Computational Astrophysics 7

Hydrodynamics with source terms

Romain Teyssier

Oscar Agertz

University of Zurich

Outline

- Optically thin radiative hydrodynamics
- Relaxation towards the diffusion limit
- Hydrodynamics with gravity source term
- Relaxation towards the Burger's equation

The Euler equations with a cooling/heating source term

$$\partial_t(\rho) + \partial_x(\rho u) = 0$$

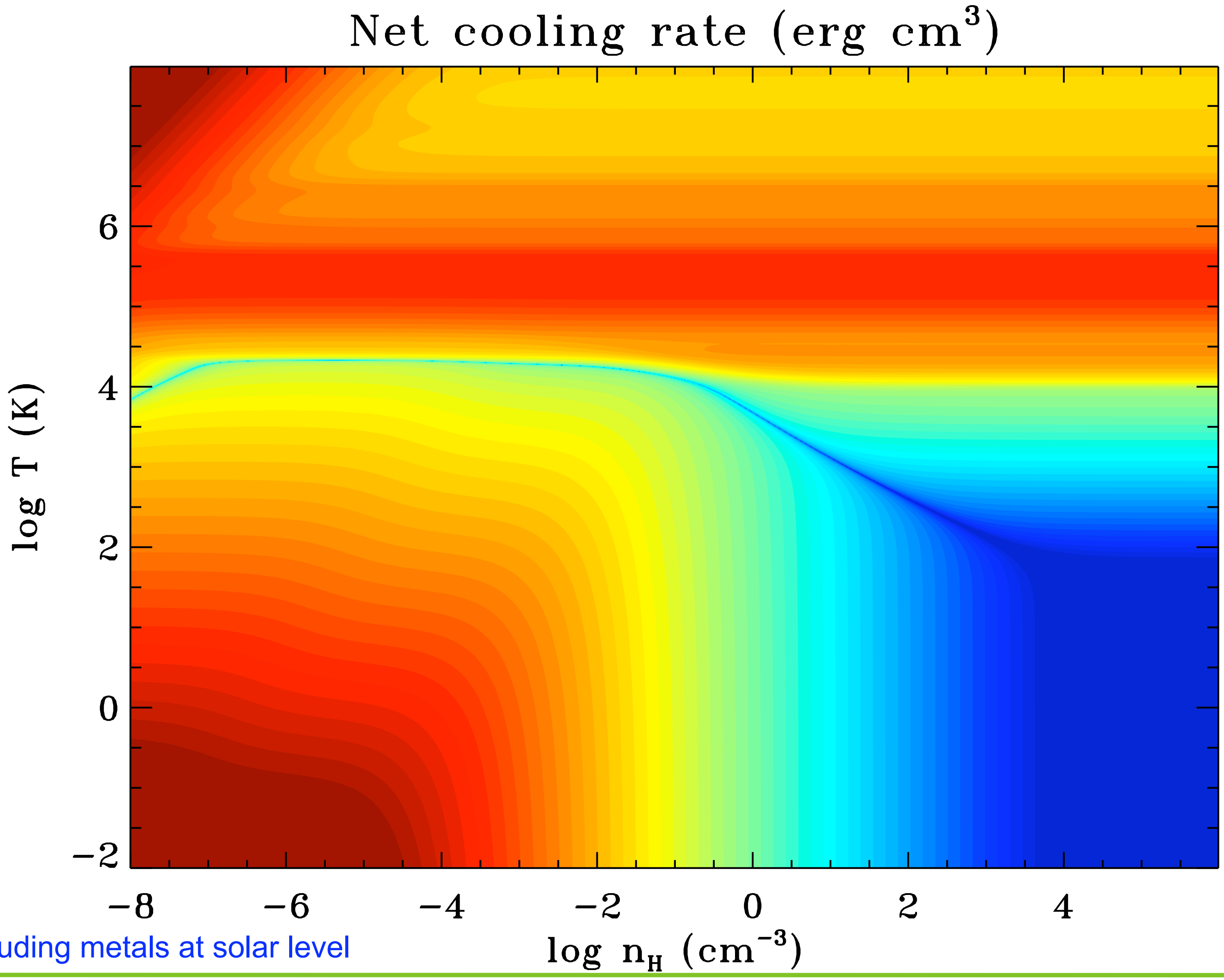
$$\partial_t(\rho u) + \partial_x(\rho u^2 + P) = 0$$

$$\partial_t(E) + \partial_x(E + P)u = \Gamma(\rho, T) - \Lambda(\rho, T)$$

Total Fluid Energy: $$E = \frac{1}{2}\rho u^2 + \rho\epsilon$$

Equation-Of-State: $$P = (\gamma - 1)\rho\epsilon \qquad P = \frac{\rho}{\mu m_H} k_B T$$

Cooling and heating in a coronal plasma



Including metals at solar level

Relaxation towards the isothermal Euler equations

We approximate the heating/cooling source term as a relaxation term:

$$\Gamma(\rho, T) - \Lambda(\rho, T) \simeq \rho k_B \frac{T_{eq}(\rho) - T}{\tau_{cool}}$$

The equilibrium temperature for solar metallicity is roughly given by:

$$T_{eq} \simeq 10^4 \quad \mathrm{K} \qquad n_H < 0.3\ \mathrm{H/cc}$$

$$T_{eq} \simeq 10^4 \left(\frac{n_H}{1\ \mathrm{H/cc}}\right)^{-1/2} \quad \mathrm{K} \qquad n_H > 0.3\ \mathrm{H/cc}$$

For very short cooling time, the previous system relaxes towards a new one:

$$\partial_t(\rho) + \partial_x(\rho u) = 0$$

$$\partial_t(\rho u) + \partial_x(\rho u^2 + P) = 0$$

with the isothermal pressure: $$P = \frac{\rho}{\mu m_H} k_B T_{eq}$$

Sub-characteristics condition

Adiabatic Euler system, sound speed:

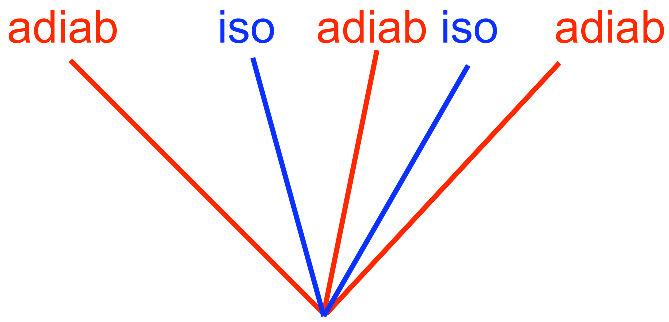
3 eigenvalues (*u-c, u, u+c)*

$$c^2 = \frac{\gamma P}{\rho}$$

Isothermal Euler system, sound speed:

2 eigenvalues (*u-c, u+c*)

$$c^2 = \frac{P}{\rho} \simeq c_{eq}^2$$



The solution of the adiabatic Euler system with source terms will converge uniformly towards the solution of the isothermal Euler system because the eigenvalues of the isothermal system follow:

$$u-c_{ad} < u-c_{iso} < u < u+c_{iso} < u+c_{ad}$$

Sub-characteristics condition

Hyperbolic system of conservation laws with source terms:

$$\partial_t \mathbf{U} + \partial_x \mathbf{F} = \mathbf{S}(\mathbf{U})$$

Equilibrium state is defined by $\mathbf{S}(\mathbf{U_{eq}}) = 0$

We defined a sub-system on the sub-space $\mathbf{u} = \mathbf{U_{eq}}$

$$\partial_t \mathbf{u} + \partial_x \mathbf{f} = 0$$

where the new flux function is defined by $\mathbf{f}(\mathbf{u}) = \mathbf{F}(\mathbf{U_{eq}})$

If the sub-system is also hyperbolic, then the main system with source term will relax towards the sub-system solution if the following sub-characteristic condition is full-filled:

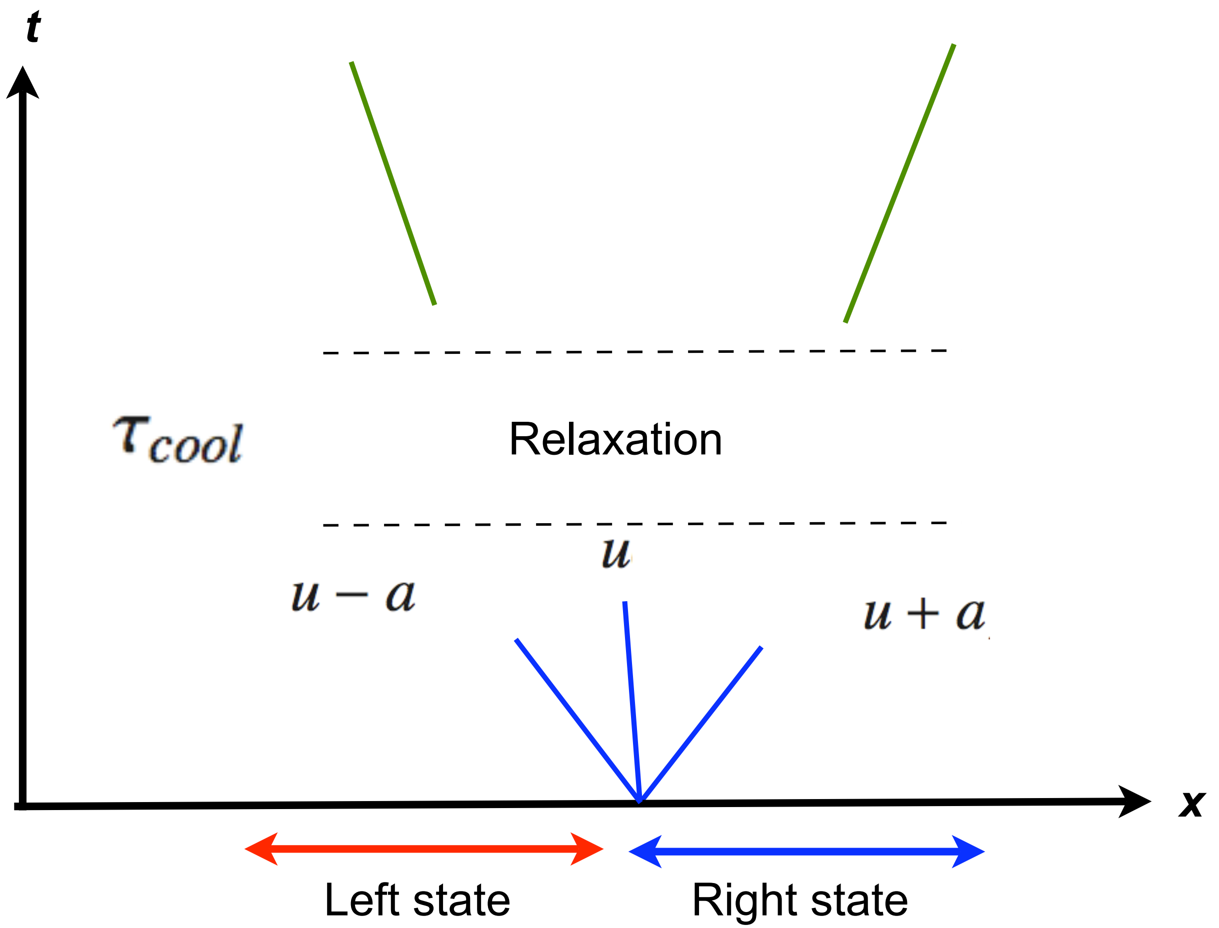
$$\min(\Lambda_j) < \lambda_i < \max(\Lambda_j)$$

Strong sub-characteristics condition:

$$\Lambda_1 < \lambda_1 < \Lambda_2 < \lambda_i < \Lambda_{N-1} < \lambda_{N-1} < \Lambda_N$$

Hyperbolic systems with source terms

We need to solve Generalized Riemann Problem: wave speeds are not constant anymore and the Riemann solution is not self-similar anymore.



Hyperbolic systems with source terms

Numerical implementation of the MUSCL Godunov scheme with source terms:

1- Modify the predictor step to account for the source term

$$\mathbf{W}_{i+1/2,L}^{n+1/2} = \mathbf{W}_i^n + (\mathbf{I} - \mathbf{A}\frac{\Delta t}{\Delta x})\frac{(\Delta \mathbf{W})_i^n}{2} + \mathbf{S}(\mathbf{W}_i^n)\frac{\Delta t}{2}$$

2- Use the Riemann solver of the original hyperbolic system.

$$\mathbf{F}_{i+1/2}^{n+1/2} = \mathbf{F}^*(\mathbf{W}_{i+1/2,L}^{n+1/2}, \mathbf{W}_{i+1/2,R}^{n+1/2})$$

3- Update conservative variables using original flux and source term.

$$\frac{\mathbf{U}_i^{n+1} - \mathbf{U}_i^n}{\Delta t} + \frac{\mathbf{F}_{i+1/2}^{n+1/2} - \mathbf{F}_{i-1/2}^{n+1/2}}{\Delta x} = \mathbf{S}_i^{n+1/2}$$

Computing the source term is the main difficulty:

- Use fully implicit method (first order accurate with operator splitting)

- Use second order accurate source term (Crank-Nicholson)

- Problem of well-balanced scheme (satisfy exactly the stationary regime)

Randall J. LeVeque, "*Balancing source terms and flux gradients in high-resolution Godunov methods: the quasi-steady wave-propagation algorithm*", 1998, Journal of Computational Physics, 146, 346,

Sod test with cooling source term

Use RAMSES to solve the Euler equations with a source term.

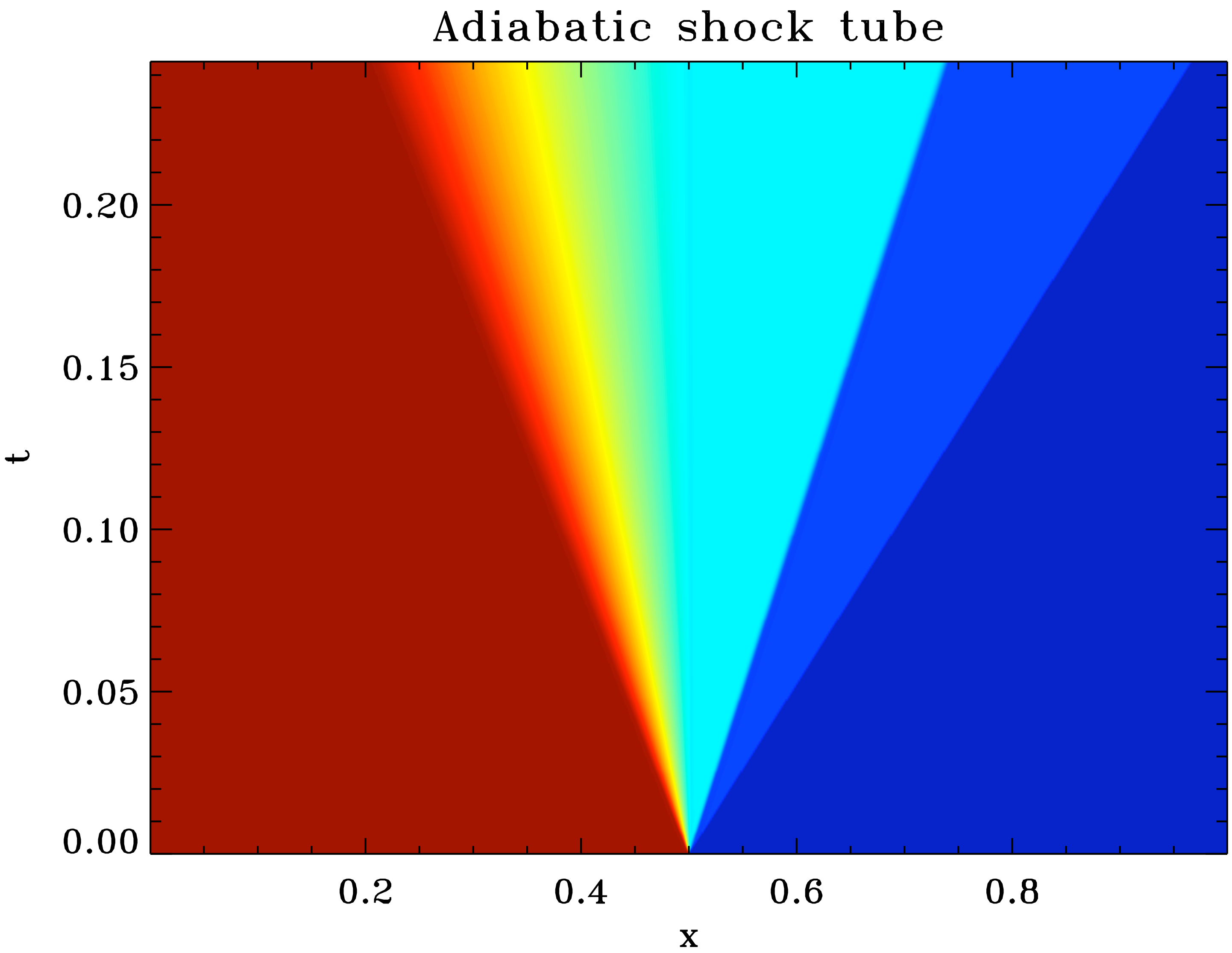
```
! Compute pressure
do i=1,nleaf
   T2(i)=uold(ind_leaf(i),ndim+2)
end do
do i=1,nleaf
   ekk(i)=0.0d0
end do
do idim=1,ndim
   do i=1,nleaf
      ekk(i)=ekk(i)+0.5*uold(ind_leaf(i),idim+1)**2/nH(i)
   end do
end do
do i=1,nleaf
   T2(i)=(gamma-1.0)*(T2(i)-ekk(i))
end do

! Compute T2=T/mu in Kelvin
do i=1,nleaf
   T2(i)=T2(i)/nH(i)
end do


! Compute cooling time step in second
dtcool = dtnew(ilevel)

! Compute net energy sink
do i=1,nleaf
   delta_T2(i) = nH(i)/(gamma-1.0)*(1.0-T2(i))*(1.0-exp(-dtcool/0.02))
end do

! Update total fluid energy
do i=1,nleaf
   T2(i) = uold(ind_leaf(i),ndim+2)
end do
if(cooling)then
   do i=1,nleaf
      T2(i) = T2(i)+delta_T2(i)
   end do
endif
do i=1,nleaf
   uold(ind_leaf(i),ndim+2) = T2(i)
end do
```

Patch cooling_fine.f90

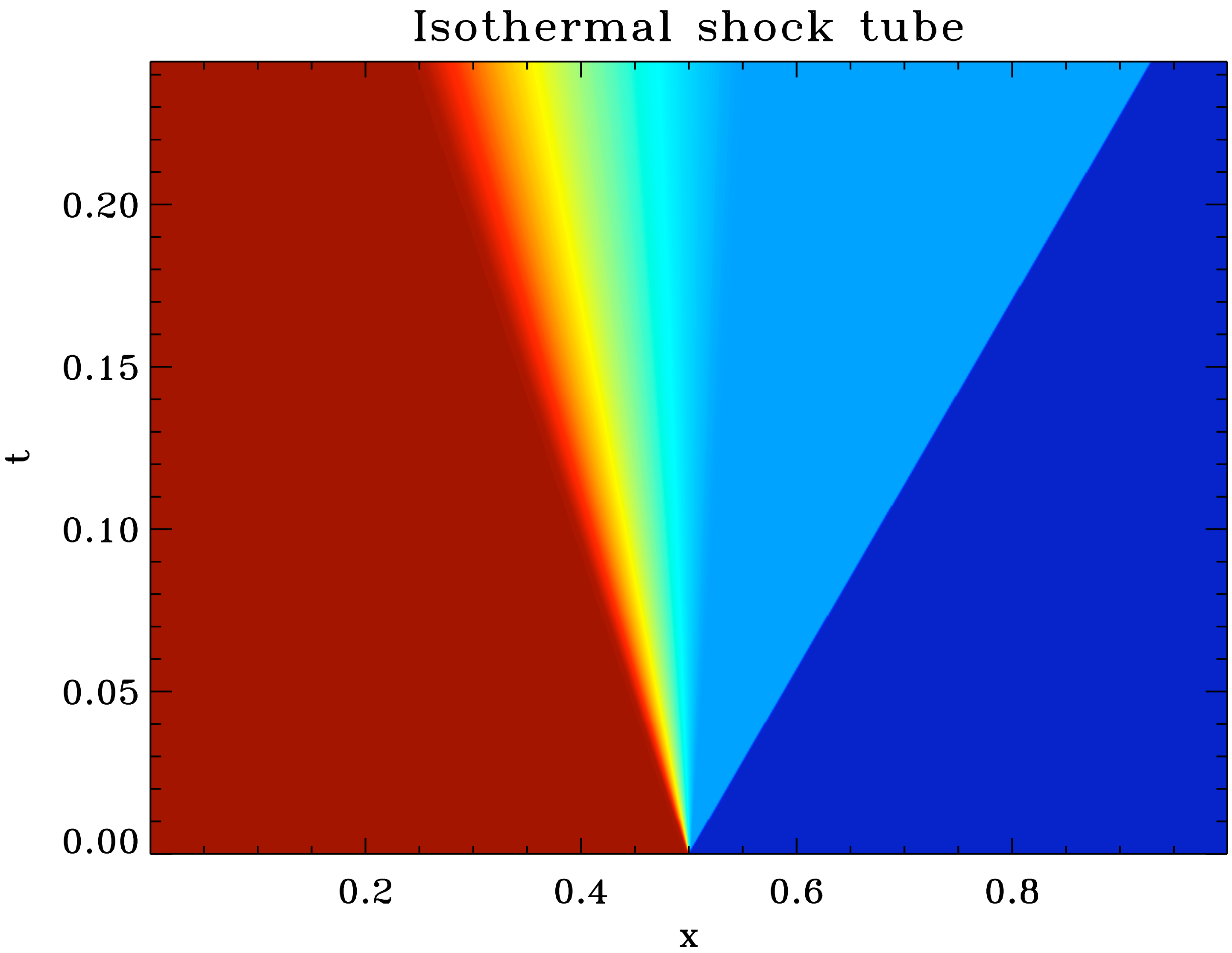
Modify the namelist file.

```
&INIT_PARAMS
nregion=2
region_type(1)='square'
region_type(2)='square'
x_center=0.25,0.75
length_x=0.5,0.5
d_region=1.0,0.1
u_region=0.0,0.0
p_region=1.0,0.1
/

&HYDRO_PARAMS
gamma=1.4
courant_factor=0.8
slope_type=2
scheme='muscl'
riemann='hllc'
/

&PHYSICS_PARAMS
cooling=.true.
/
```

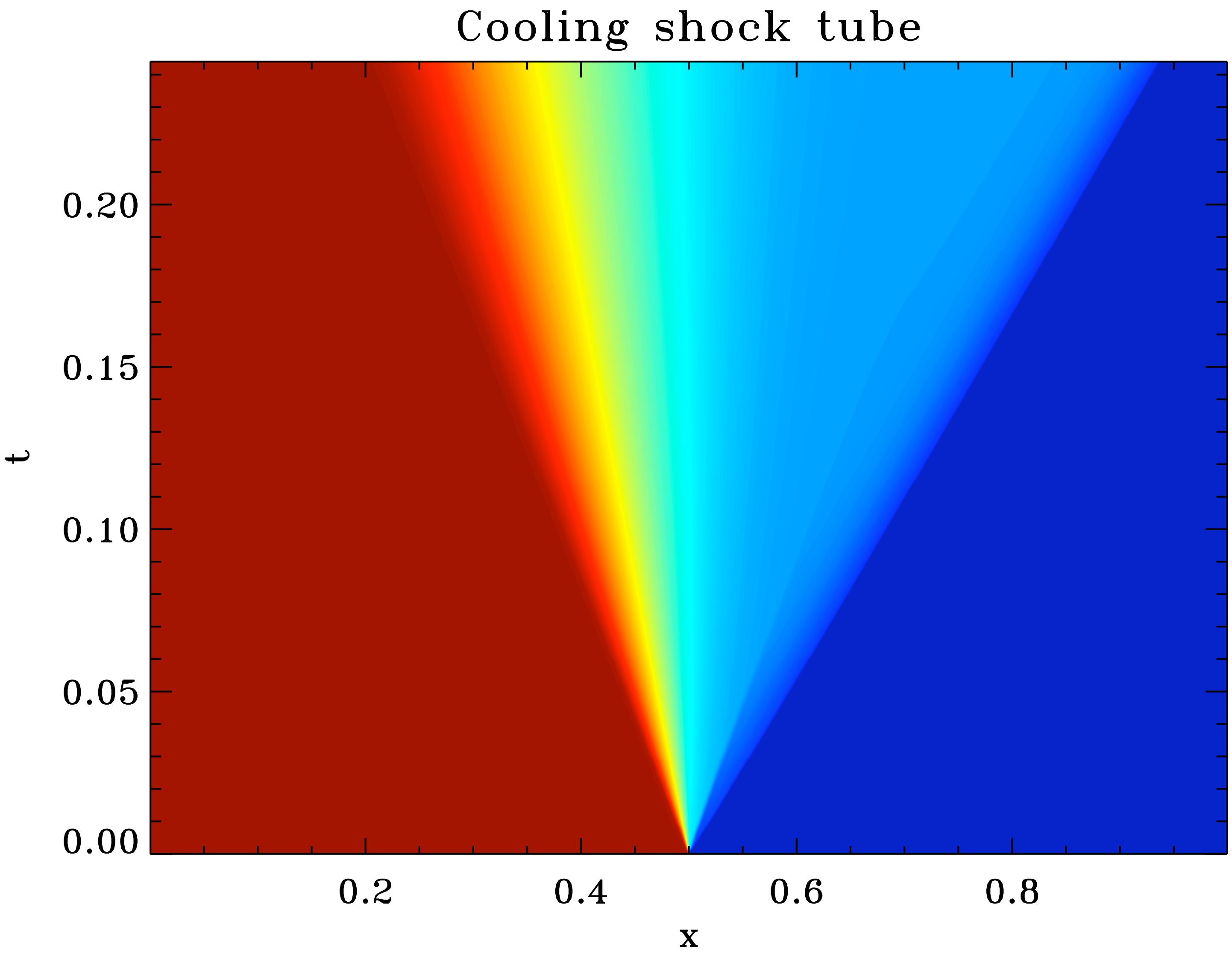
Sod test with cooling source term



Sod test with cooling source term



Sod test with cooling source term



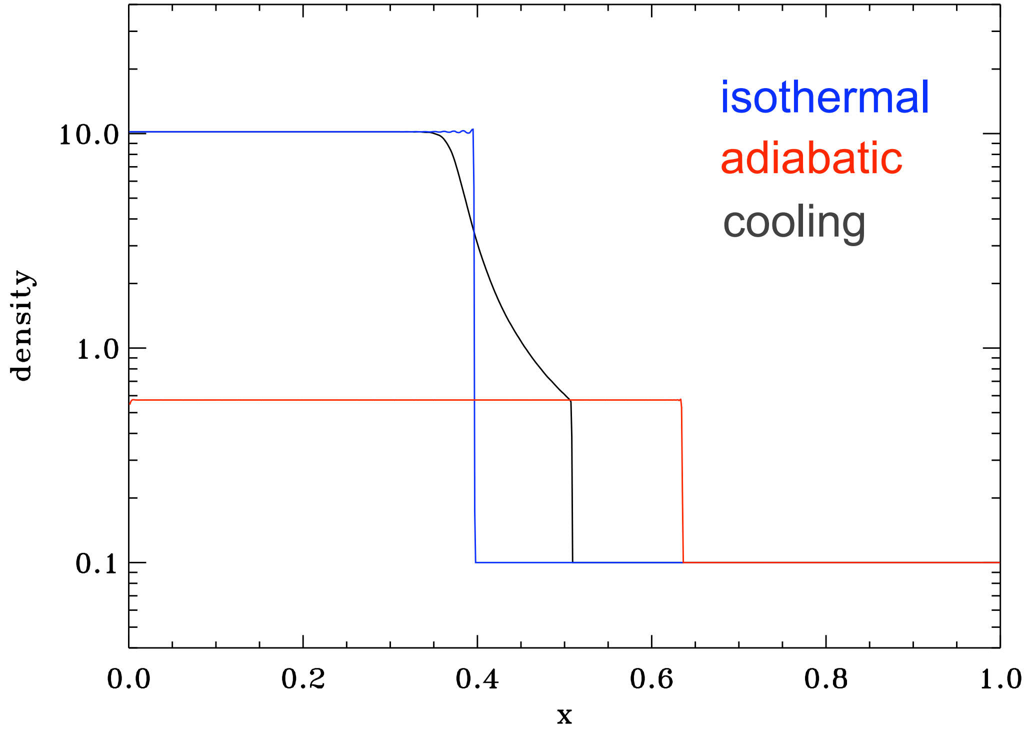
Radiative shock waves

```
&BOUNDARY_PARAMS
nboundary=2
ibound_min=-1,+1
ibound_max=-1,+1
bound_type= 1, 3
d_bound=0.1,0.1
u_bound=-10.0,-10.0
P_bound=0.1,0.1

/

&INIT_PARAMS
nregion=1
region_type(1)='square'
x_center=0.5
length_x=2.0
d_region=0.1
u_region=-10.0
p_region=0.1
/

&HYDRO_PARAMS
gamma=1.4
courant_factor=0.8
slope_type=1
scheme='muscl'
riemann='hllc'
/

&PHYSICS_PARAMS
cooling=.true.
/
```

Use RAMSES to create a shock wave, reflecting on a wall.

Cooling with T_{eq} = 1 and tau=0, 0.05 and infinity.

Radiative layer of thickness L=u_{PS}.tau



Do we resolve the cooling wave ?

Yes if $\Delta x < c\tau$ (Peclet number less than one).

Hyperbolic systems with source terms

A problem arises in the previous numerical scheme.

The equilibrium hyperbolic system (isothermal Euler equations) has a different Riemann solver than the original one (adiabatic Euler equations).

Exemple: the Lax-Friedrich Riemann solver, gives

$$(P + \rho u^2)^* = \frac{P_L + \rho_L u_L^2 + P_R + \rho_R u_R^2}{2} - (|u| + c)\frac{\rho_R u_R - \rho_L u_L}{2}$$

Righter-most term is a numerical diffusion term with coefficient $\nu = (|u| + c)\dfrac{\Delta x}{2}$

Adiabatic sound speed $c^2 = \dfrac{\gamma P}{\rho}$ is larger than the isothermal one $c^2 = \dfrac{P}{\rho}$,

so that the resulting scheme is more diffusive than the equilibrium one.

Radiative transfer in the diffusion limit

We solve the first 2 moments equation of radiative transfer in the grey LTE limit:

$$\frac{\partial e}{\partial t} = \frac{E - aT^4}{\tau} \qquad \frac{\partial E}{\partial t} + \nabla \cdot \mathbf{F} = \frac{aT^4 - E}{\tau}$$

$$\frac{\partial \mathbf{F}}{\partial t} + c^2 \nabla \cdot \mathbf{P} = \frac{-\mathbf{F}}{\tau}$$

In the diffusion limit, we have $E \simeq aT^4 \qquad \mathbf{F} \simeq -\frac{c^2}{3}\tau \nabla E$

so that the previous system relax to the following equilibrium problem:

$$\frac{\partial e}{\partial t} = \nabla \cdot \left(\frac{c^2}{3} \tau \nabla a T^4 \right)$$

The equilibrium system is not hyperbolic but parabolic !

We relax from an hyperbolic system (with eigenvalues between -c and +c) with source terms to a parabolic system of conservation law with no real eigenvalues.

Numerical scheme for stiff relaxation systems

Using a Godunov solver for the radiation transport step, we have the following approximation for the Lax-Friedrich numerical flux:

$$\mathbf{F} = \frac{F_L + F_R}{2} - c\frac{E_R - E_L}{2}$$

To leading order, we have: $$\mathbf{F}_{adv} = \mathbf{F}_{true} - \frac{c\Delta x}{2}\nabla E$$

In the diffusion limit, numerical diffusion is larger than radiation diffusion if

$\frac{c^2\tau}{3} < \frac{c\Delta x}{2}$ or the Peclet number $\mathrm{Pe} = \frac{\Delta x}{c\tau} > \frac{2}{3}$

A stable and accurate numerical scheme valid in the diffusion limit is:

$$\mathbf{F}_{diff} = -\frac{c^2\tau}{3\Delta x}(E_R - E_L)$$

Jin & Levermore, 1996, JCP, 126, 449, proposed the following *hybrid* numerical flux:

$$\mathbf{F}_{num} = \omega\mathbf{F}_{adv} + (1-\omega)\mathbf{F}_{diff} \quad \text{with} \quad \omega = \tanh\left(\frac{1}{\mathrm{Pe}}\right)$$

The Euler equations with a gravity source term

$$\partial_t(\rho) + \partial_x(\rho u) = 0$$

$$\partial_t(\rho u) + \partial_x(\rho u^2 + P) = \rho \mathbf{g}$$

$$\partial_t(E) + \partial_x(E + P)u = \rho \mathbf{u} \cdot \mathbf{g}$$

Gravitational acceleration $\mathbf{g} = -\nabla \Phi$ from the Poisson equation $\Delta \Phi = 4\pi G \rho$

By analogy with the previous analysis, we can define the characteristic time scale for gravitational collapse as the isothermal free-fall time:

$$\tau_{ff} = \sqrt{\frac{\pi}{G\rho}}$$

We can define the gravitational Peclet number as: $\mathrm{Pe} = \dfrac{\Delta x}{c\tau_{ff}} = \dfrac{\Delta x}{\lambda_J}$

Homogeneous collapse

Consider the isothermal collapse of an self-gravitating gas sphere.

Velocity field: $\mathbf{u} = -H(t)\mathbf{r}$ with $H(t)^2 = \frac{8\pi}{3}G\rho(t)\left(1 - \frac{R(t)}{R_0}\right)$

Using the Lax-Friedrich Riemann solver, we have the following flux:

$$(P + \rho u^2)^* \simeq \rho(t)\left(a^2 + H(t)^2 r^2 - \frac{(H(t)r + a)}{2}\Delta x H(t)\right)$$

At the origin, numerical diffusion is larger than thermal pressure if:

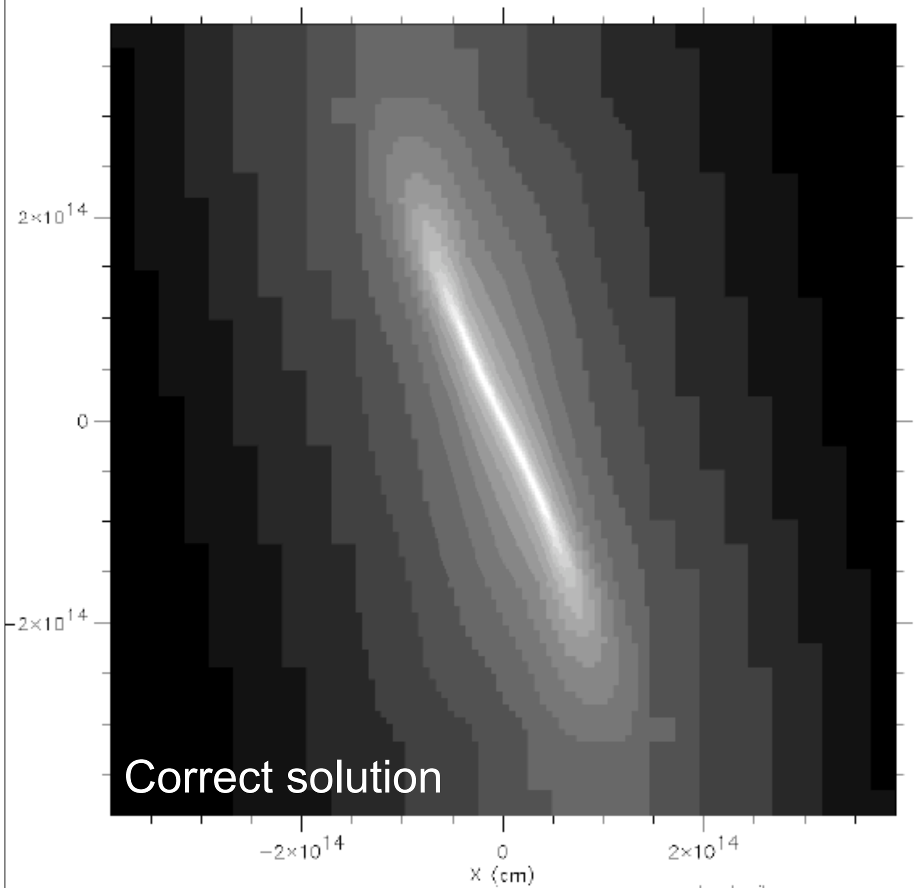
$$a < \frac{H(t)\Delta x}{2} \simeq \frac{12}{\tau_{ff}}\Delta x \quad \text{or} \quad \Delta x > \frac{\lambda_J}{12}$$

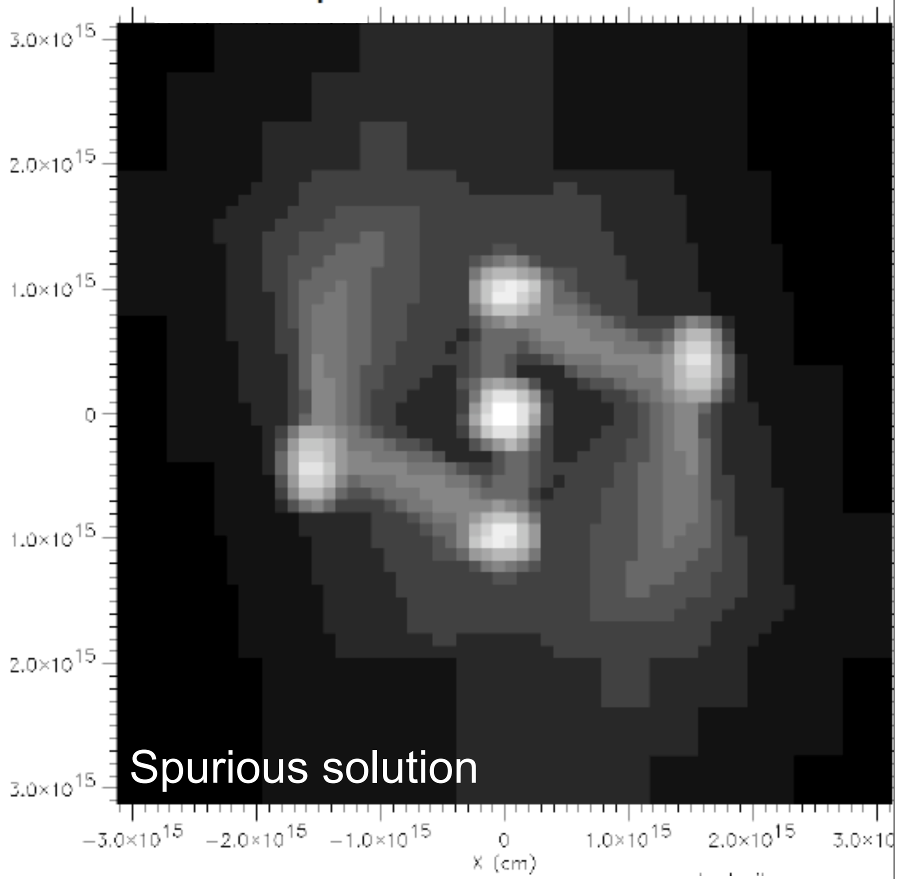
We need to resolve the Jeans length by at least ten cells in order to minimise numerical diffusion.

Otherwise, spurious fragmentation of the cloud occurs before collapse.

Numerical test with a collapsing cloud

Truelove *et al.* (1997) considered an initial m=2 perturbation for the spherical collapse of the homogeneous cloud. Using a PPM solver, they found that spurious fragmentation is avoided for $\Delta x < \frac{\lambda_J}{4}$





J. K. Truelove *et al.*, "The Jeans condition: a new constraint on spatial resolution in simulation of isothermal self-gravitational hydrodynamics", ApJ, 1997, 489, L179

Cold sine wave collapse

Use RAMSES to create a cold sine wave velocity perturbation (Zeldovich pancake)

```
!=================================================================
integer::ivar,i,id,iu,ip
real(dp)::twopi
real(dp),dimension(1:nvector,1:nvar),save::q   ! Primitive variables

id=1; iu=2; ip=ndim+2
twopi=2.0*acos(-1.0)
do i=1,nn
   q(i,id)=1.0
   q(i,iu)=sin(twopi*(x(i,1)))
   q(i,ip)=1e-5
end do

! Convert primitive to conservative variables
```

Patch condinit.f90

```
&AMR_PARAMS
levelmin=7
levelmax=7
ngridmax=20000
nexpand=1
boxlen=1.0
/

&INIT_PARAMS
nregion=0
/

&HYDRO_PARAMS
gamma=1.66667
courant_factor=0.8
slope_type=1
scheme='muscl'
riemann='hllc'
/
```

Periodic BCs.

Before shell crossing and shock formation, we know the analytical solution.

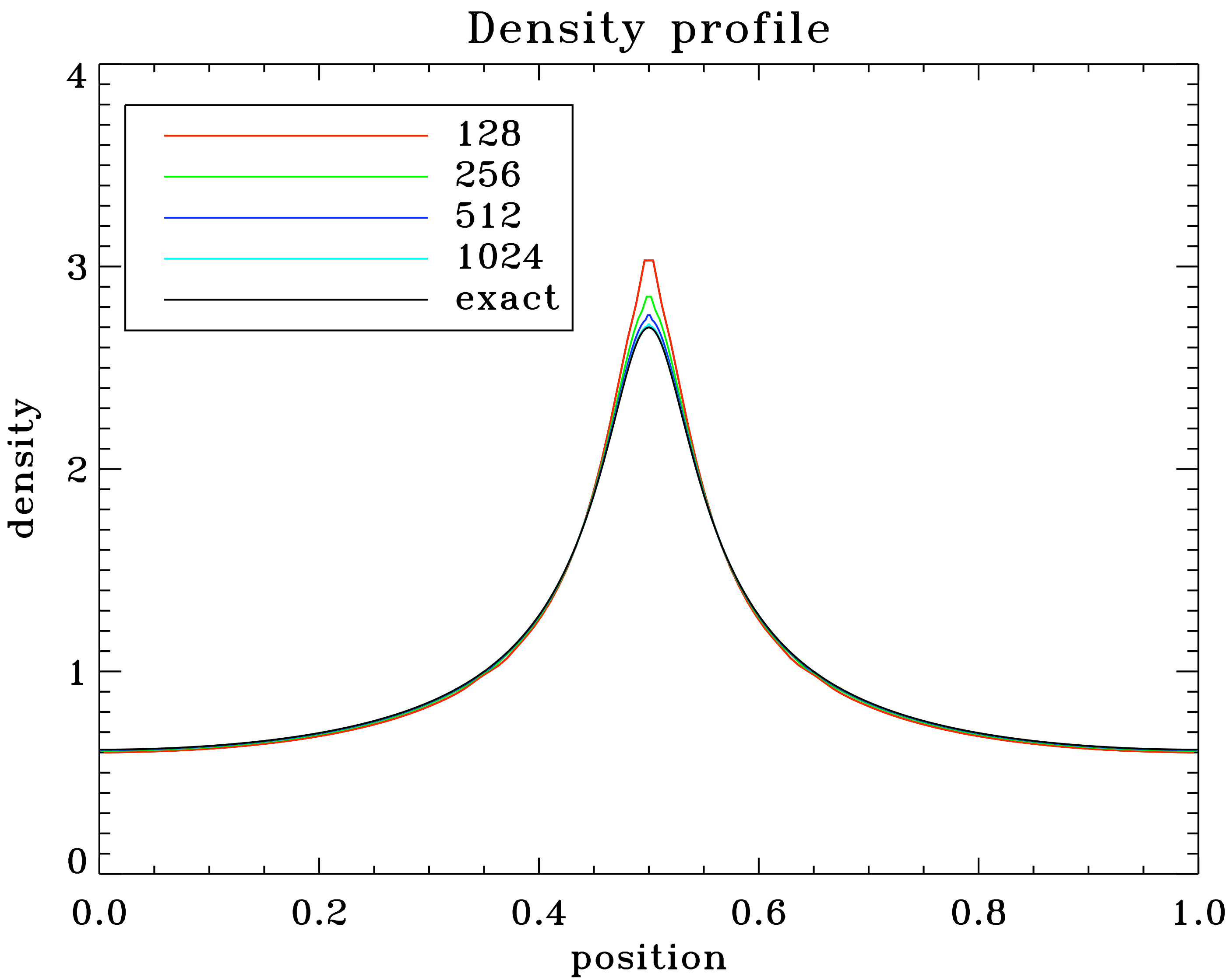
Because the initial temperature is very low, we have spurious heating.

We define a compression time: $$\frac{1}{\tau_{comp}} = \left|\frac{\partial u}{\partial x}\right| \simeq \frac{1}{H(t)}$$

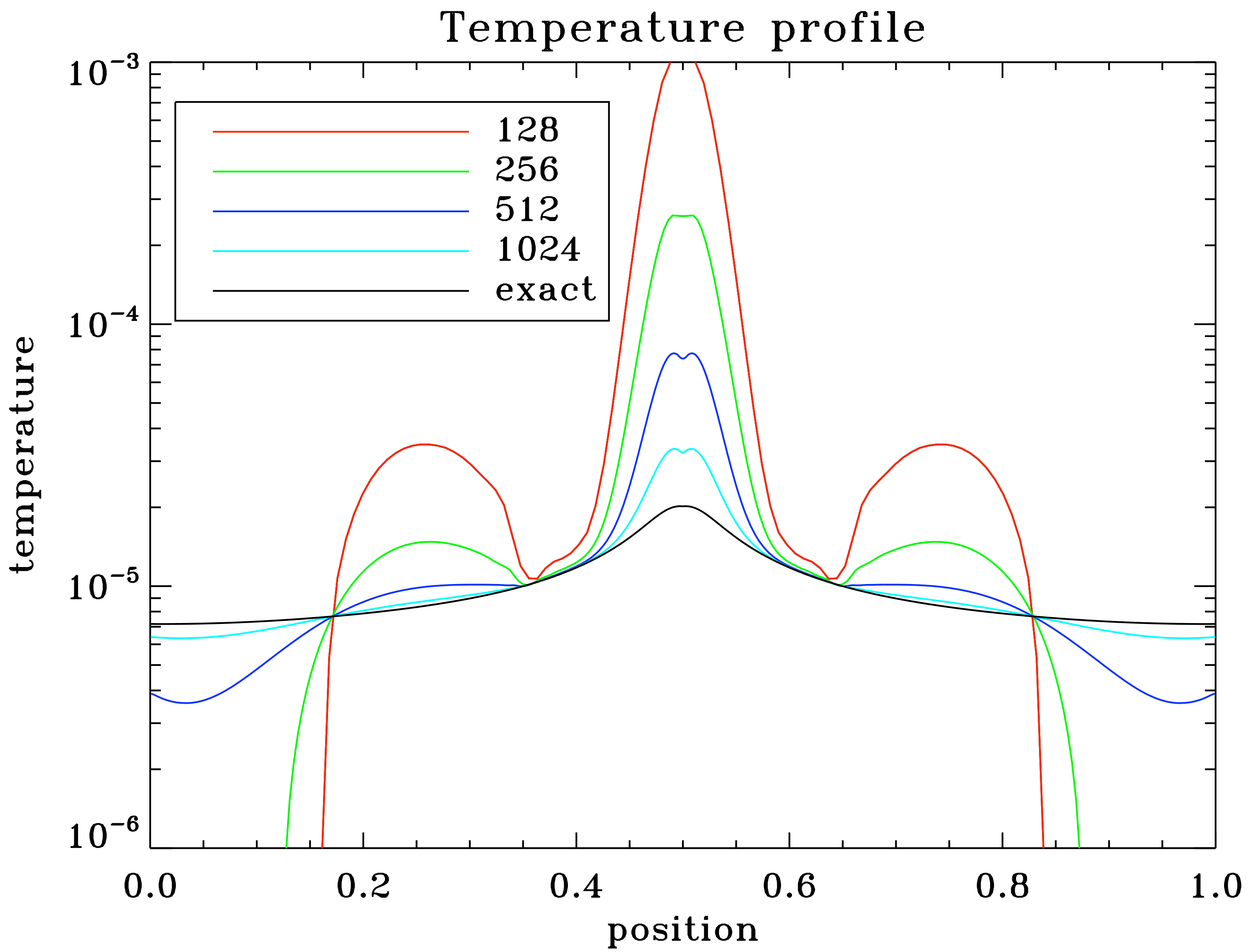
Spurious effects arise if:

$$c\tau_{comp} < \Delta x$$

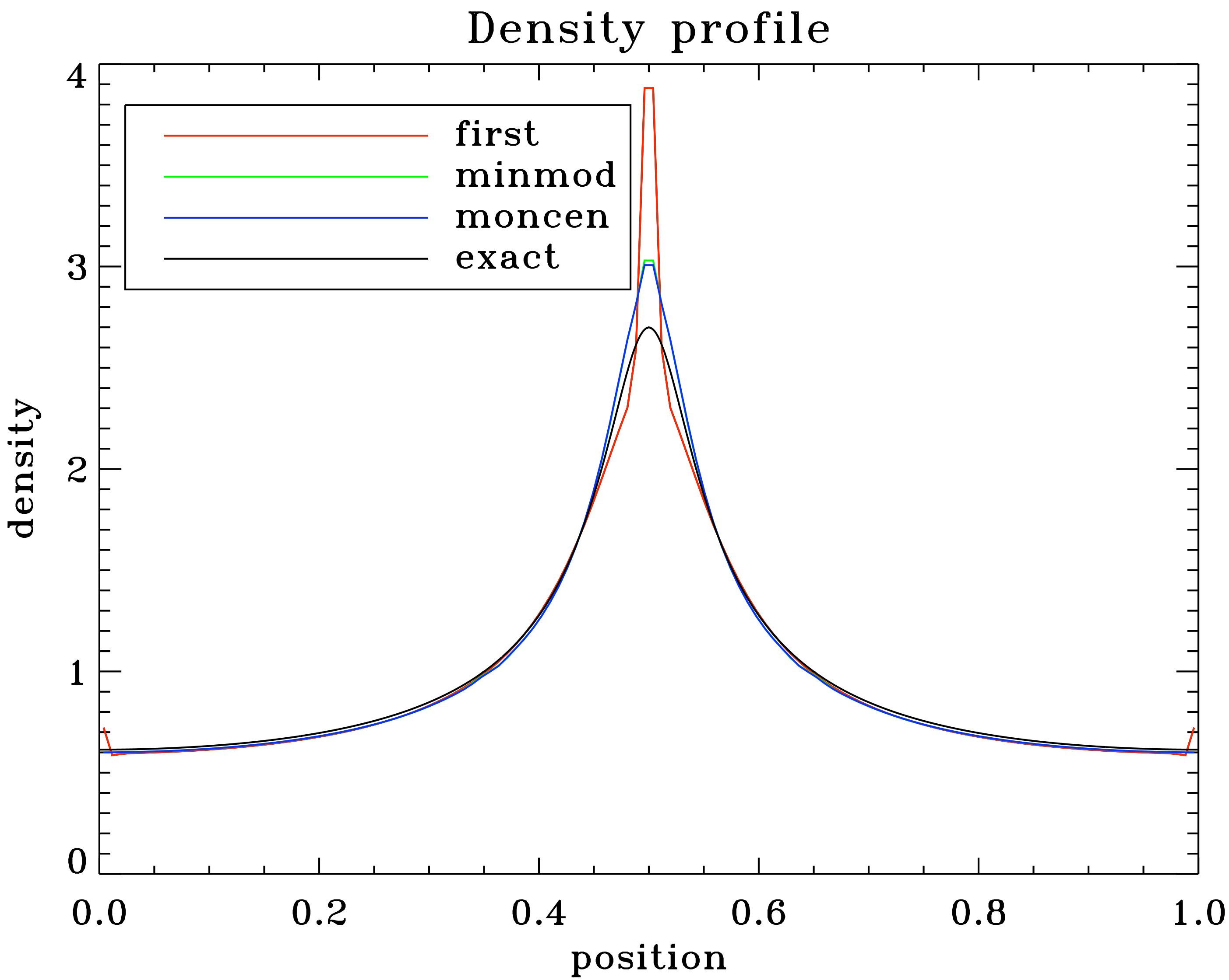
Cold sine wave collapse at t=0.1



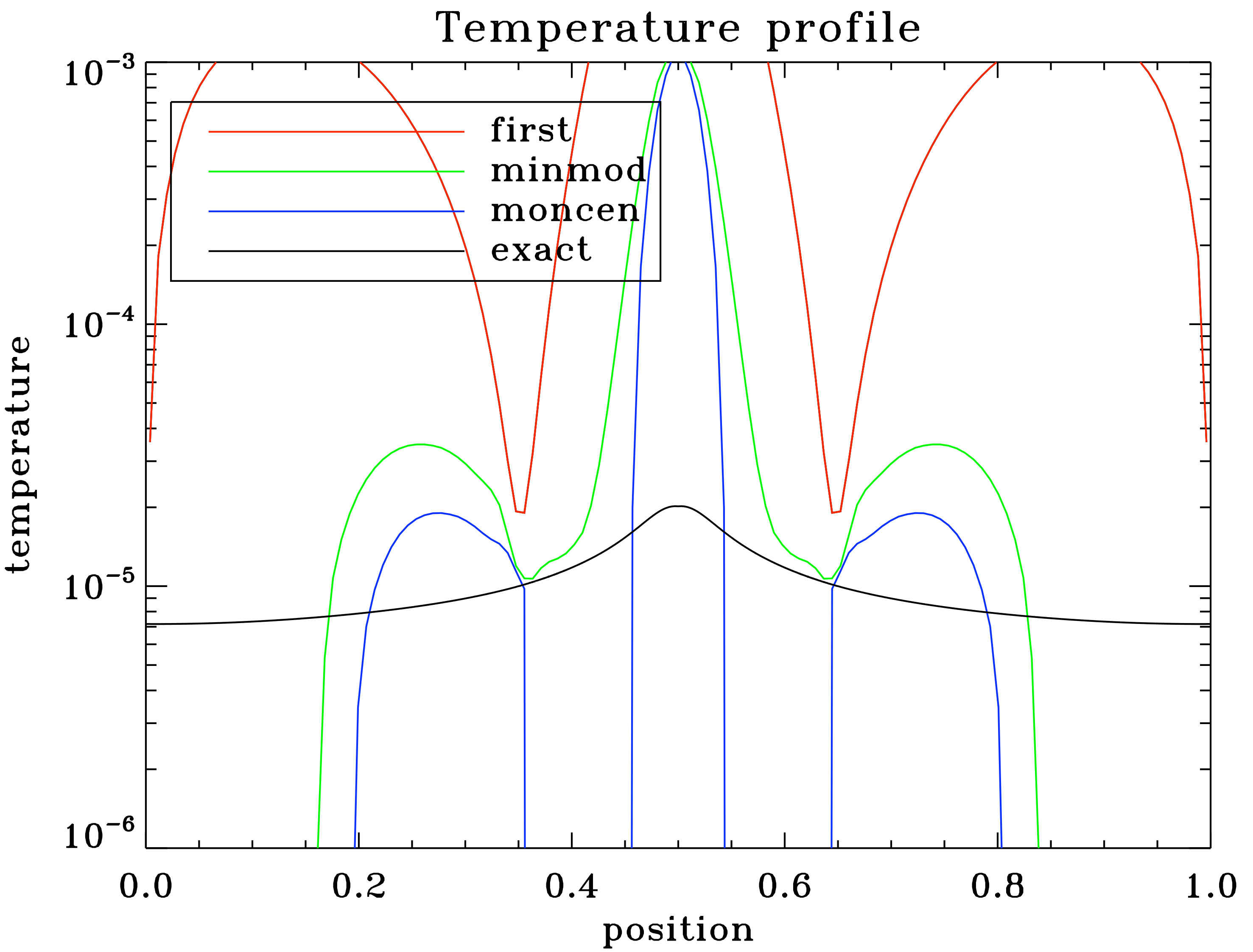
Cold sine wave collapse at t=0.1



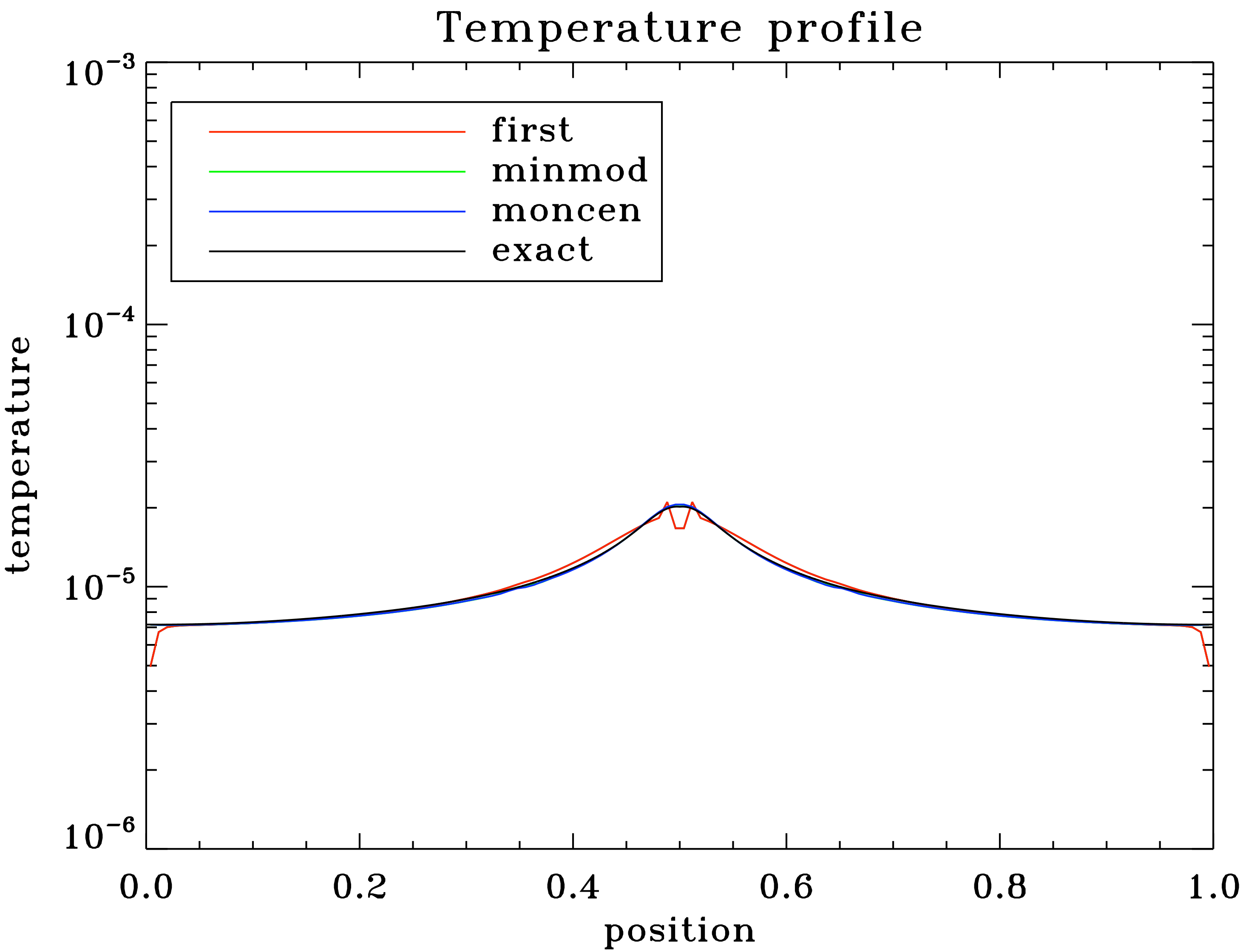
Cold sine wave collapse: effect of the solver



Cold sine wave collapse: effect of the solver



Cold sine wave collapse: primitive scheme solution



Hybrid scheme for high-Mach-number flows

Conservative scheme: total energy flux and pressure evaluation

$$\partial_t(E) + \partial_x(E+P)u = \rho \mathbf{u} \cdot \mathbf{g} \qquad P = (\gamma - 1)\left(E - \frac{1}{2}\rho u^2\right)$$

Primitive scheme: internal energy flux and pressure evaluation

$$\partial_t(e) + \partial_x e u = -P\partial_x u \qquad P = (\gamma - 1)e$$

For high-Mach-number flows, compression is stiff with respect to sound waves.

Cold hydrodynamics is better described by Burger's equation.

Following Jin & Levermore fix for stiff problems, we define the hybrid scheme:

Use total energy update if: $c > \beta \Delta x \, |\partial_x u|$

and internal energy update if: $c < \beta \Delta x \, |\partial_x u|$

```
&HYDRO_PARAMS
gamma=1.66667
courant_factor=0.8
slope_type=1
scheme='muscl'
riemann='hllc'
pressure_fix=.true.
beta_fix=1.0
/
```

See also V. Springel, G. Bryan

Conclusion

- An hyperbolic systems with source term relaxes to another equilibrium system
- Euler equations: adiabatic with cooling $\longrightarrow$ isothermal (hyperbolic)
- Radiative transfer: transport with absorption $\longrightarrow$ diffusion (parabolic)
- When source terms are stiff, numerical diffusion in the original hyperbolic system can dominate the equilibrium solution and lead to spurious results.
- This depends on the Peclet number $Pe = \frac{c\tau}{\Delta x}$
- You can either refine like hell (using AMR)
- You can use hybrid schemes !

Next lecture: Hyperbolic systems with source terms