Computational Astrophysics 6

Radiation Hydrodynamics

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Outline

- The Radiation Transfer Equation
- Microscopic Processes (TE, LTE, CE)
- Moments of the RT Equation
- Optically Thin and Thick Limits
- Closure Relation and the M1 Approximation
- A Godunov Solver for Radiative Transfer
- Stiff Source Terms and Implicit Schemes

The Radiative Transfer Equation

Conservation of photons in phase-space

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \mathbf{n} \cdot \nabla I_\nu = -\kappa_\nu I_\nu + \eta_\nu,$$

$I_\nu(\mathbf{x}, \mathbf{n}, t)$	radiation specific intensity
$\kappa_\nu(\mathbf{x}, \mathbf{n}, t)$	absorption coefficient
$\eta_\nu(\mathbf{x}, \mathbf{n}, t)$	source function

Source term: microscopic collisions with plasma particles.
Absorption and emission processes.
A special case: scattering processes.

Atomic Processes

Radiation is emitted or absorbed when electrons make transitions between different states

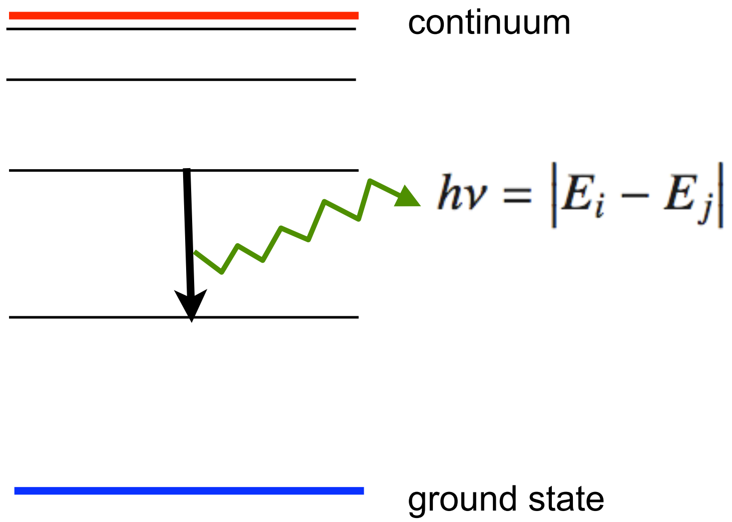
Bound-bound: electrons moves between 2 bound states in an atom or an ion. A photon is emitted or absorbed.

Bound-free: electrons move to the continuum (ionization) or a absorbed from the continuum to a bound state (recombination)

Free-free: electrons in the continuum gain or loose energy (a photon) when orbiting around ions (Bremsstrahlung).

Absorption and emission in lines

Transitions between two atomic energy levels



Boltzmann's Law (LTE):

$$\frac{n_j}{n_i} = \frac{g_j}{g_i} \exp\left(-\left(E_j - E_i\right)/k_B T\right)$$

Ionization energy: transition to the continuum $\chi_i = |E_\infty - E_0|$

Two regimes for ionization and recombination

1 Ionization Equilibrium: the Saha Distribution
Free electrons are in LTE (Maxwell distribution).
Sum over all possible configurations

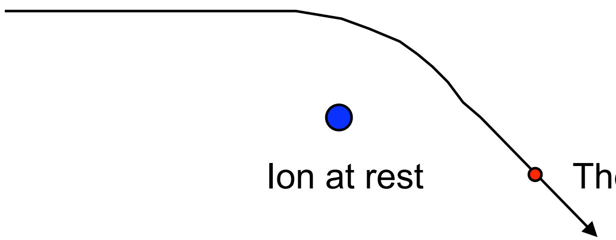
$$\frac{N_{i+1}}{N_i} = \frac{2}{n_e}\frac{Z_{i+1}}{Z_i}\left(2\pi m_e k T_e / h^2\right)^{3/2} \exp\left(-\chi_i / k_B T_i\right)$$

2 Non-equilibrium ionization balance (here for H):

$$\frac{\partial n_{\mathrm{H}_0}}{\partial t} = -n_{\mathrm{H}_0} c \sigma_\gamma N_\gamma + n_{\mathrm{e}} n_{\mathrm{H}_+} \alpha_A - n_{\mathrm{e}} n_{\mathrm{H}_0} \beta,$$

photo-ionization collisional recombination collisional ionization

Bremsstrahlung



Coulomb collisions between charged particles.
Plasma at LTE (Maxwell with temperature T_e and T_i)
Ionization balance gives n_e and n_i (ions of charge Z)

Absorbed and emitted radiation is

$$\epsilon^{ff} = 1.4 \times 10^{-27} n_e n_i Z^2 T_e^{1/2} \quad \mathrm{erg/sec/cm^3}$$

$$\epsilon_\nu^{ff} = 6.8 \times 10^{-38} n_e n_i Z^2 T_e^{-1/2} \exp\left(-h\nu/k_B T_e\right)$$

in $\mathrm{erg/sec/cm^3/Hz}$

4 regimes for matter and radiation interactions

1- Thermodynamical Equilibrium
2- Local Thermodynamical Equilibrium
3- Coronal Equilibrium
4- Non-equilibrium radiation

Thermodynamical Equiibrium

Full Thermodynamical Equilibrium:
- Maxwell's law for electrons and ions velocities.
- Boltzmann's law for bound states
- Saha's law for ionization states
- Planck's law for photon energy
- Isotropic radiation.

$$B_\nu(T) = \frac{2h\nu^3/c^2}{\exp\left(h\nu/k_B T\right) - 1}$$

A variant: assume TE but with different T_r, T_e and T_i

Local Thermodynamical Equilibrium:
- Maxwell's law for electrons and ions velocities.
- Boltzmann's law for bound states
- Saha's law for ionization states
- *Radiation field is neither Planckian nor isotropic*

Typical regime for high-density astrophysical plasmas.

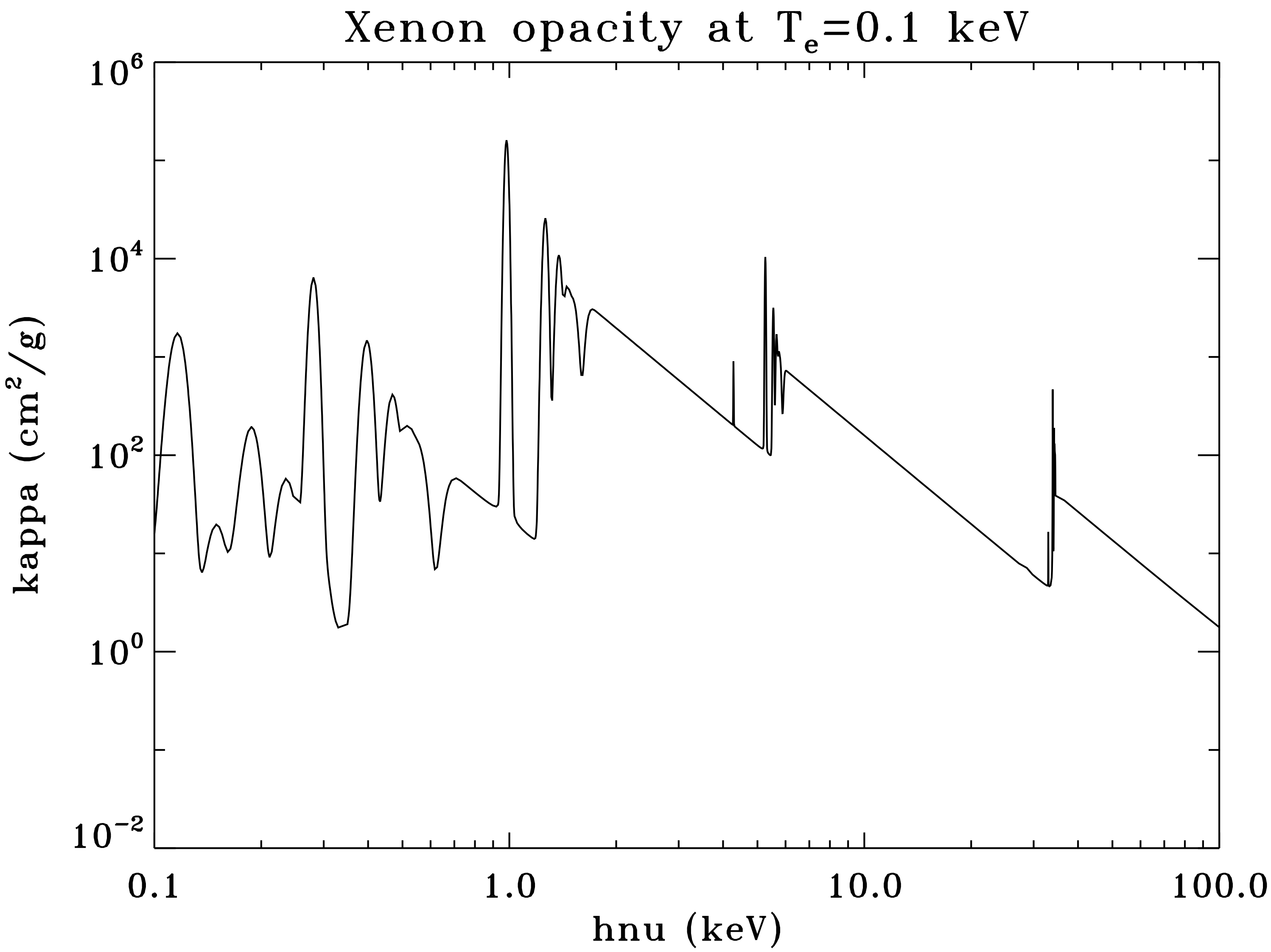
Coronal Equilibrium

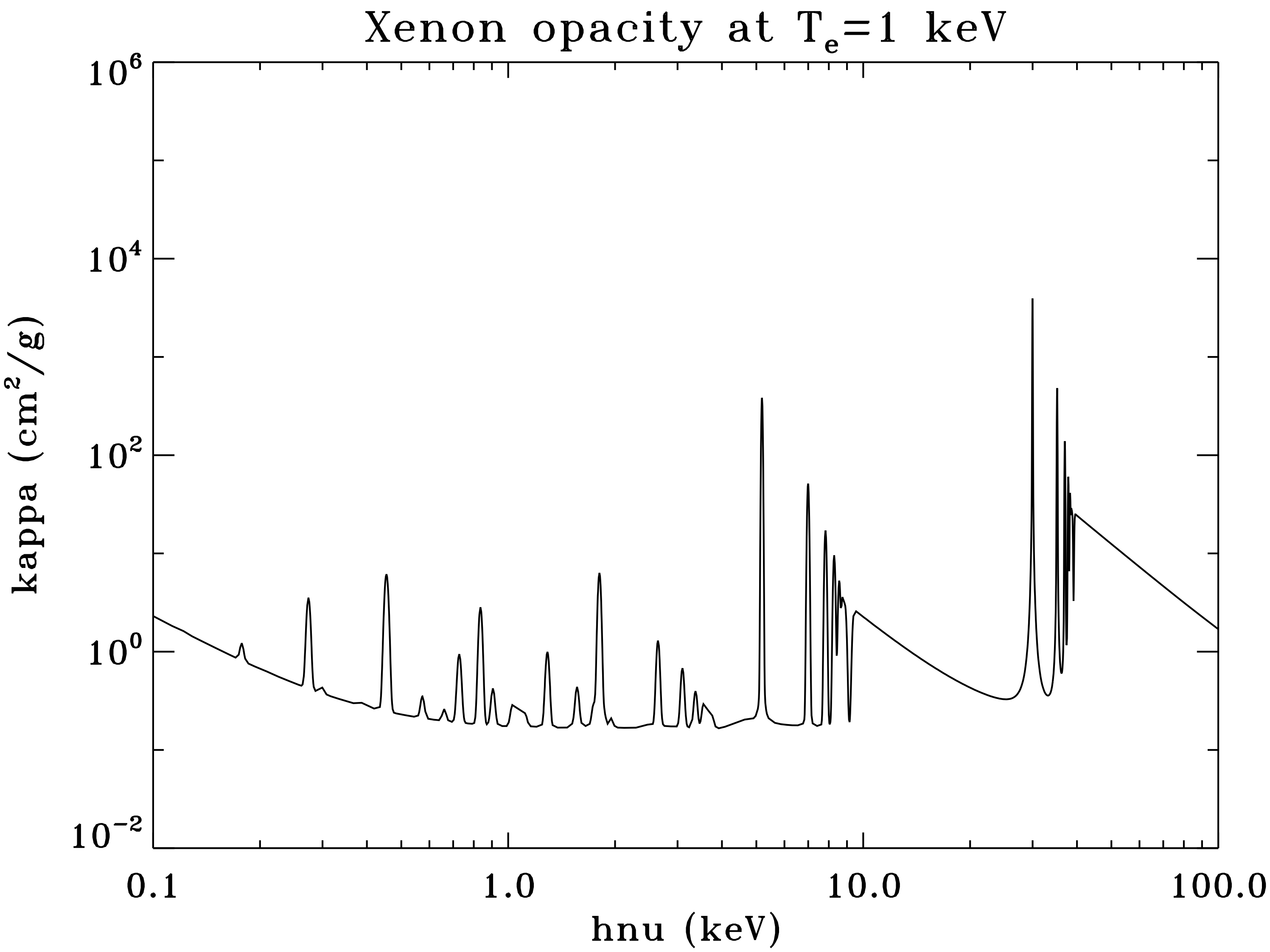
Conditions for Coronal Equilibrium:
- Maxwell's law for electrons and ions velocities.
- Boltzmann's law for bound states
- *Non-equibrium ionization balance*
- *Non-equilibrium radiation field*

We need to solve for the ionization balance equations.
Typical regime for low-density astrophysical plasmas.

Last case: non-LTE radiation field
- Maxwell's law for electrons and ions velocities.
- *Non-LTE level populations*
- *Non-equilibrium ionization balance*
- *Non-equilibrium radiation field*

We need to solve for each level populations.


Xenon opacity at T_e=0.1 keV
10^6
10^4
10^2
10^0
10^-2
kappa (cm^2/g)
0.1
1.0
10.0
100.0
hnu (keV)


Xenon opacity at T_e=1 keV
10^6
10^4
10^2
10^0
10^{-2}
kappa (cm^2/g)
0.1
1.0
10.0
100.0
hnu (keV)

Moments of the Radiative Transfer equation

Radiation energy

$$E_\nu(\mathbf{x}, t) = \int I_\nu(\mathbf{x}, \mathbf{n}, t) \frac{\mathrm{d}\mathbf{\Omega}}{c}$$

Radiation flux

$$\mathbf{F}_\nu(\mathbf{x}, t) = \int I_\nu(\mathbf{x}, \mathbf{n}, t)\mathbf{n} \frac{\mathrm{d}\mathbf{\Omega}}{c}$$

Radiation pressure tensor

$$\mathbf{P}_\nu(\mathbf{x}, t) = \int I_\nu(\mathbf{x}, \mathbf{n}, t)\mathbf{n} \times \mathbf{n} \frac{\mathrm{d}\mathbf{\Omega}}{c}$$

Moments of the Radiative Transfer Equation

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \mathbf{n}\cdot\nabla I_\nu = -\kappa_\nu I_\nu + \eta_\nu ,$$

$$\int \mathrm{d}\boldsymbol{\Omega} \quad \times \qquad \frac{\partial E_\nu}{\partial t} + \nabla \mathbf{F}_\nu \quad = \quad -\kappa_\nu c E_\nu + S_\nu ,$$

$$\int \mathbf{n}\mathrm{d}\boldsymbol{\Omega} \quad \times \qquad \frac{\partial \mathbf{F}_\nu}{\partial t} + c^2 \nabla \mathbf{P}_\nu \quad = \quad -\kappa_\nu c \mathbf{F}_\nu .$$

Fluid energy equation writes: $\dfrac{\partial E}{\partial t} + \nabla\cdot(\mathbf{u}\,(E+P)) = \Gamma - \Lambda$

Heating and cooling functions:

$$\Gamma = \int \kappa_\nu c E_\nu \mathrm{d}\nu \qquad \Lambda = \int S_\nu \mathrm{d}\nu$$

Under LTE conditions, one has $S_\nu = \kappa_\nu c B_\nu(T)$

The Eddington tensor

The Pressure tensor encapsulate the local geometry of the radiation field.

Define a dimensionless tensor called the Eddington tensor:

$$\mathbf{P}_\nu = \mathbf{D}_\nu E_\nu$$

Close to LTE, radiation field is nearly isotropic: $\mathbf{D} = \frac{1}{3}\mathbf{I}$

Single stream regime, radiation field in only one direction: $\mathbf{D} = \mathbf{n}_0 \times \mathbf{n}_0$

We need a radiation model to specify the Eddington tensor.

Infinite hierarchy: we need a closure relation.

Optically thick or optically thin

$\lambda_\nu = \frac{1}{\kappa_\nu}$ is the mean free path of the photons

Define a radiation time scale $\tau_\nu = \frac{\lambda_\nu}{c}$

Two limiting cases:

$\lambda \gg L$: pure radiation transport regime
free streaming
no source terms (collision-less ?)

$\lambda \ll L$: diffusion limit (close to LTE)

The diffusion limit

Chapman-Enskog expansion in term of $\tau_\nu = \frac{\lambda_\nu}{c}$

$$\mathbf{F}_\nu = -\frac{c^2}{3}\tau_\nu \nabla E_\nu \qquad E_\nu = B_\nu(T) + \frac{c^2}{3}\tau_\nu^2 \nabla E_\nu$$

Integrate radiation over frequencies using $\nabla E_\nu \simeq \partial_T B_\nu \nabla T$

We get the total radiation flux: $\mathbf{F} = -\frac{4}{3}c^2 a T^3 \tau_R \nabla T$

The Rosseland average opacity: $\tau_R = \frac{1}{4aT^3}\int \tau_\nu \partial_T B_\nu \mathrm{d}\nu$

The fluid energy equation now features a new diffusion term:

$\Gamma - \Lambda = -\nabla \cdot \mathbf{F} = \nabla \cdot (\kappa_R \nabla T)$ with $\kappa_R = 4/3 c^2 a T^3 \tau_R$

Optically thin astrophysical plasmas

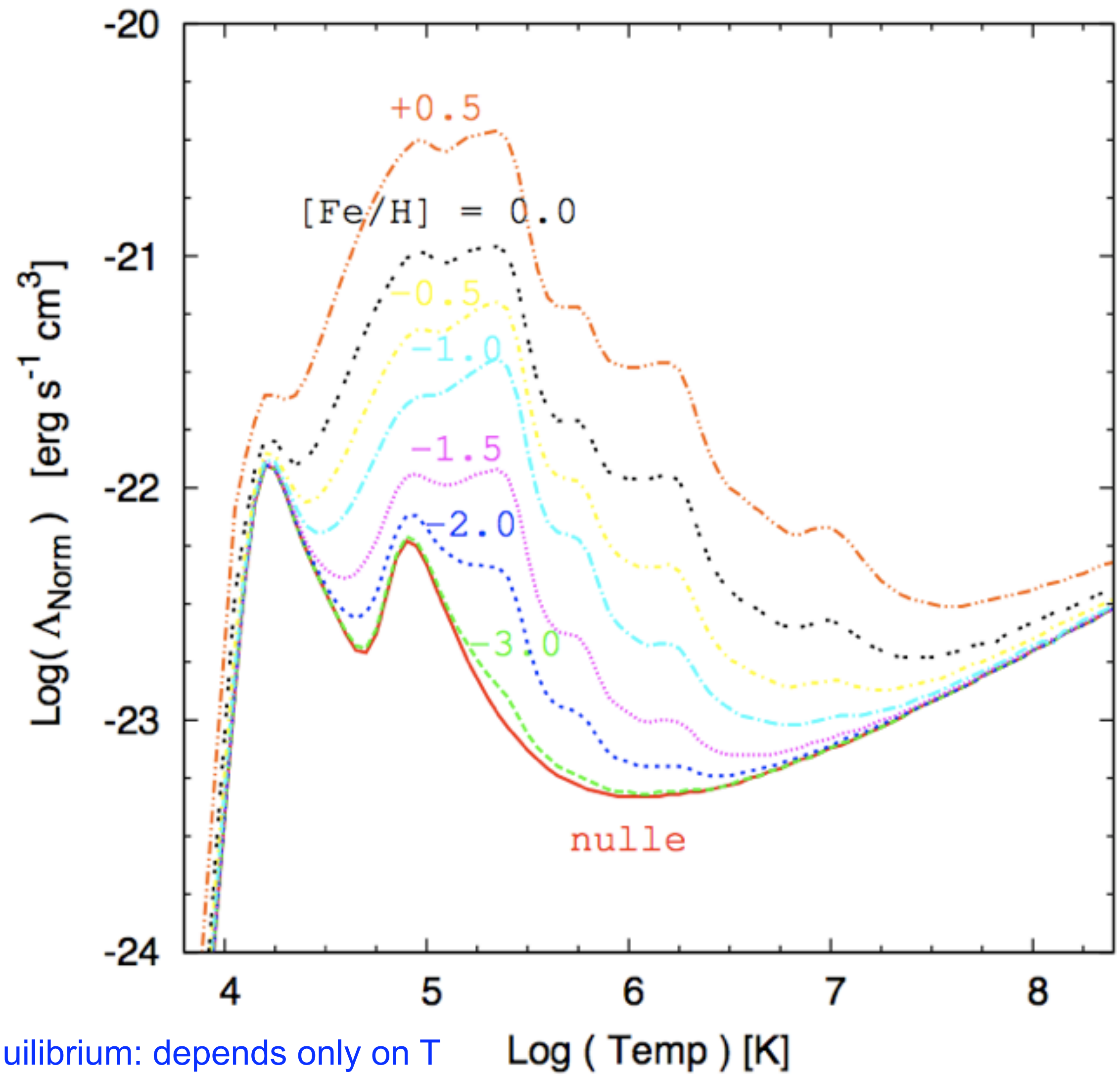
Mean-free path much larger than the system size.
The radiation flux is homogeneous (no absorption).
Photons instantaneously leave the system.
We assume Boltzmann's law for bound states.
We need to compute the ionization state.

$$\frac{\partial n_{\mathrm{H}_0}}{\partial t} = -n_{\mathrm{H}_0} c \sigma_\gamma N_\gamma + n_{\mathrm{e}} n_{\mathrm{H}_+} \alpha_A - n_{\mathrm{e}} n_{\mathrm{H}_0} \beta,$$

The radiation energy is given as input variable.
2 equilibrium cases:
- Collisional equilibrium at high-density (Saha's laws)
- Coronal equilibrium at low-density

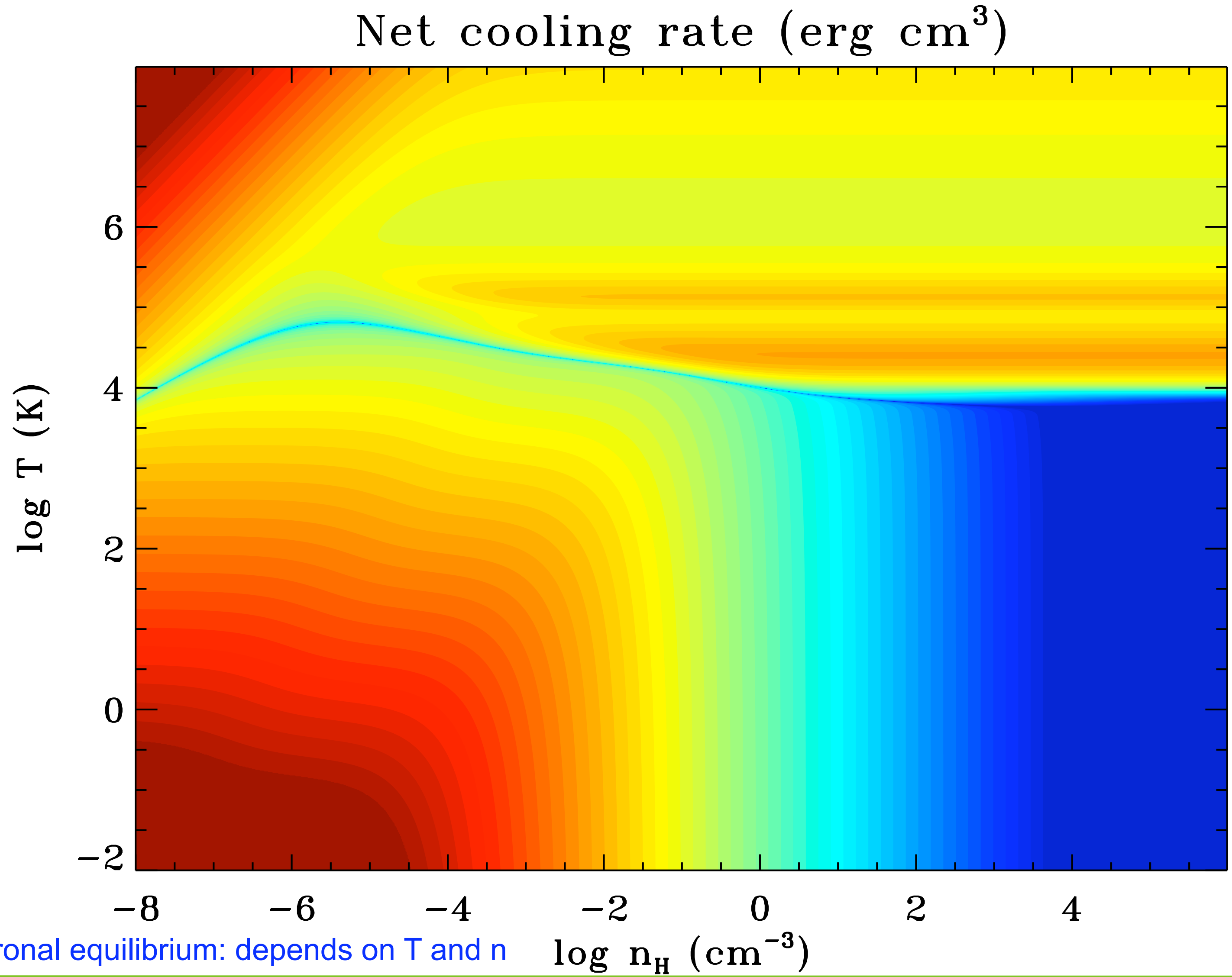
Non-equilibrium chemistry at very low density.

Cooling function for astrophysical plasmas



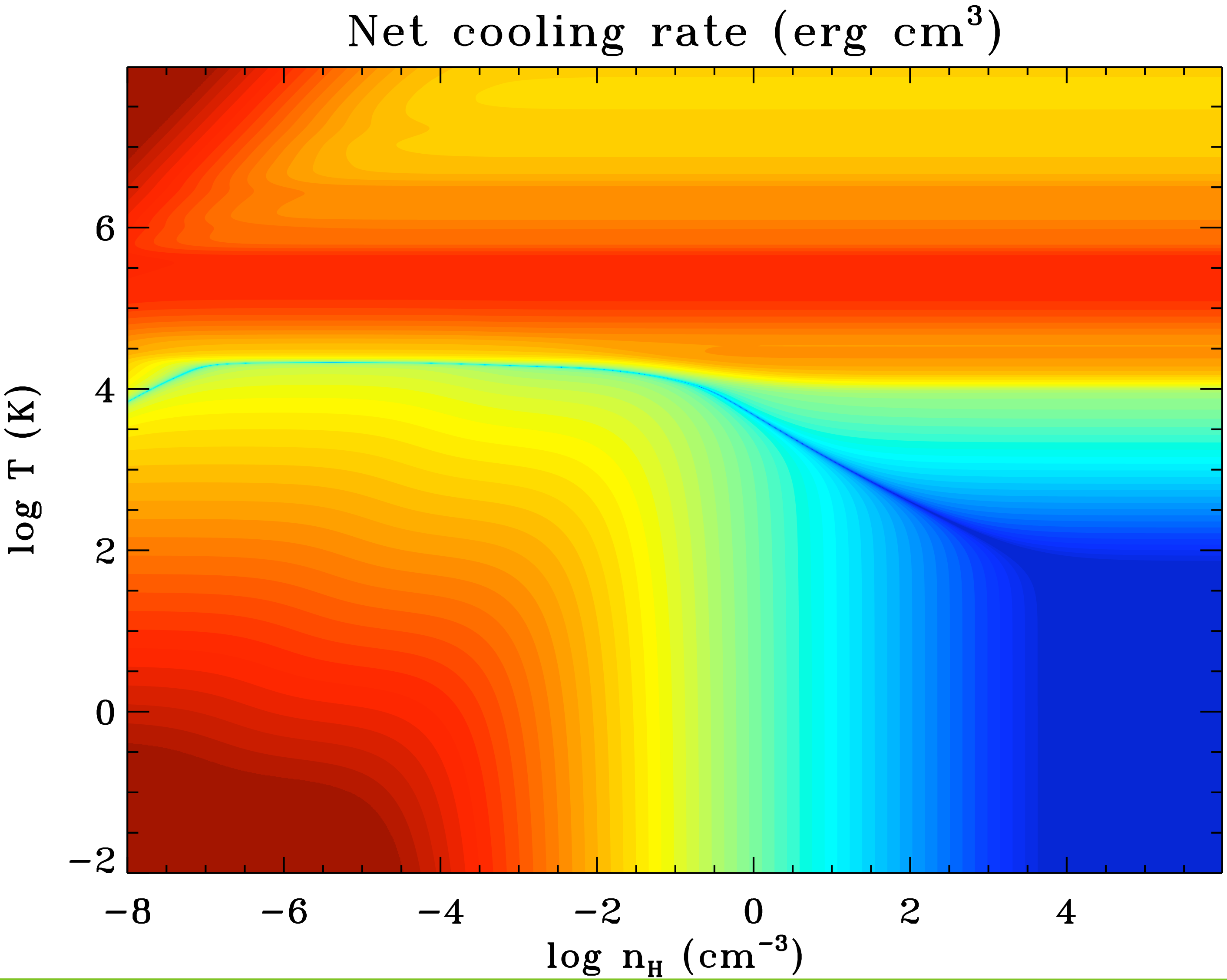
Collisional equilibrium: depends only on T

Cooling function for astrophysical plasmas



Coronal equilibrium: depends on T and n

Cooling function for astrophysical plasmas



The M1 approximation

Assume that the radiation field is locally a dipole.
The photon distribution function is the Lorentz transform of a Planckian distribution.
Dipole is aligned locally to the radiation flux.

The M1 Eddington tensor writes:

$$\mathbf{D} = \frac{1-\chi}{2}\mathbf{I} + \frac{3\chi - 1}{2}\mathbf{u}\otimes\mathbf{u}, \quad \chi = \frac{3+4|\mathbf{f}|^2}{5+2\sqrt{4-3|\mathbf{f}|^2}}.$$

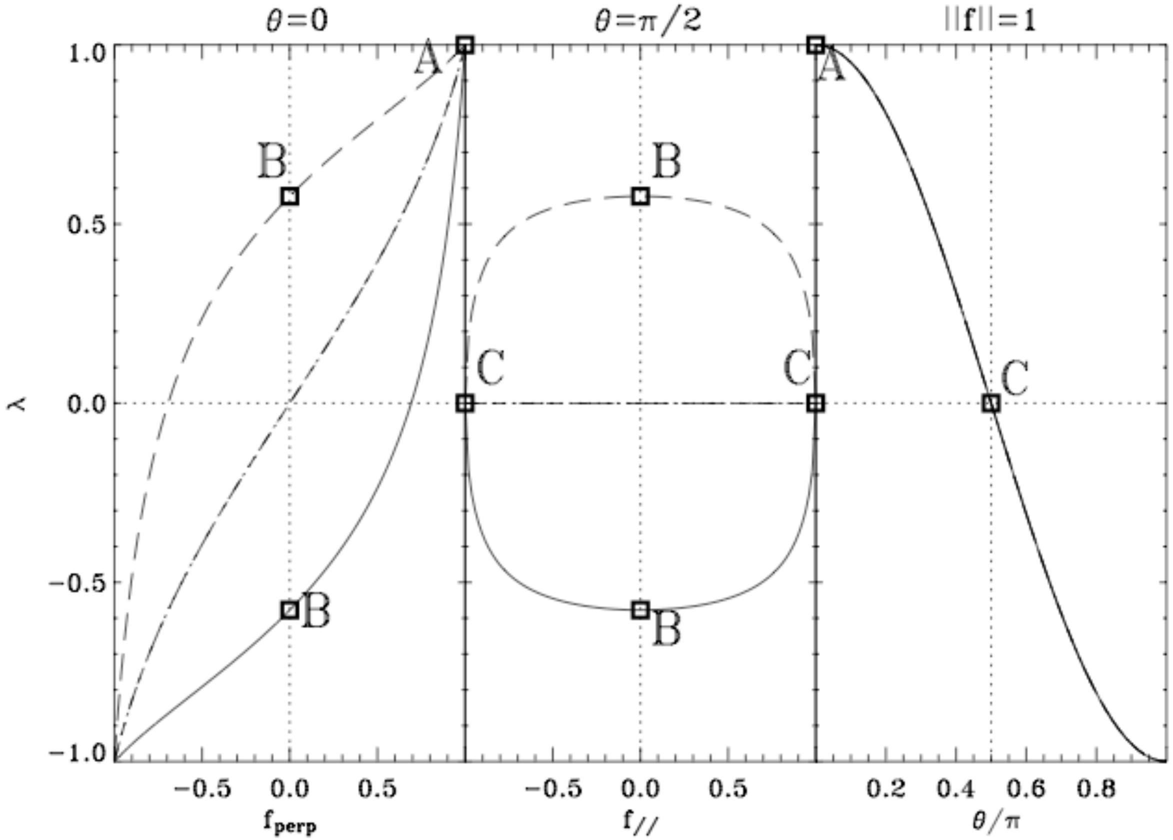
where we define the reduced flux: $\mathbf{f} = \dfrac{\mathbf{F}}{cN} = f\mathbf{u}.$

Asymptotics: f=0 gives D=1/3 and f=1 gives D=1

Levermore C. D., 1984, Journal of Quantitative Spectroscopy and Radiative Transfer, 31, 149

The M1 approximation

Under M1, the radiative transfer equation is turned into an hyperbolic system of conservation laws with source terms.



The 4 eigenvalues of the M1 hyperbolic sub-system from Gonzalez, Audit & Huynes, 2007, A&A, 464, 429

A Godunov scheme for M1 radiation transport

A system of 4 conservations laws:

$$\frac{\partial N_\gamma}{\partial t} + \nabla \mathbf{F}_\gamma = 0,$$
$$\frac{\partial \mathbf{F}_\gamma}{\partial t} + c^2 \nabla \mathbf{P}_\gamma = 0,$$

Vector of conservative variables: $\mathcal{U} = (N_\gamma, F_\gamma)^T$

Vector of flux functions: $\mathcal{F} = (F_\gamma, P_\gamma)^T$

Godunov's method:

$$\frac{(N_\gamma)_i^{n+1} - (N_\gamma)_i^n}{\Delta t} + \frac{(F_\gamma)_{i+1/2}^m - (F_\gamma)_{i-1/2}^m}{\Delta x} = 0,$$
$$\frac{(F_\gamma)_i^{n+1} - (F_\gamma)_i^n}{\Delta t} + c^2 \frac{(P_\gamma)_{i+1/2}^m - (P_\gamma)_{i-1/2}^m}{\Delta x} = 0.$$

A Godunov scheme for M1 radiation transport

Use 2 different Riemann solver for radiation fluxes.

The HLL Riemann solver:

$$(\mathcal{F}_{HLL})^m_{i+1/2} = \frac{\lambda^+ \mathcal{F}^m_i - \lambda^- \mathcal{F}^m_{i+1} + \lambda^+ \lambda^- (\mathcal{U}^m_{i+1} - \mathcal{U}^m_i)}{\lambda^+ - \lambda^-},$$

with HLL wave speeds:

$$\lambda^- = \min(0, \lambda_{min})$$
$$\lambda^+ = \max(0, \lambda_{max})$$

The Global Lax-Friedrich Riemann solver:

$$(\mathcal{F}_{GLF})^m_{i+1/2} = \frac{\mathcal{F}^m_i + \mathcal{F}^m_{i+1}}{2} - \frac{c}{2}(\mathcal{U}^m_{i+1} - \mathcal{U}^m_i).$$

If m=n, we use explicit time integration (the simpler the better !)
If m=n+1, we need an implicit solver (matrix inversion).

Other approximations

Flux-limited diffusion: $\mathbf{F}_\nu = -\frac{c^2}{3}\tau_\nu \nabla E_\nu$
and in the optically thin regime, limits: $\mathbf{f} = \max\left(F/cN, 1\right)\mathbf{u}$

The OTVET approximation: compute the Eddington tensor assuming the system is optically thin. Solve a Poisson equation with radiation sources and then solve the moment equations.

The “Ray=Tracing” approximation: solve the exact radiative transfer equation along light rays (usually assume infinite speed of light).

Monte-Carlo approximation: send photons packets that interact with the fluid (noisy but good spectral description).

OTVET: Gnedin & Abel (2001); M1 in the ATON code: Aubert & Teyssier (2008)

Stiff source terms and the implicit method

Even in the optically thin regime, radiation is stiff: $\tau \ll \Delta t$

Explicit time integration is unappropriate.

$$\frac{T^{n+1}-T^{n}}{\Delta t}=\Gamma(T^{n})-\Lambda(T^{n})\simeq\frac{T_{eq}-T}{\tau}$$

Fully implicit (non-linear) method: stable for all time steps.

$$\frac{T^{n+1}-T^{n}}{\Delta t}=\Gamma(T^{n+1})-\Lambda(T^{n+1})$$

Accuracy ? Convergence ?

Linearized implicit method: $\Delta T \ll T$

$$\frac{T^{n+1}-T^{n}}{\Delta t}=\Gamma(T^{n})-\Lambda(T^{n})+\frac{\partial(\Gamma-\Lambda)}{\partial T}(T^{n+1}-T^{n})$$

Need to control the time step: this can be tricky !

Radiation transport and cosmic reionization

Number of H ionizing photons $$N_{\gamma \mathrm{H}_0} = \int_{\nu_{\mathrm{H}_0}}^{\infty} N_\nu \mathrm{d}\nu.$$

$$\begin{aligned}
\frac{\partial N_\gamma}{\partial t} + \nabla \mathbf{F}_\gamma &= -n_{\mathrm{H}_0} c \sigma_N N_\gamma + \dot{N}_\gamma^* + \dot{N}_\gamma^{rec}, \\
\frac{\partial \mathbf{F}_\gamma}{\partial t} + c^2 \nabla \mathbf{P}_\gamma &= -n_{\mathrm{H}_0} c \sigma_F \mathbf{F}_\gamma,
\end{aligned}$$

Recombination photons (isotropic radiation)

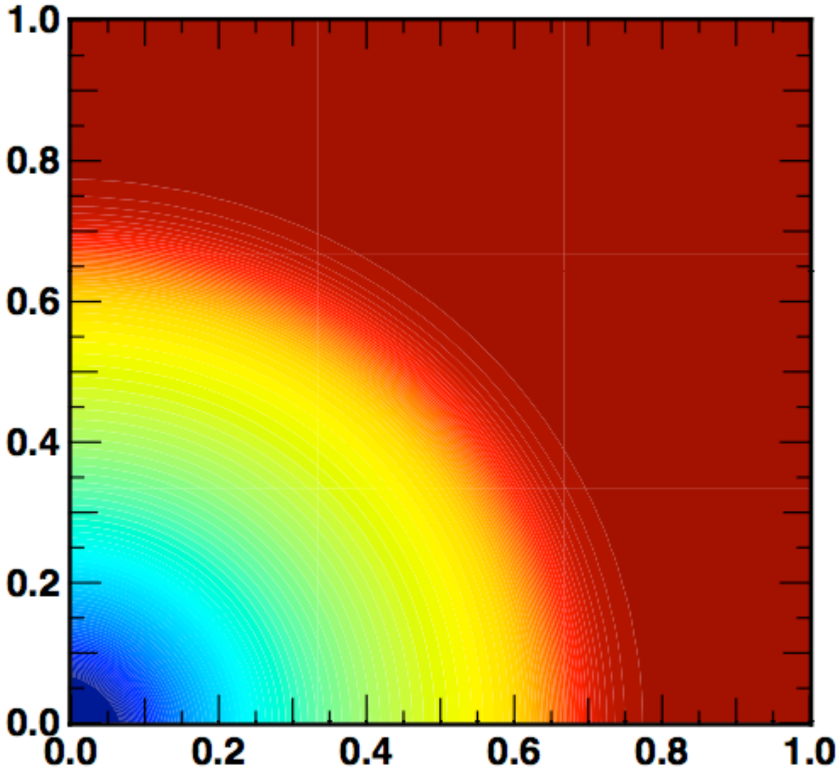
$$\dot{N}_\gamma^{rec} = \int_{\nu_{\mathrm{H}_0}}^{\infty} n_e n_{\mathrm{H}_+} \dot{\epsilon}_{\mathrm{H}_+}(\nu, T) = n_e n_{\mathrm{H}_+} (\alpha_A - \alpha_B).$$

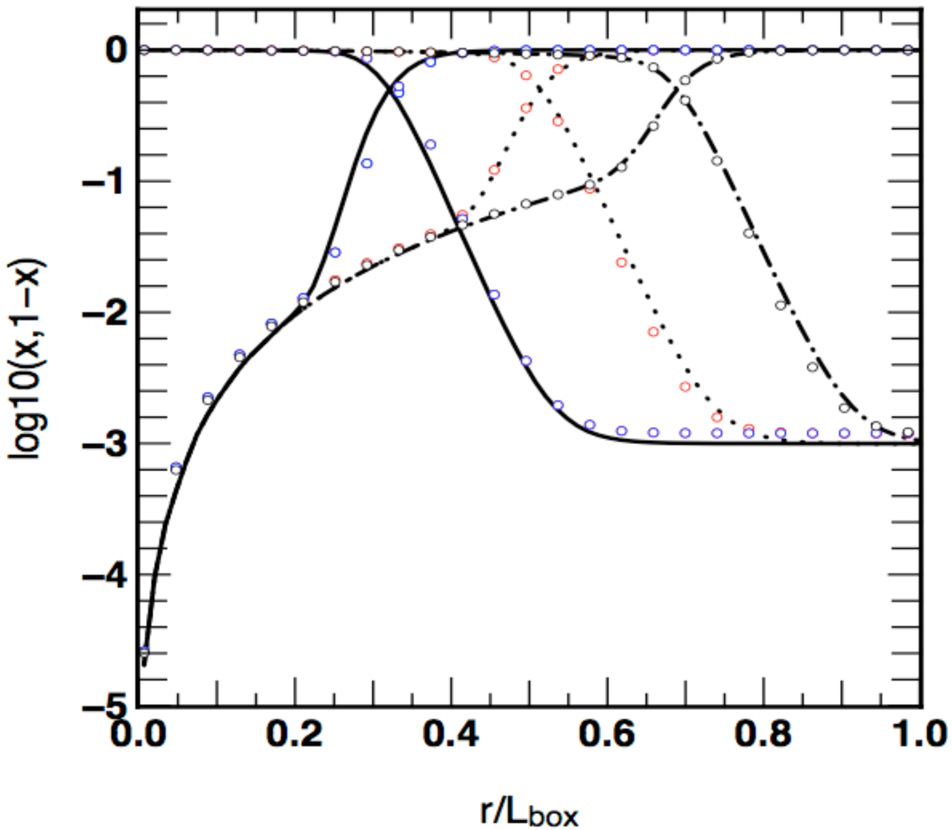
Photo-ionization cross sections:

$$\sigma_N N_\gamma = \int_{\nu_{\mathrm{H}_0}}^{\infty} \sigma_\nu N_\nu \mathrm{d}\nu \quad \text{and} \quad \sigma_{\mathrm{F}} \mathrm{F}_\gamma = \int_{\nu_{\mathrm{H}_0}}^{\infty} \sigma_\nu \mathrm{F}_\nu \mathrm{d}\nu.$$

$$\sigma_F \simeq \sigma_N \simeq \sigma_\gamma = \int_{\nu_{\mathrm{H}_0}}^{\infty} \sigma_\nu \frac{4\pi J_0(\nu)}{h\nu} \mathrm{d}\nu \Big/ \int_{\nu_{\mathrm{H}_0}}^{\infty} \frac{4\pi J_0(\nu)}{h\nu} \mathrm{d}\nu,$$

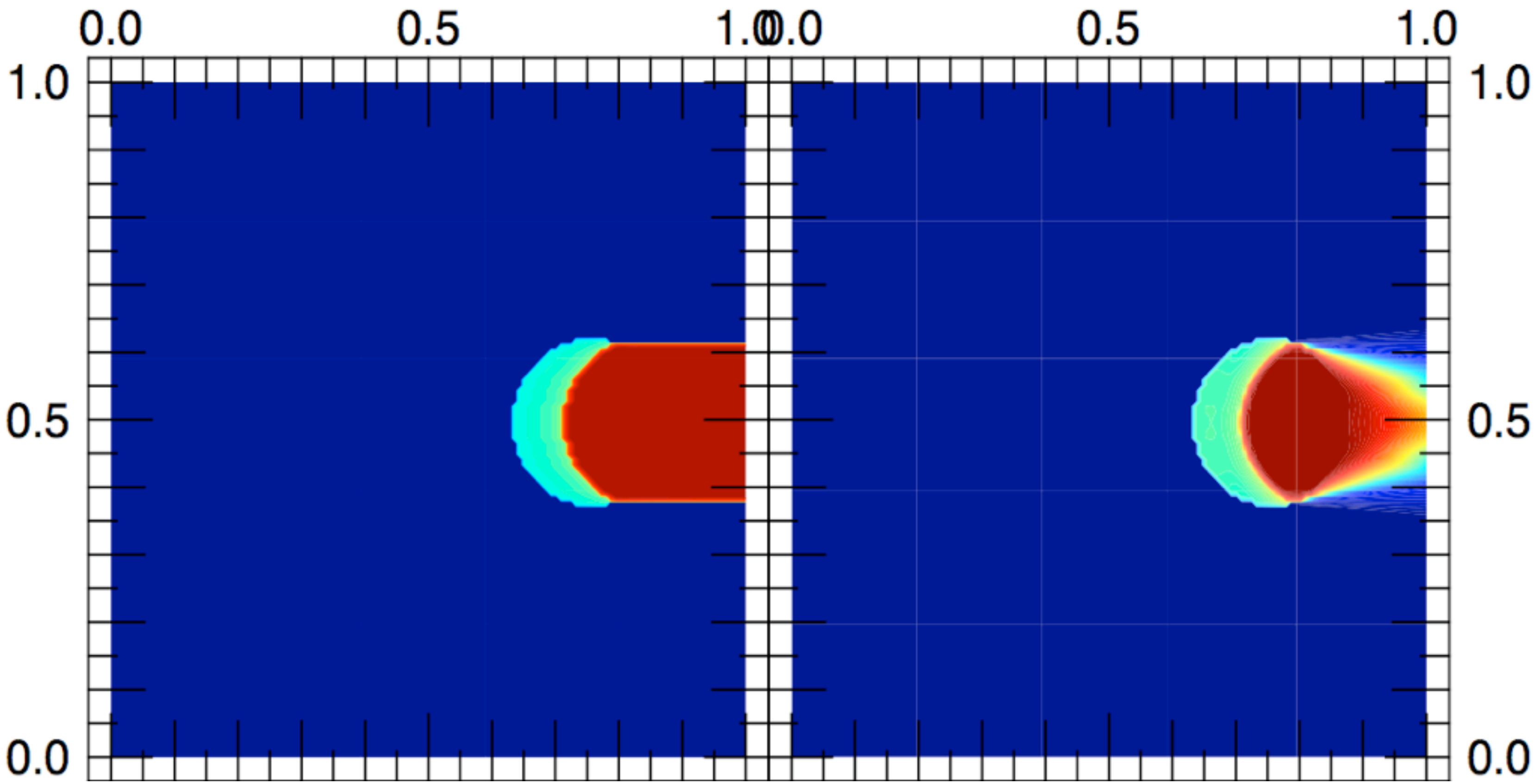
The Stromgen sphere test





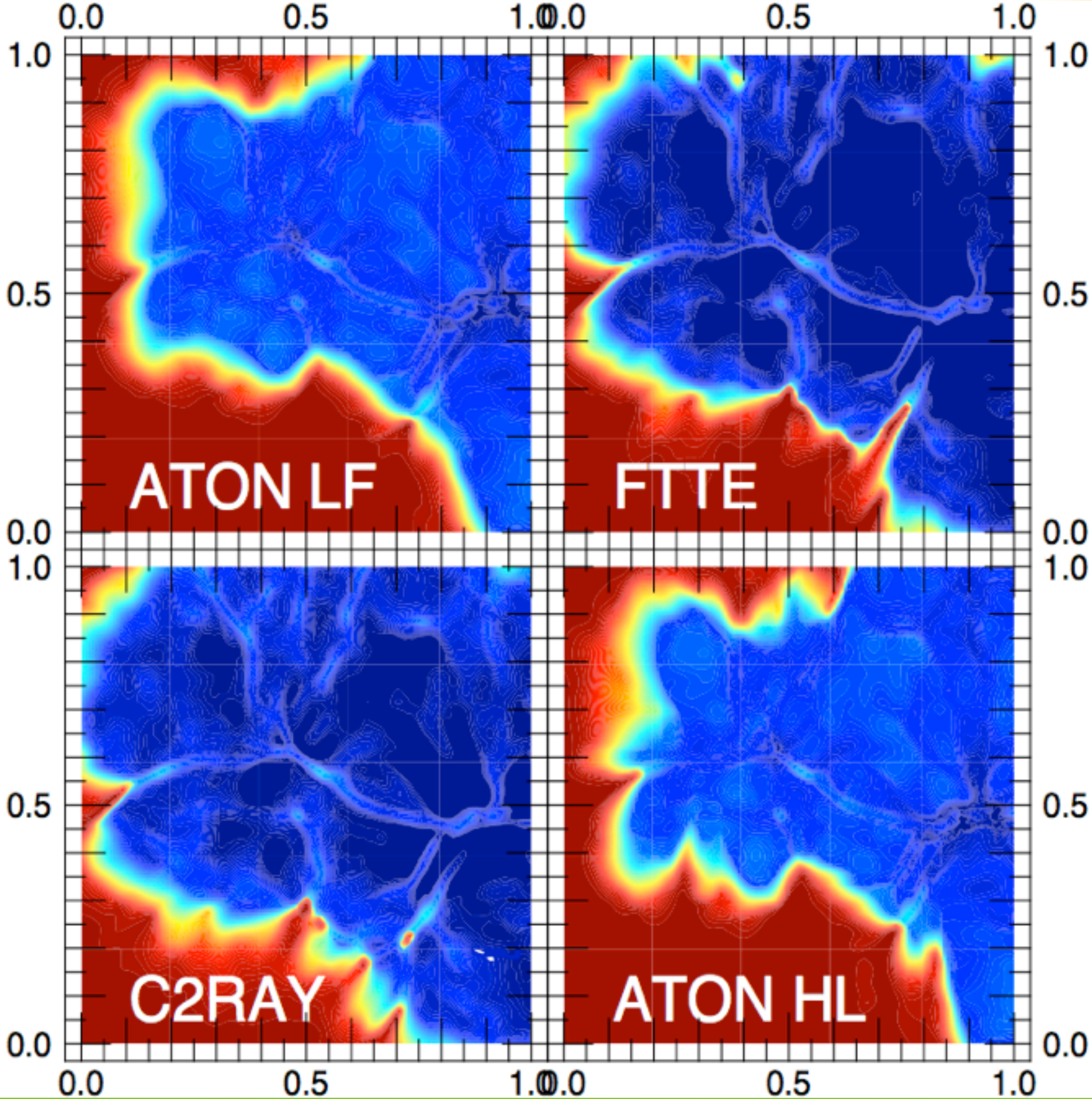
Shadowing by a dense clump

Assume anisotropic recombination (ray-tracing-like)



HLL Riemann solver LLF Riemann solver

Cosmic reionization



Conclusion

- Moments of the radiation transfer equation
- Diffusion limit as a deviation from LTE (adds up to the thermal flux)
- The M1 approximation as an hyperbolic system with source terms
- The M1 approximation for photo-ionization problems

Coupling an hyperbolic solver with stiff source terms ?

The Peclet number: $$Pe = \frac{c\tau}{\Delta x}$$

Next lecture: Hyperbolic solvers with source terms