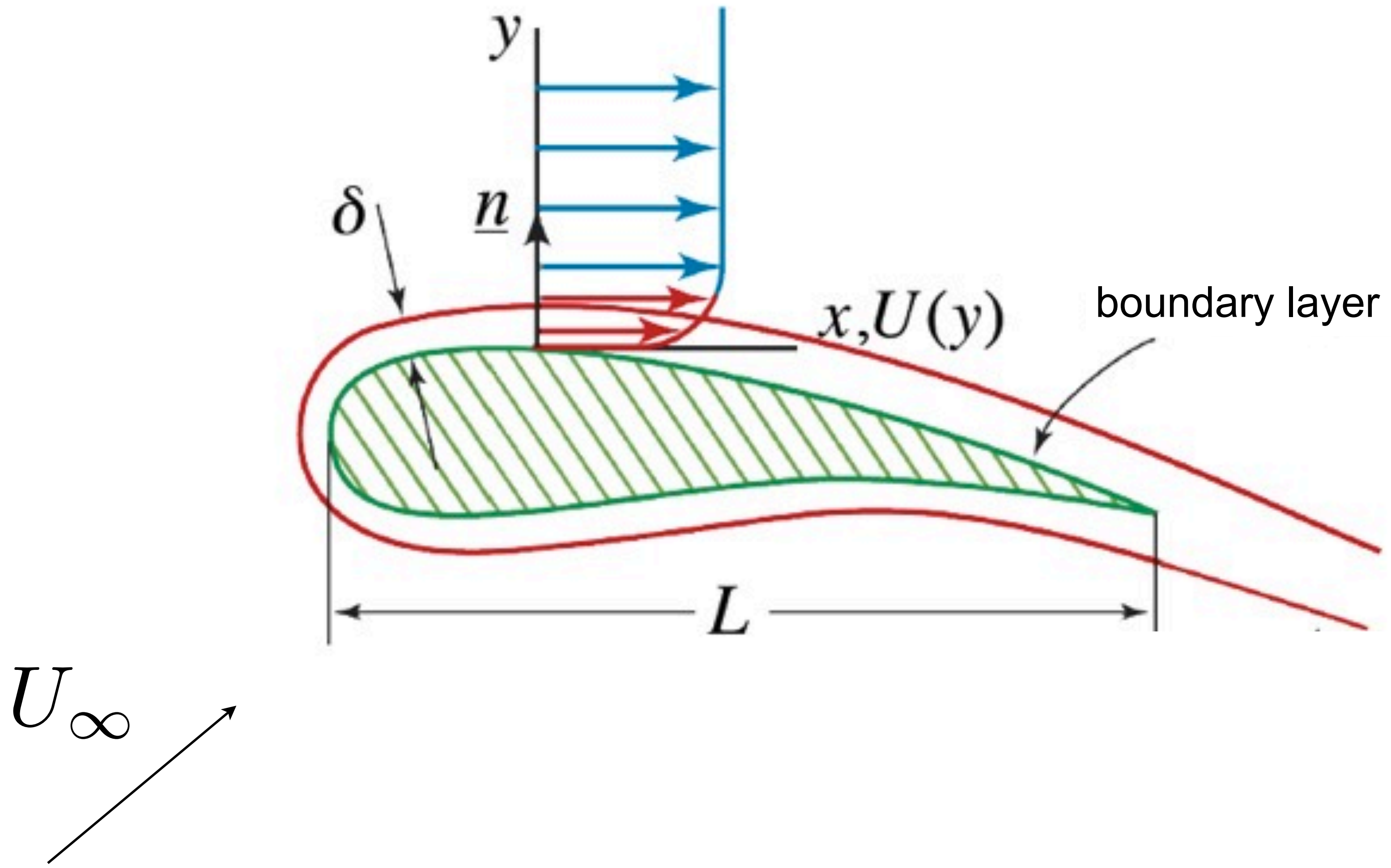
It does not converge uniformly towards 1.

For $x \leq \epsilon$, the flow will never reach the ideal limit.

This thin layer, called the boundary layer, is always present.

It can be ignored only if its properties do not perturb the ideal flow for $x > \epsilon$.

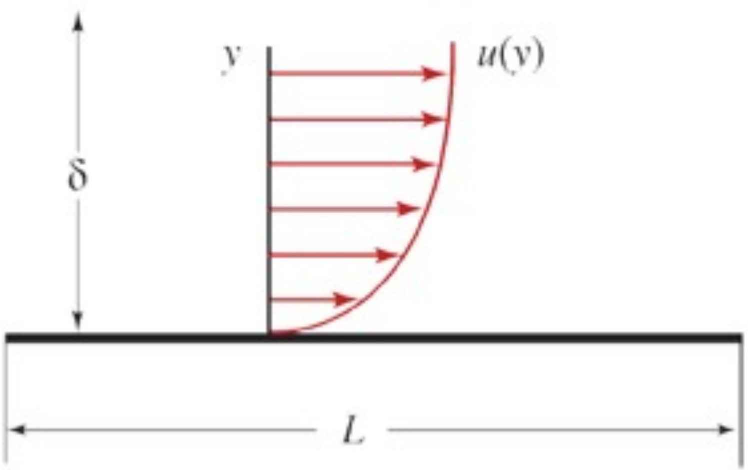
The boundary layer problem



Boundary layer on a finite plate

For an infinite plate and an impulsive start, we got a non-stationary boundary layer of thickness $\delta = 4\sqrt{\nu t}$.

We now consider a finite plate of size L, with upwind velocity U_∞.



The stationary Navier-Stokes equations write for $\vec{v} = (u(x,y), v(x,y))$

$$u\partial_x u + v\partial_y u = -\frac{1}{\rho}\partial_x p + \nu\Delta u$$

$$u\partial_x v + v\partial_y v = -\frac{1}{\rho}\partial_y p + \nu\Delta v$$

$$\partial_x u + \partial_y v = 0$$

with boundary conditions $u(x,0) = v(x,0) = 0 \text{ for } 0 \leq x \leq L$

$$u \rightarrow U_\infty \text{ and } v \rightarrow 0 \text{ for } (x,y) \rightarrow +\infty$$

The thin layer approximation

When we move with a fluid element starting at the leading edge of the plate,

we postulate a boundary layer of thickness $\delta \simeq \sqrt{\nu t}$

where t is the elapsed time since it first passed the edge $t \simeq \dfrac{x}{U_\infty}$

An approximation of the thickness is therefore (with $Re = U_\infty L/\nu$)

$$\boxed{\delta \simeq \sqrt{\frac{\nu x}{U_\infty}} = \frac{4L}{\sqrt{Re}}\sqrt{\frac{x}{L}}}$$

We now estimate what are the leading order terms in the equations.

$u \sim U_\infty \quad x \sim L \quad y \sim \delta \simeq \sqrt{\dfrac{\nu L}{U_\infty}}$ From $\partial_y v = -\partial_x u$, we get $v \sim \dfrac{\delta}{L} U_\infty \ll U_\infty$

$$u\partial_x u + v\partial_y u = -\frac{1}{\rho}\partial_x p + \nu(\cancel{\partial_x^2 u} + \partial_y^2 u)$$

$$\mathcal{O}(\frac{U_\infty^2}{L}) \quad \mathcal{O}(\frac{U_\infty^2}{L}) \quad \mathcal{O}(\frac{U_\infty^2}{L}) \quad \mathcal{O}(\nu\frac{U_\infty}{L^2}) \quad \mathcal{O}(\nu\frac{U_\infty}{\delta^2})$$

$$u\cancel{\partial_x v} + v\cancel{\partial_y v} = -\frac{1}{\rho}\partial_y p + \nu(\cancel{\partial_x^2 v} + \cancel{\partial_y^2 v})$$

$$\mathcal{O}(\frac{U_\infty^2}{L}\frac{\delta}{L}) \quad \mathcal{O}(\frac{U_\infty^2}{L}\frac{\delta}{L}) \quad \mathcal{O}(\frac{U_\infty^2}{L}\frac{L}{\delta}) \quad \mathcal{O}(\frac{U_\infty^2}{L}\frac{\delta}{L}\frac{1}{Re}) \quad \mathcal{O}(\frac{U_\infty^2}{L}\frac{\delta}{L})$$

The thin layer approximation

We have now simplified the previous system to:

$$u\partial_x u + v\partial_y u = -\frac{1}{\rho}\partial_x p + \nu\partial_y^2 u \qquad 0 = \partial_y p \qquad \partial_x u + \partial_y v = 0$$

p depends only on x: $p(x, y) = p_\infty(x)$

As before, the transverse pressure gradient is a free parameter and is related to the properties of the external flow at infinity. If the external flow is a potential flow, it satisfies the Bernoulli relation

$$\frac{1}{\rho}p_\infty(x) + \frac{1}{2}U_\infty^2(x) = \text{constant}$$

To satisfy automatically the divergence free condition, we use the stream function

$$u = \partial_y\psi \qquad v = -\partial_x\psi \qquad \psi = \mathcal{O}(U_\infty\delta)$$

$$\boxed{\partial_y\psi(\partial_{xy}^2\psi) - \partial_x\psi(\partial_y^2\psi) = \nu\partial_y^3\psi - \frac{1}{\rho}\partial_x p_\infty(x)}$$

Because the finite plate is infinitely thin, we look for self-similar solutions of the form

$$\boxed{\psi(x, y) = U_\infty(x)\delta(x)f(\xi) \text{ with } \xi = \frac{y}{\delta(x)} \text{ and } \delta(x) = \sqrt{\frac{\nu x}{U_\infty(x)}}}$$

The Blasius boundary layer solution

We consider in this case a flow with constant velocity and pressure at infinity.

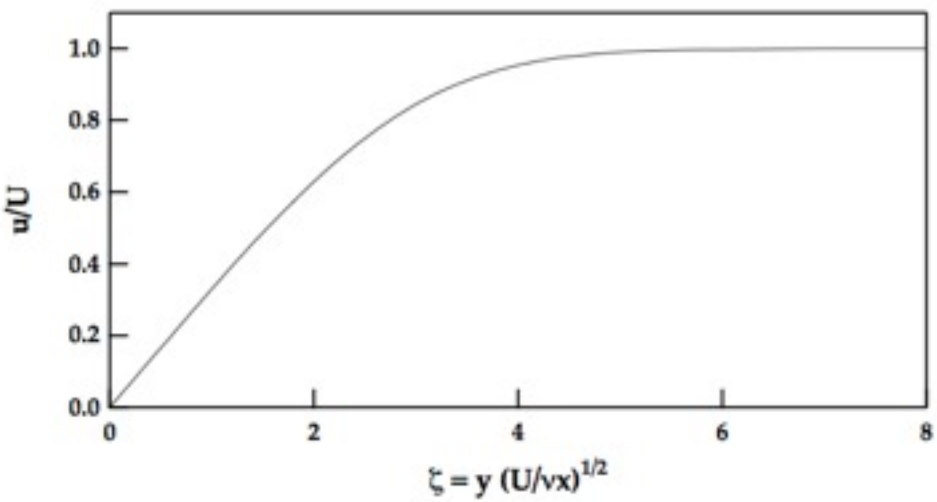
$$\partial_x \psi = U_\infty \delta' \left(f - \xi f'\right) \qquad \partial_y \psi = U_\infty f' \qquad \delta' = \frac{1}{2x}\delta$$

$$\partial^2_{xy}\psi = -U_\infty \frac{\delta'}{\delta}\xi f'' \qquad \partial^2_y \psi = U_\infty \frac{1}{\delta} f'' \qquad \partial^3_y \psi = U_\infty \frac{1}{\delta^2} f'''$$

We found the following simple ODE:

$$2f''' + ff'' = 0$$

$$f(0) = 0,\ \ f'(0) = 0 \text{ and } f'(+\infty) = 1.$$



The transverse velocity profile is

$$u(x, y) = U_\infty f'(\xi)$$

The normal velocity profile is

$$v(x, y) = -U_\infty \delta'(f - \xi f')$$

In this case, the thickness is also $4\delta(x)$

The stress field at the wall is given by $\vec{T} = \sigma_{xy}\vec{e}_x + (-p_e(x) + \sigma_{yy})\vec{e}_y$

The pressure contribution cancels and $\partial_x v = 0 \text{ and } \partial_y v = 0 \text{ at } y = 0$

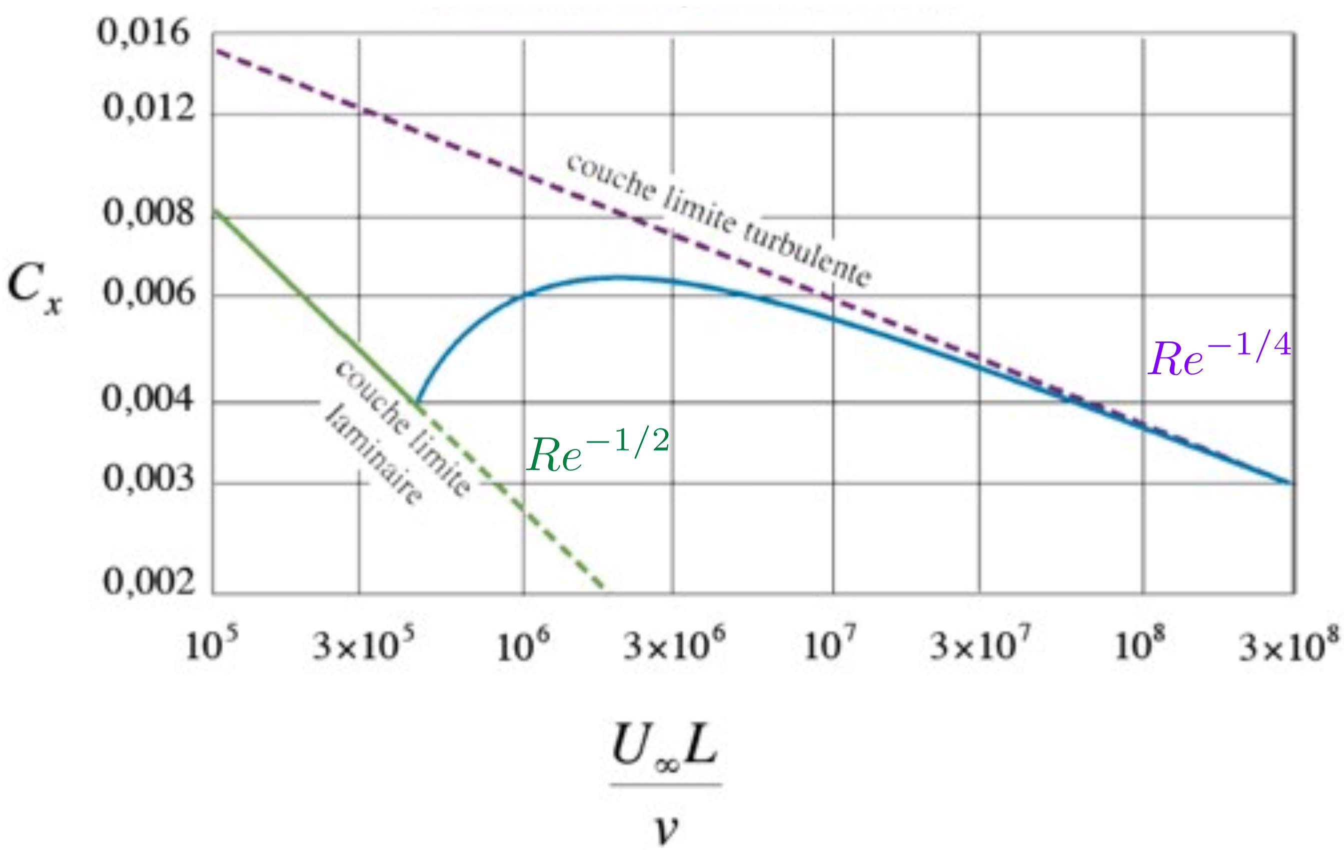
$$F_x = \int_{x=0}^{L} \eta \partial_y u(x, 0)\mathrm{d}x = 2\eta U_\infty f''(0)\sqrt{Re}$$

$$\boxed{C_x = \frac{F_x}{\frac{1}{2}\rho U_\infty^2 L} = \frac{4f''(0)}{\sqrt{Re}}}$$

Laminar versus turbulent boundary layers

The Blasius solution is also a laminar shear flow.

For large Reynolds numbers, it is unstable (Kelvin-Helmholtz instability for example) and becomes a turbulent flow.



Boundary layers with pressure gradients

We are now considering cases for which the velocity at infinity depends on x.

We will be able to study the effect of fluid accelerations or deceleration on the solution.

We consider self-similar velocity profiles such that $U_\infty(x) = Cx^m \quad \rightarrow \quad \frac{U'}{U} = \frac{m}{x}$

Using Bernoulli, we have also $\frac{1}{\rho}\partial_x p_\infty = -C^2 m x^{2m-1}$

The thickness new scales as $\delta(x) \propto x^{\frac{1-m}{2}} \quad \rightarrow \quad \frac{\delta'}{\delta} = \frac{1-m}{2x}$

We have $\partial_x \psi = U\delta\left[\frac{1+m}{2x}f - \frac{1-m}{2x}\xi f'\right] \qquad \partial^2_{xy}\psi = U\left[\frac{m}{x}f' - \frac{1-m}{2x}\xi f''\right]$

$$\partial_y \psi = f'U \qquad \partial_y^2\psi = f''\frac{U}{\delta} \qquad \partial_y^3\psi = f'''\frac{U}{\delta^2}$$

We obtain the Falkner-Skan solution: $m(f')^2 - \frac{m+1}{2}ff'' = f''' + m$

$$f(0) = 0,\ f'(0) = 0 \text{ and } f'(+\infty) = 1.$$

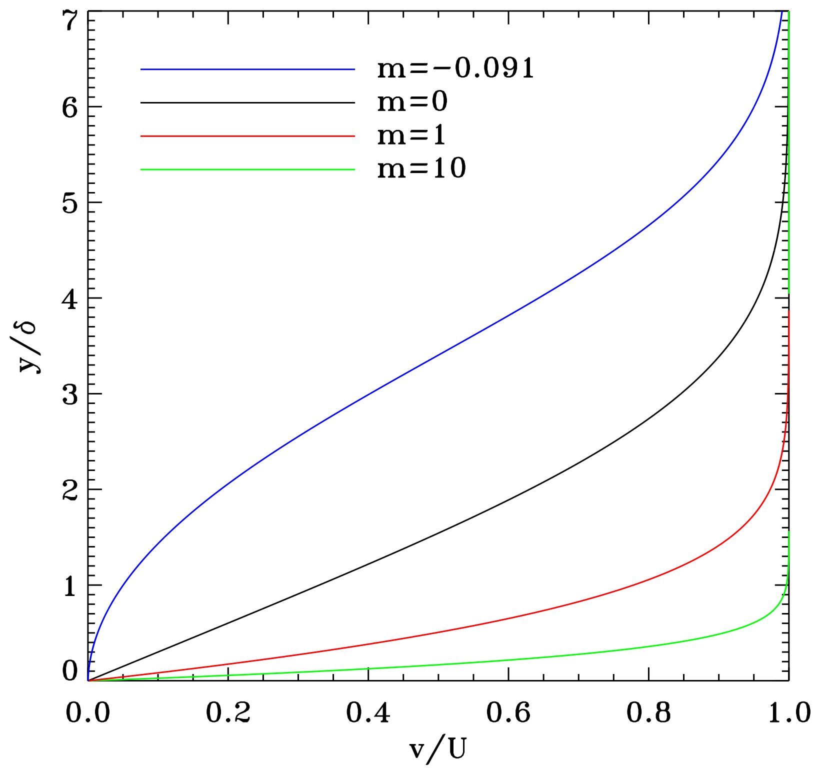
The solution is solved numerically using shooting techniques.

Falkner-Skan solutions

External potential flows with complex potential $F(z) = z^m$

For m=0, we recover the Blasius solution.

For m>0, the flow is accelerating along the wall (flow past a corner). The pressure gradient is said to be *favorable* and the boundary layer is confined to the wall.



For m=1, the boundary layer has a constant thickness (stagnation point).

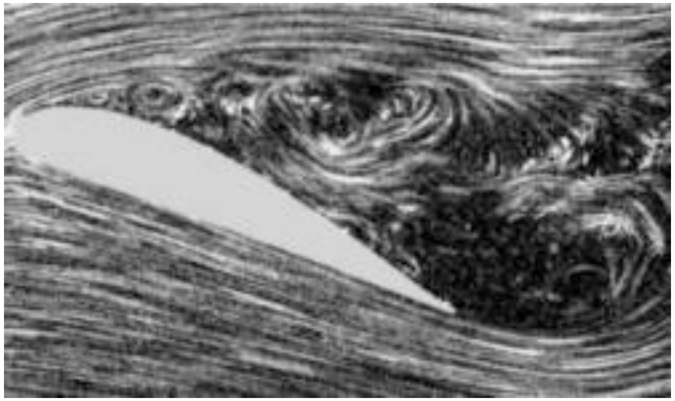
For m<0, the flow is decelerating and we have an *adverse* pressure gradient (flow past an edge). For m=-0.091, $f''(0) = 0$ and there is no more drag force.

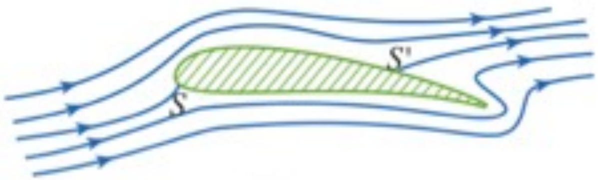
For m<-0.091, the flow is reversed (similar to Couette flows) and the boundary layer separates from the wall. The flow is detached from the wall .

When the flow is detached, the viscous layer increases dramatically and starts to modify the external flow. Usually, strong turbulence develops.

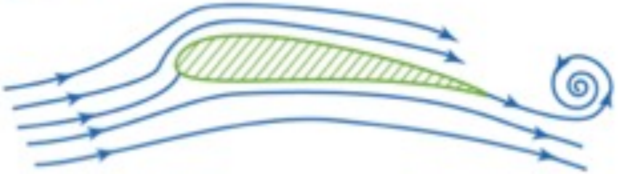
Flow separation

For large angle of attack, the flow separates from the body. Large eddies and vortices are generated and carry away vorticity, leaving behind an opposite circulation around the body.

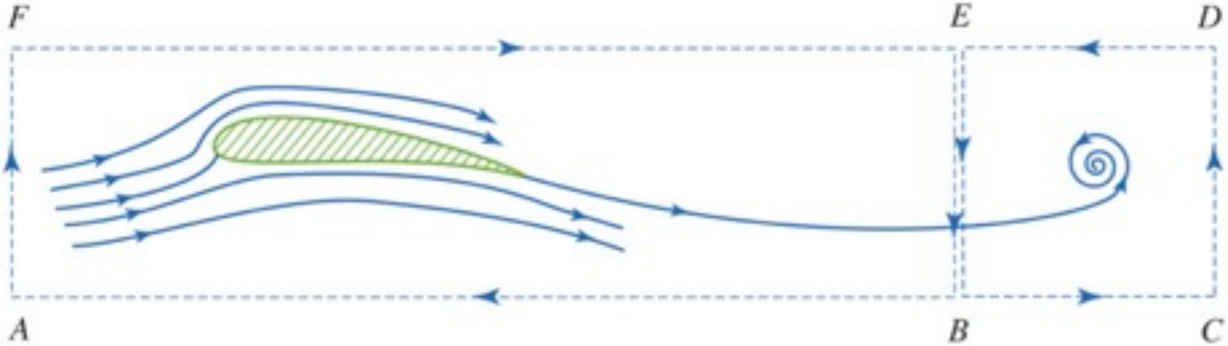




Flow past the trailing edge: adverse pressure gradients result in the flow separation. The boundary layer expands and modify the initial potential flow solution.



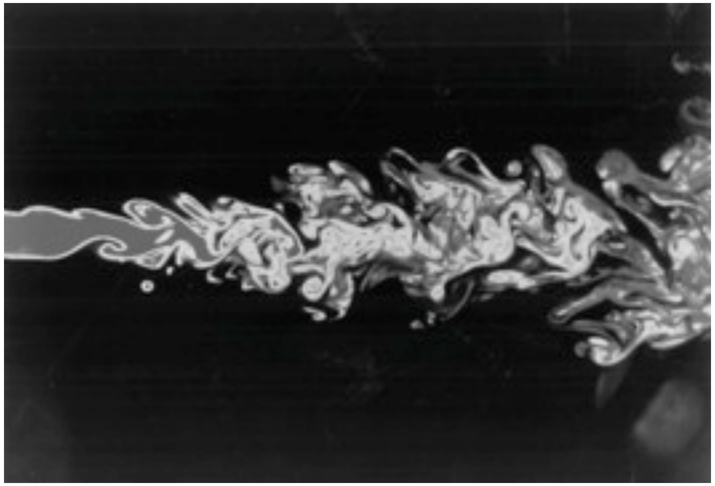
The vorticity of the boundary layer is advected away, resulting in a new potential flow with favorable pressure gradients and a laminar boundary layer around the body.



Turbulence

When turbulence develops, we see velocity fluctuations both in time and in space.

This chaotic velocity field can be decomposed into a smooth, average, laminar field and a fluctuating field, for which only the statistical properties are known.



$$v_i = V_i + v_i'$$

true velocity = average velocity + fluctuation

The fluctuation is described by a Probability Density Function (PDF) such as

$$\int f(v')\mathrm{d}v' = 1 \qquad \int v' f(v')\mathrm{d}v' = 0 \qquad \int (v')^2 f(v')\mathrm{d}v' = \sigma^2$$

Average quantities can be average in time or space.

We will consider here instead *ensemble averages* between N independent realizations. We have the following results for ensemble averages:

$$\overline{v} = V \qquad \overline{v'} = 0 \qquad \overline{v'^2} = \sigma^2 \qquad \overline{\partial_x A} = \partial_x \overline{A} \qquad \overline{\partial_t A} = \partial_t \overline{A}$$

Reynolds stress and turbulent pressure

The Navier-Stokes equation writes $\partial_t v_i + (\vec{v}\cdot\vec{\nabla})v_i = -\frac{1}{\rho}\partial_i p + \nu\Delta v_i$ with $\vec{\nabla}\cdot\vec{v} = 0$

Taking the ensemble average of the divergence $\overline{\vec{\nabla}\cdot\vec{v}} = \vec{\nabla}\cdot\left(\vec{V} + \overline{\vec{v}'}\right) = \vec{\nabla}\cdot\vec{V} = 0$

tells us that the mean flow should also be divergence free.

From $\vec{\nabla}\cdot\vec{v} = \vec{\nabla}\cdot\vec{V} + \vec{\nabla}\cdot\vec{v}'$, we get that the fluctuations also follows $\vec{\nabla}\cdot\vec{v}' = 0$

For a divergence free velocity field, we have $(\vec{v}\cdot\vec{\nabla})v_i = \vec{\nabla}\cdot(v_i\vec{v})$

The average of the inertial term can be decomposed into 2 terms

$$\overline{(\vec{v}\cdot\vec{\nabla})v_i} = \vec{\nabla}\cdot(\overline{v_i\vec{v}}) = \vec{\nabla}\cdot(V_i V_j) + \vec{\nabla}\cdot\overline{v_i' v_j'} = (\vec{V}\cdot\vec{\nabla})V_i + \vec{\nabla}\cdot\sigma_{ij}^2$$

The average Navier-Stokes equation $\partial_t\vec{V} + (\vec{V}\cdot\vec{\nabla})\vec{V} = -\frac{1}{\rho}\vec{\nabla}P + \nu\Delta\vec{V} + \vec{\nabla}\cdot\overline{\overline{R}}$

where we introduced a new tensor called the *Reynolds stress* $\boxed{R_{ij} = -\sigma_{ij}^2 = -\overline{v_i' v_j'}}$

We have $\mathrm{Tr}(\overline{\overline{R}}) = -\sigma^2 = -\sigma_{xx}^2 - \sigma_{yy}^2 - \sigma_{zz}^2$ We define $\overline{\overline{T}} = \overline{\overline{R}} + \frac{\sigma^2}{3}\overline{\overline{1}}$ so $\mathrm{Tr}(\overline{\overline{T}}) = 0$

Using these definitions, we finally get, with $P_T = \frac{1}{3}\rho\sigma^2$ the turbulent pressure,

$$\boxed{\partial_t\vec{V} + (\vec{V}\cdot\vec{\nabla})\vec{V} = -\frac{1}{\rho}\vec{\nabla}(P + P_T) + \nu\Delta\vec{V} + \vec{\nabla}\cdot\overline{\overline{T}}} \qquad \boxed{T_{ij} = \frac{\sigma^2}{3}\delta_{ij} - \sigma_{ij}^2}$$

Properties of the Reynolds stress

The Reynolds stress is symmetric because it is based on 2nd order moments.

Like the viscous stress has been interpreted as a momentum flux due to microscopic forces at the atomic level (surface forces), the Reynolds stress expresses the momentum transported by eddies and vortices in the flow.

We now want to model the Reynolds stress R or the turbulent shear stress T as a function of the mean flow properties: *the closure problem*.

In analogy to Newtonian fluids, where the stress tensor is proportional to the rate of strain tensor, we introduce the *Boussinesq approximation*

$$T_{ij} = \nu_T \left(\partial_{x_j} V_i + \partial_{x_i} V_j\right)$$

From kinetic theory, we know that the fluid viscosity is $\nu \simeq c_s \lambda$ where c_s is the sound speed and λ is the mean free path of the molecules.

The *eddy-viscosity model* assumes that $\nu_T \simeq v_T \ell_m$ where v_T is the typical velocity of vortices and l_m is called the *mixing length*. They are free parameters of the theory. The turbulent pressure is usually approximated by $P_T = \frac{1}{3}\rho v_T^2$

In shear flows, a good approximation is $v_T = \ell_m \left|\partial_y V_x\right|$ so only one free parameter remains, the mixing length.

Effect of turbulence on laminar flows

We use the mixing length approach for our boundary layer problems.

We assume that the mixing length is proportional to the thickness of the layer.

We consider a (mean) flow that follows the Blasius solution.

$$\nu_T = \delta(x)^2 \left|\partial_y u(x,0)\right| = \delta(x) U_\infty f''(0)$$

Working at x=L, we can compute the turbulent Reynolds number $\boxed{Re_T = \frac{U_\infty L}{\nu_T}}$

We find $Re_T \simeq 3\sqrt{Re}$, therefore much smaller than the laminar value.

Turbulence increases the effective viscosity and associated momentum transport.

Inserting this new Reynolds number into the drag force formulae, we find

$$\boxed{(C_x)_T \simeq \frac{4}{3\sqrt{Re_T}} = \frac{4}{3\sqrt{3}Re^{1/4}}}$$

The thickness of the boundary layer at x=L will also grow significantly,

from its laminar value $\delta \simeq \frac{4}{\sqrt{Re}}$ to its turbulent value $\boxed{\delta_T \simeq \frac{4}{\sqrt{3}Re^{1/4}}}$.