Computational Astrophysics 4

The Godunov method

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Outline

- Hyperbolic system of conservation laws
- Finite difference approximation
- The Modified Equation
- The Upwind scheme
- Von Neumann Analysis
- The Godunov Method
- Riemann solvers
- 2D Godunov schemes

HS of CL

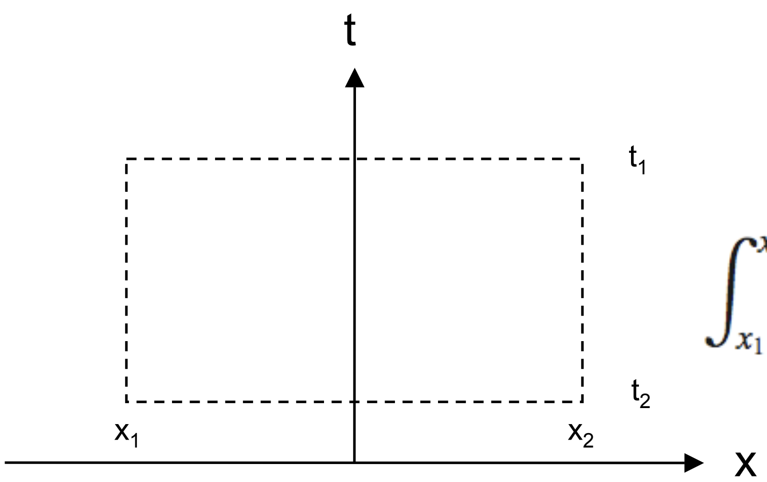
System of conservation laws

$$\partial_t \mathbf{U} + \partial_x \mathbf{F} = 0$$

- Vector of conservative variables $\quad {}^T\mathbf{U} = (\rho, \rho u, E)$

- Flux function $\quad {}^T\mathbf{F} = (\rho u, \rho u^2 + P, (E+P)u)$

Integral form

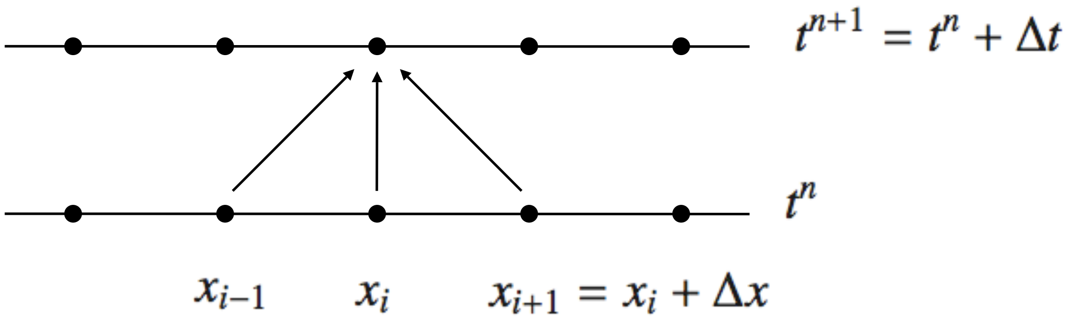


$$\int_{x_1}^{x_2} \int_{t_1}^{t_2} \mathrm{dxdt}\,(\partial_t \mathbf{U} + \partial_x \mathbf{F}) = 0$$

$$\int_{x_1}^{x_2} \mathrm{dx}\,(\mathbf{U}(t_2) - \mathbf{U}(t_1)) + \int_{t_1}^{t_2} \mathrm{dt}\,(\mathbf{F}(x_2) - \mathbf{F}(x_1)) = 0$$

$$\mathcal{U}(t_2) - \mathcal{U}(t_1) + \mathcal{F}(x_2) - \mathcal{F}(x_1) = 0$$

Finite difference scheme



$$u_i^n = u(x_i, t^n) \qquad \partial_x u \simeq \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} \qquad \partial_t u \simeq \frac{u_i^{n+1} - u_i^n}{\Delta t}$$

Finite difference approximation of the advection equation

$$\partial_t u + a\partial_x u = 0 \quad \Longrightarrow \quad \frac{u_i^{n+1} - u_i^n}{\Delta t} + a\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} = 0$$

The Modified Equation

$$\frac{u_i^{n+1}-u_i^n}{\Delta t}+a\frac{u_{i+1}^n-u_{i-1}^n}{2\Delta x}=0$$

Taylor expansion in time up to second order

$$u_i^{n+1}=u_i^n+\Delta t\left(\frac{\partial u}{\partial t}\right)+\frac{(\Delta t)^2}{2}\left(\frac{\partial^2 u}{\partial t^2}\right)$$

Taylor expansion in space up to second order

$$u_{i+1}^n=u_i^n+\Delta x\left(\frac{\partial u}{\partial x}\right)+\frac{(\Delta x)^2}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)$$

$$u_{i-1}^n=u_i^n-\Delta x\left(\frac{\partial u}{\partial x}\right)+\frac{(\Delta x)^2}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)$$

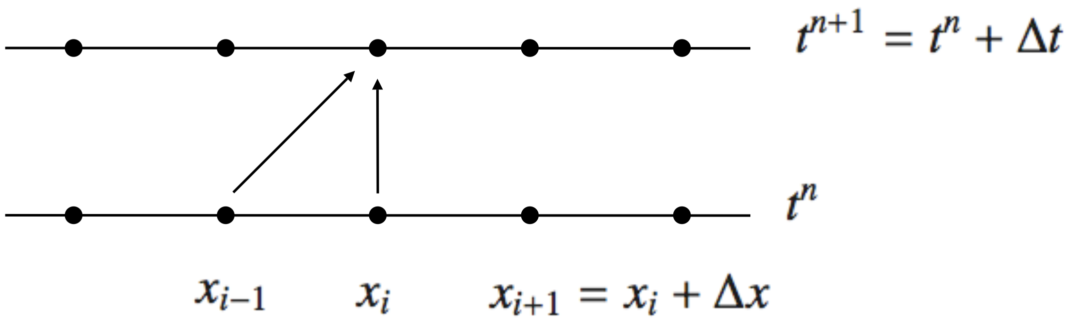
The advection equation becomes the advection-diffusion equation

$$\left(\frac{\partial u}{\partial t}\right)+a\left(\frac{\partial u}{\partial x}\right)=-\frac{\Delta t}{2}\left(\frac{\partial^2 u}{\partial t^2}\right)+O(\Delta t^2,\Delta x^2)$$

$$\left(\frac{\partial u}{\partial t}\right)+a\left(\frac{\partial u}{\partial x}\right)=-a^2\frac{\Delta t}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)+O(\Delta t^2,\Delta x^2)$$

Negative diffusion coefficient: the scheme is *unconditionally unstable*

The Upwind scheme



a>0: use only upwind values, discard downwind variables

$$\partial_x u \simeq \frac{u_i^n - u_{i-1}^n}{\Delta x} \quad \Rightarrow \quad \frac{u_i^{n+1} - u_i^n}{\Delta t} + a\frac{u_i^n - u_{i-1}^n}{\Delta x} = 0$$

Taylor expansion up to second order:

$$\left(\frac{\partial u}{\partial t}\right) + a\left(\frac{\partial u}{\partial x}\right) = -\frac{\Delta t}{2}\left(\frac{\partial^2 u}{\partial t^2}\right) + a\frac{\Delta x}{2}\left(\frac{\partial^2 u}{\partial x^2}\right) + O(\Delta t^2, \Delta x^2)$$

Upwind scheme is stable if C<1, with $C = a\dfrac{\Delta t}{\Delta x}$

$$\left(\frac{\partial u}{\partial t}\right) + a\left(\frac{\partial u}{\partial x}\right) = a\frac{\Delta x}{2}(1 - C)\left(\frac{\partial^2 u}{\partial x^2}\right) + O(\Delta t^2, \Delta x^2)$$

Von Neumann analysis

Fourier transform the current solution: $$u_i^n = \sum_k A_k^n \exp(-ikx_i)$$

Evaluate the amplification factor of the 2 schemes.

Fromm scheme: $$u_i^{n+1} = u_i^n - \frac{C}{2}u_{i+1}^n + \frac{C}{2}u_{i-1}^n$$

$$A_k^{n+1} = A_k^n\left(1 - \frac{C}{2}\exp(-ik\Delta x) + \frac{C}{2}\exp(ik\Delta x)\right)$$

$$\omega^2 = \frac{|A_k^{n+1}|^2}{|A_k^n|^2} = 1 + C^2\sin(k\Delta x)^2$$

$\omega>1$: the scheme is unconditionally unstable

Upwind scheme: $$u_i^{n+1} = u_i^n(1-C) + Cu_{i-1}^n$$

$$A_k^{n+1} = A_k^n\left(1 - C + C\exp(ik\Delta x)\right)$$

$$\omega^2 = \frac{|A_k^{n+1}|^2}{|A_k^n|^2} = 1 - 2C(1-C)(1-\cos(k\Delta x))$$

$\omega<1$ if $C<1$: the scheme is stable under the Courant condition.

The advection-diffusion equation

Finite difference approximation of the advection equation:

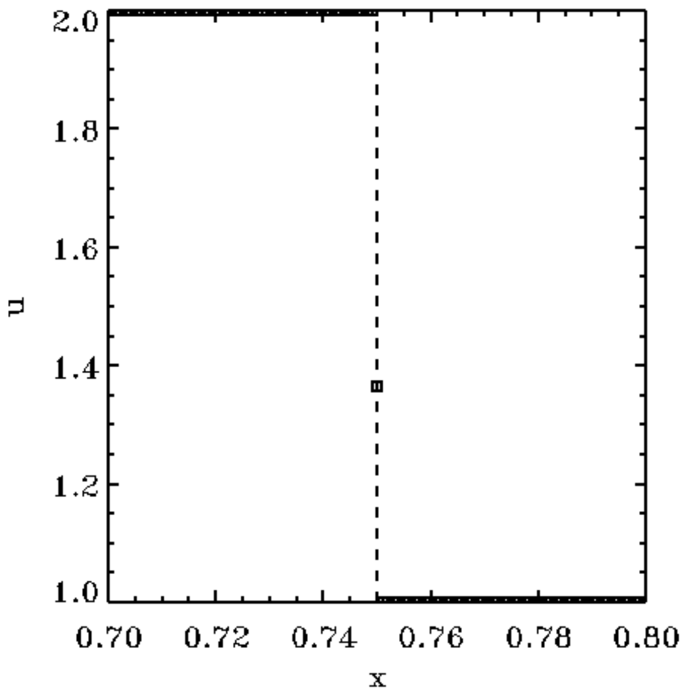
$$\left(\frac{\partial u}{\partial t}\right) + a\left(\frac{\partial u}{\partial x}\right) = \eta\left(\frac{\partial^2 u}{\partial x^2}\right)$$

Central differencing unstable: $\eta < 0$

Upwind differencing is stable: $\eta > 0$ $\quad \eta = a\frac{\Delta x}{2}(1 - C)$

Smearing of initial discontinuity:

"numerical diffusion"



Thickness increases as $\sqrt{\eta t}$

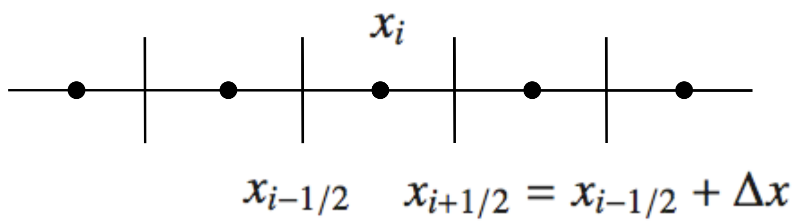
The Godunov method

Sergei Konstantinovich Godunov

Sergei Konstantinovich Godunov

Born	17th July, 1929 Moscow

Finite volume scheme



Finite volume approximation of the advection equation:

$$u_i^n = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} u(x, t^n)\mathrm{d}x$$

Use integral form of the conservation law:

$$\int_{x_{i-1/2}}^{x_{i+1/2}}\int_{t^n}^{t^{n+1}} \mathrm{d}x\mathrm{d}t\,(\partial_t u + a\partial_x u) = 0$$

Exact evolution of volume averaged quantities:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\frac{u_{i+1/2}^{n+1/2} - u_{i-1/2}^{n+1/2}}{\Delta x} = 0$$

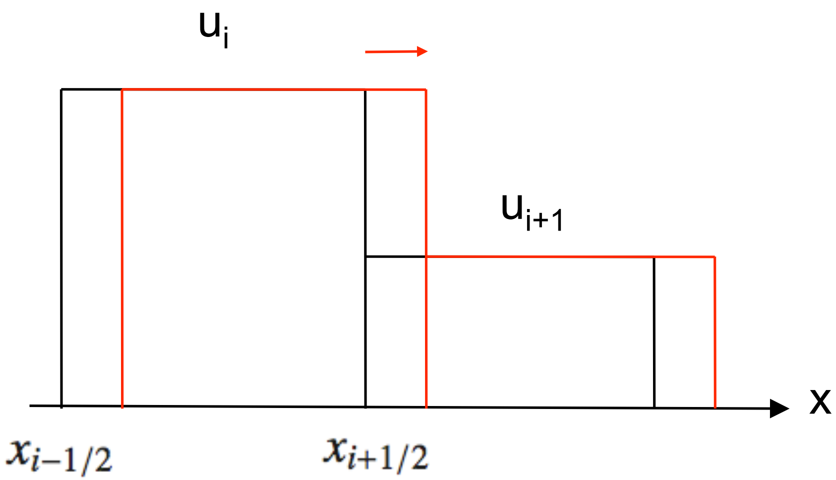
Time averaged flux function: $$u_{i+1/2}^{n+1/2} = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} u(x_{i+1/2}, t)\mathrm{d}t$$

Godunov scheme for the advection equation

The time averaged flux function: $u_{i+1/2}^{n+1/2} = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} u(x_{i+1/2}, t)\mathrm{d}t$

is computed using the solution of the Riemann problem defined

at cell interfaces with piecewise constant initial data.



For all t>0:

$$u(x_{i+1/2}, t) = u_i^n \quad \text{if} \quad a > 0$$
$$u(x_{i+1/2}, t) = u_{i+1}^n \quad \text{if} \quad a < 0$$

The Godunov scheme for the advection equation is identical to the upwind finite difference scheme.

Godunov scheme for hyperbolic systems

The system of conservation laws

$$\partial_t \mathbf{U} + \partial_x \mathbf{F} = 0$$

is discretized using the following integral form:

$$\frac{\mathbf{U}_i^{n+1} - \mathbf{U}_i^n}{\Delta t} + \frac{\mathbf{F}_{i+1/2}^{n+1/2} - \mathbf{F}_{i-1/2}^{n+1/2}}{\Delta x} = 0$$

The time average flux function is computed using the self-similar solution of the inter-cell Riemann problem:

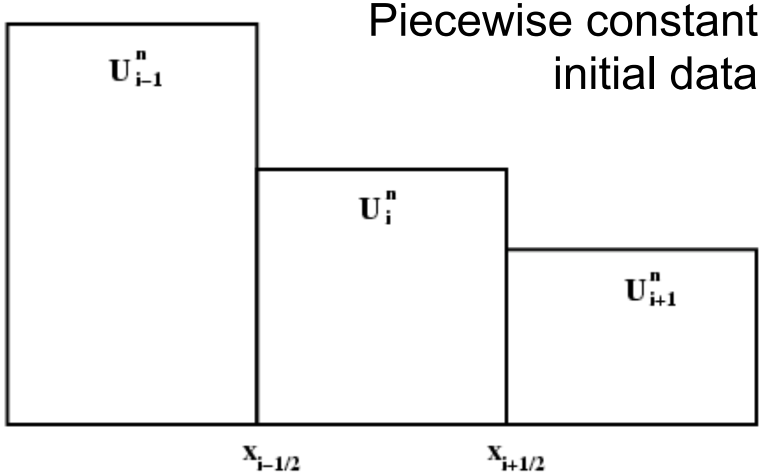
$$\mathbf{U}_{i+1/2}^*(x/t) = \mathcal{RP}\left[\mathbf{U}_i^n, \mathbf{U}_{i+1}^n\right]$$

$$\mathbf{F}_{i+1/2}^{n+1/2} = \mathbf{F}(\mathbf{U}_{i+1/2}^*(0))$$

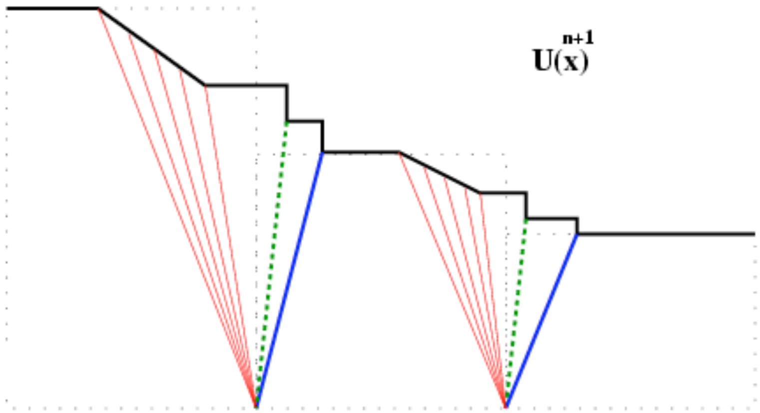
This defines the Godunov flux:

$$\mathbf{F}_{i+1/2}^{n+1/2} = \mathbf{F}^*(\mathbf{U}_i^n, \mathbf{U}_{i+1}^n)$$

Piecewise constant initial data



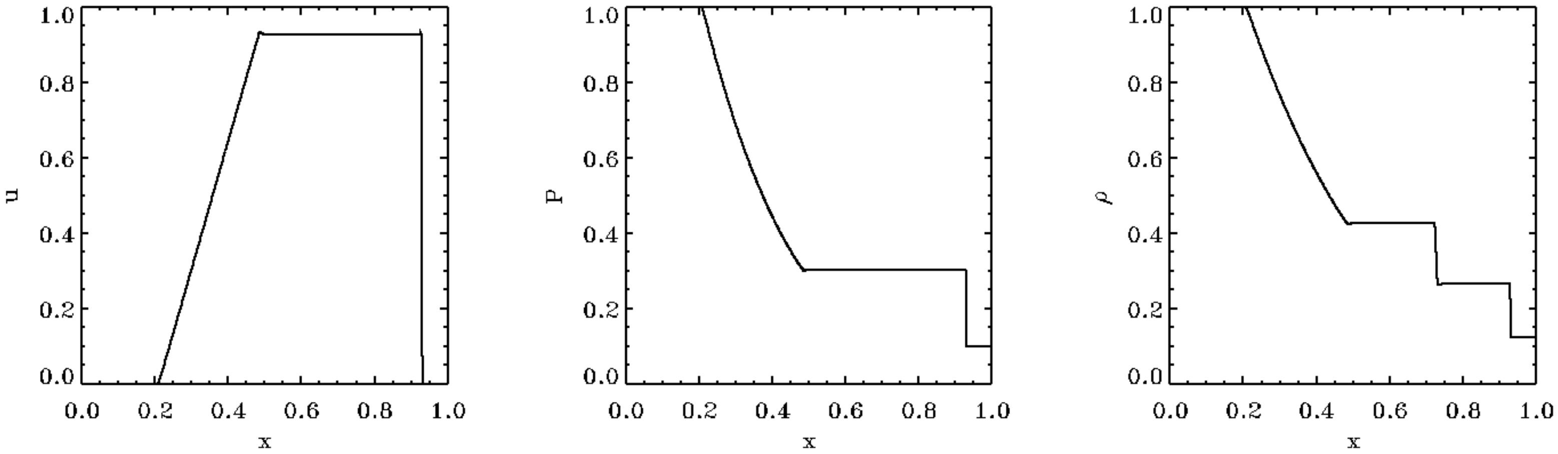
- **Godunov, S. K.** (1959), A Difference Scheme for Numerical Solution of Discontinuos Solution of Hydrodynamic Equations, *Math. Sbornik, **47**, 271-306*, translated US Joint Publ. Res. Service, JPRS 7226, 1969.



Advection: 1 wave, Euler: 3 waves, MHD: 7 waves

Riemann solvers

Exact Riemann solution is costly: involves Raphson-Newton iterations and complex non-linear functions.



Approximate Riemann solvers are more useful.

Two broad classes:

- Linear solvers

- HLL solvers

- **Toro, E. F.** (1999), *Riemann Solvers and Numerical Methods for Fluid Dynamics*, Springer-Verlag.

Linear Riemann solvers

Define a reference state as the arithmetic average or the Roe average

$$\mathbf{U}_{ref} = \frac{\mathbf{U}_L + \mathbf{U}_R}{2} \qquad \mathbf{U}_{ref} = \mathrm{Roe}\left[\mathbf{U}_L, \mathbf{U}_R\right]$$

Evaluate the Jacobian matrix at this reference state. $\mathbf{A} = \frac{\partial \mathbf{F}}{\partial \mathbf{U}}\left(\mathbf{U}_{ref}\right)$

Compute eigenvalues and (left and right) eigenvectors $\mathbf{A} = \mathbf{L}^T \Lambda \mathbf{R}$

The interface state is obtained by combining all upwind waves

$$\mathbf{A}\mathbf{U}_* = \mathbf{A}\frac{\mathbf{U}_L + \mathbf{U}_R}{2} - \mathbf{L}^T|\Lambda|\mathbf{R}\frac{\mathbf{U}_R - \mathbf{U}_L}{2}$$ where $|\Lambda| = (|\lambda_1|, |\lambda_2|, \ldots)$

Non-linear flux function with a linear diffusive term.

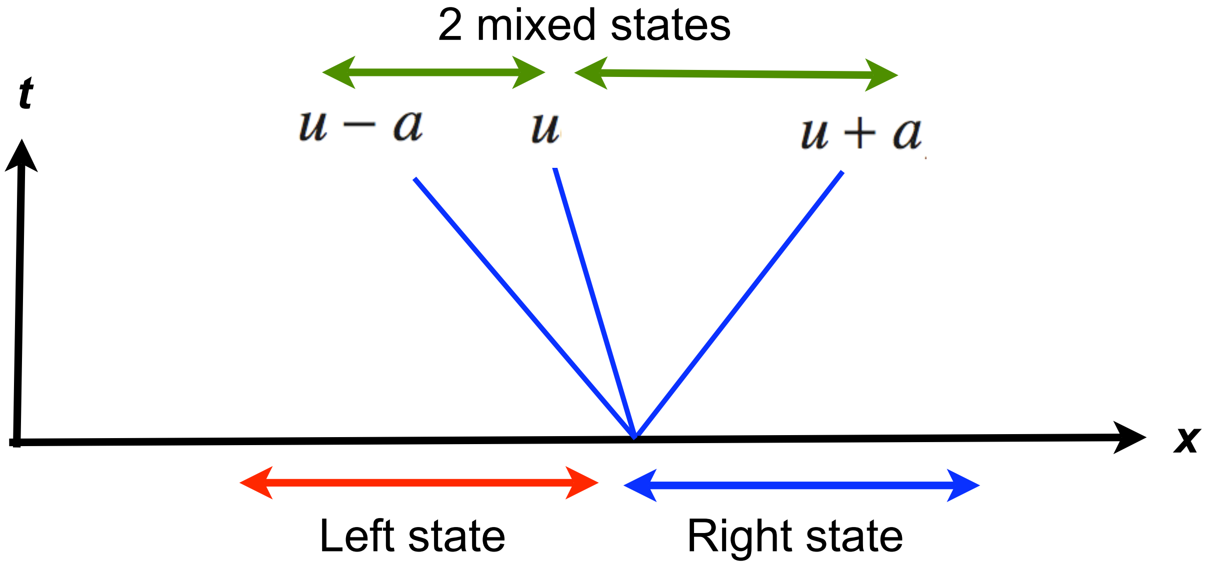
$$\mathbf{F}^*(U_L, U_R) = \frac{\mathbf{F}_L + \mathbf{F}_R}{2} - \mathbf{L}^T|\Lambda|\mathbf{R}\frac{\mathbf{U}_R - \mathbf{U}_L}{2}$$

A simple example, the *upwind* Riemann solver:

$$\mathbf{F}^*(U_L, U_R) = a\frac{\mathbf{U}_L + \mathbf{U}_R}{2} - |a|\frac{\mathbf{U}_R - \mathbf{U}_L}{2}$$

Riemann problem for adiabatic waves

Initial conditions are defined by 2 semi-infinite regions with piecewise constant initial states $(\Delta\rho_R, \Delta u_R, \Delta P_R)$ and $(\Delta\rho_L, \Delta u_L, \Delta P_L)$.



Left "star" state: (-,0,+)=(R,L,L) and right "star" state: (-,0,+)=(R,R,L).

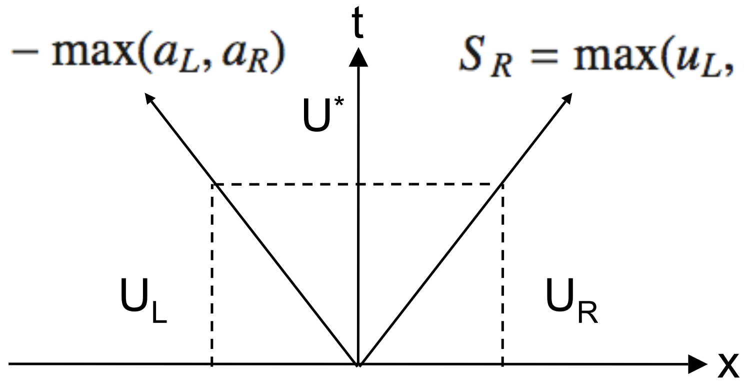
$$\Delta u^*_{L,R} = \frac{a}{\rho}(\Delta\alpha^+_L - \Delta\alpha^-_R) \qquad \Delta\rho^*_R = \Delta\alpha^+_L + \Delta\alpha^0_R + \Delta\alpha^-_R$$

$$\Delta P^*_{L,R} = \frac{a}{\rho}(\Delta\alpha^+_L + \Delta\alpha^-_R) \qquad \Delta\rho^*_L = \Delta\alpha^+_L + \Delta\alpha^0_L + \Delta\alpha^-_R$$

HLL Riemann solver

Approximate the true Riemann fan by 2 waves and 1 intermediate state:

$$S_L = \min(u_L, u_R) - \max(a_L, a_R) \qquad S_R = \max(u_L, u_R) + \max(a_L, a_R)$$



Compute U* using the integral form between $S_L t$ and $S_R t$

$$\mathbf{U}^*(\mathbf{U}_L, \mathbf{U}_R) = \frac{S_R \mathbf{U}_R - S_L \mathbf{U}_L - (\mathbf{F}_R - \mathbf{F}_L)}{S_R - S_L}$$

Compute F* using the integral form between $S_L t$ and 0.

$$S_L > 0 \quad \mathbf{F}^*(U_L, U_R) = \mathbf{F}_L$$

$$S_R < 0 \quad \mathbf{F}^*(U_L, U_R) = \mathbf{F}_R$$

$$S_L < 0 \ \text{ and } \ S_R > 0 \quad \mathbf{F}^*(U_L, U_R) = \frac{S_R \mathbf{F}_L - S_L \mathbf{F}_R + S_L S_R (\mathbf{U}_R - \mathbf{U}_L)}{S_R - S_L}$$

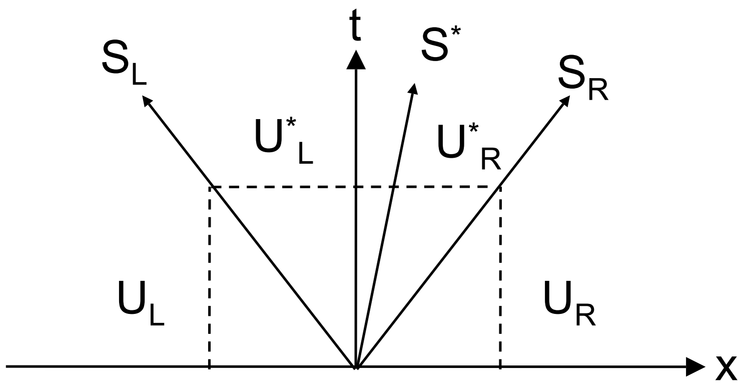
Other HLL-type Riemann solvers

Lax-Friedrich Riemann solver: $S_* = S_R = -S_L = \max(|u_L| + a_L, |u_R| + a_R)$

$$\mathbf{F}^*(U_L, U_R) = \frac{\mathbf{F}_L + \mathbf{F}_R}{2} - S_* \frac{\mathbf{U}_R - \mathbf{U}_L}{2}$$

HLLC Riemann solver: add a third wave for the contact (entropy) wave.

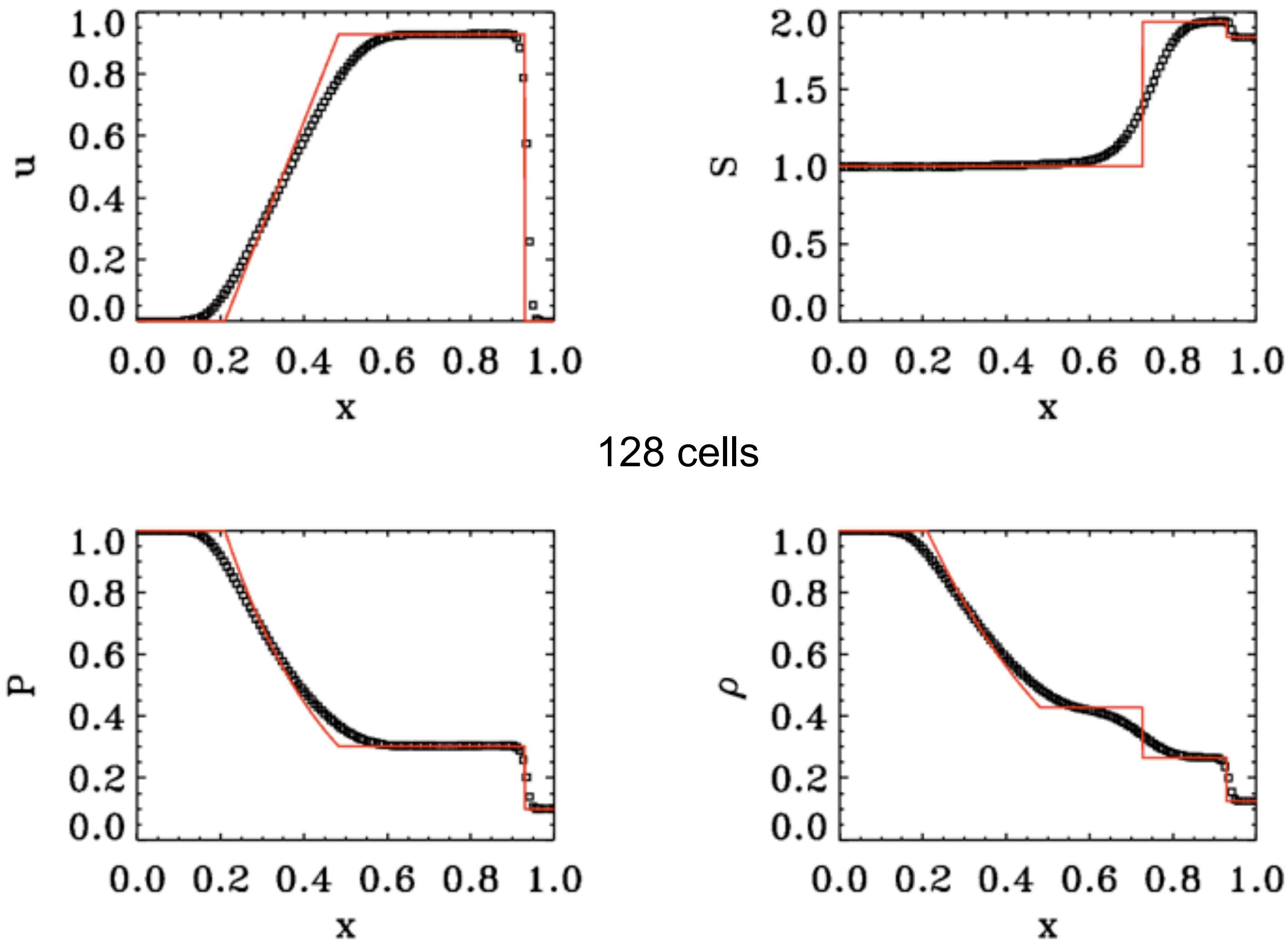
$$S_* = \frac{(\rho u)^*_{HLL}}{\rho^*_{HLL}}$$



See Toro (1997) for details.

Sod test with the Godunov scheme

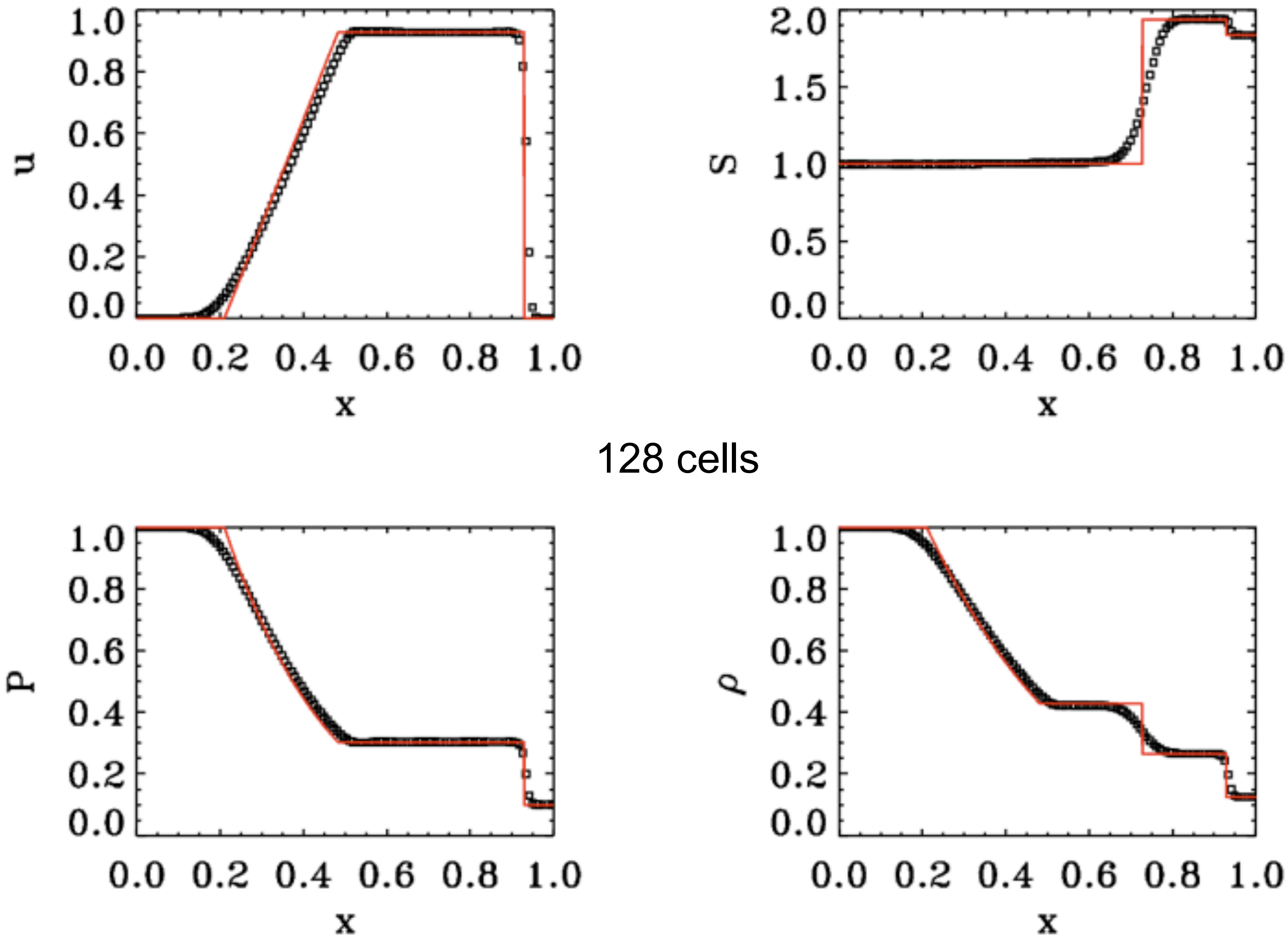
Lax-Friedrich Riemann solver



128 cells

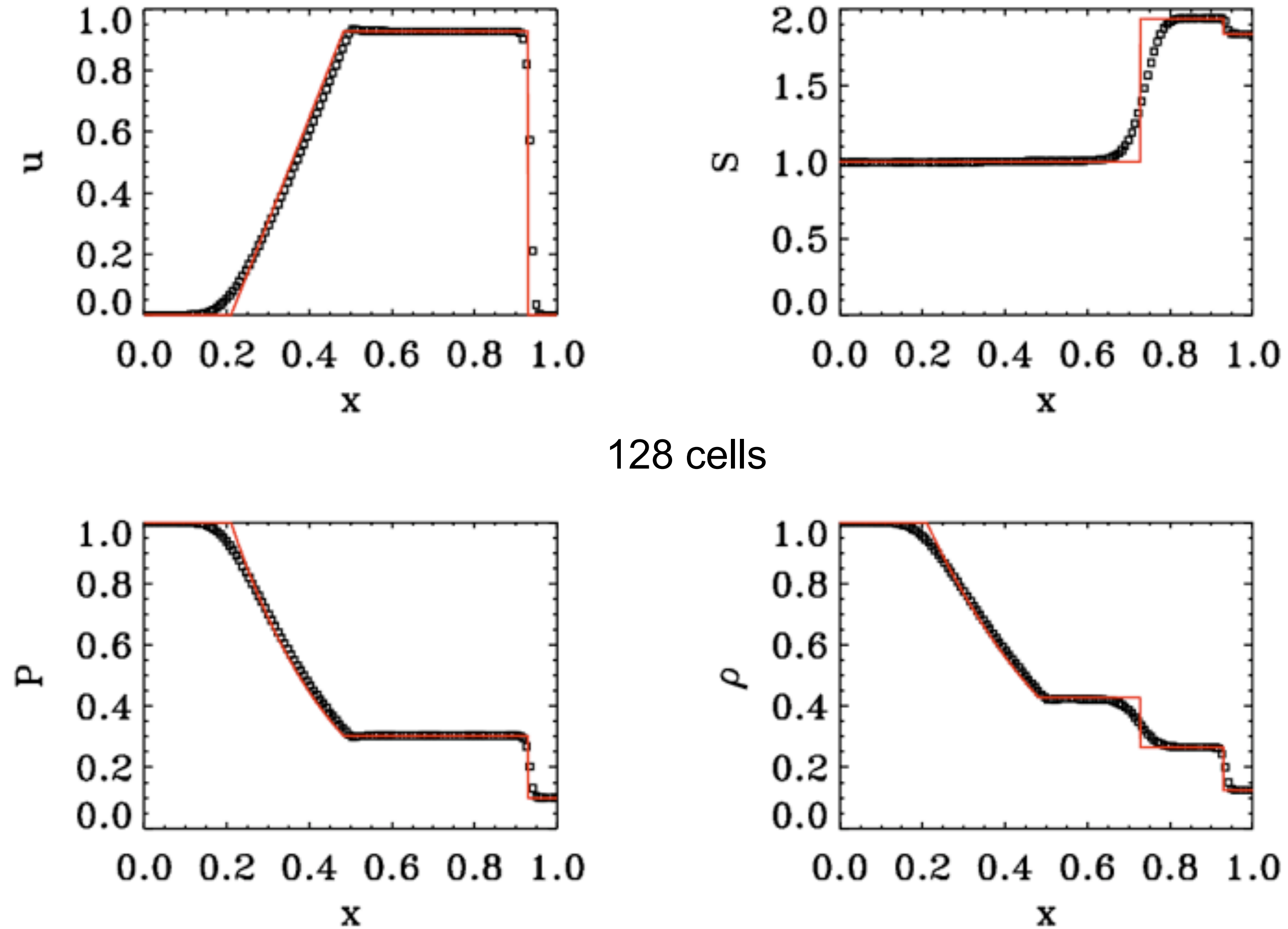
Sod test with the Godunov scheme

HLLC Riemann solver



Sod test with the Godunov scheme

Exact Riemann solver



Multidimensional Godunov schemes

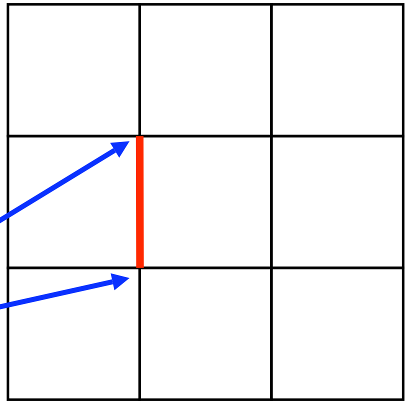
2D Euler equations in integral (conservative) form

$$\mathbf{U}_{i,j}^{n+1} - \mathbf{U}_{i,j}^{n} + \frac{\Delta t}{\Delta x}\left(\mathbf{F}_{i+1/2,j}^{n+1/2} - \mathbf{F}_{i-1/2,j}^{n+1/2}\right) + \frac{\Delta t}{\Delta y}\left(\mathbf{G}_{i,j+1/2}^{n+1/2} - \mathbf{G}_{i,j-1/2}^{n+1/2}\right) = 0$$

Flux functions are now time and space average.

$$\mathbf{F}_{i+1/2,j}^{n+1/2} = \frac{1}{\Delta t}\frac{1}{\Delta y}\int_{t^n}^{t^{n+1}}\int_{y_{j-1/2}}^{y_{j+1/2}} \mathbf{F}(x_{i+1/2}, y, t)\,\mathrm{d}t\,\mathrm{d}y$$

$$\mathbf{G}_{i,j+1/2}^{n+1/2} = \frac{1}{\Delta t}\frac{1}{\Delta x}\int_{t^n}^{t^{n+1}}\int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{G}(x, y_{j+1/2}, t)\,\mathrm{d}t\,\mathrm{d}x$$



2D Riemann problems interact along cell edges:

$$\mathbf{U}_{i+1/2,j+1/2}^{*}(x/t, y/t) = \mathcal{RP}\left[\langle\mathbf{U}\rangle_{i,j}^{n}, \langle\mathbf{U}\rangle_{i+1,j}^{n}\, \langle\mathbf{U}\rangle_{i,j+1}^{n}\, \langle\mathbf{U}\rangle_{i+1,j+1}^{n}\right]$$

Even at first order, self-similarity does not apply to the flux functions anymore.

Predictor-corrector schemes ?

Directional (Strang) splitting

Perform 1D Godunov scheme along each direction in sequence.

X step: $$\mathbf{U}_{i,j}^{n+1} - \mathbf{U}_{i,j}^{n} + \frac{\Delta t}{\Delta x}\left(\mathbf{F}_{i+1/2,j}^{n+1/2} - \mathbf{F}_{i-1/2,j}^{n+1/2}\right) = 0$$

Y step: $$\mathbf{U}_{i,j}^{n+2} - \mathbf{U}_{i,j}^{n+1} + \frac{\Delta t}{\Delta y}\left(\mathbf{G}_{i,j+1/2}^{n+3/2} - \mathbf{G}_{i,j-1/2}^{n+3/2}\right) = 0$$

Change direction at the next step using the same time step.

Compute Δt, X step, Y step, t=t+Δt Y step, X step t=t+Δt

Courant factor per direction: $C_x = (|u| + a)\dfrac{\Delta t}{\Delta x} \qquad C_y = (|v| + a)\dfrac{\Delta t}{\Delta y}$

Courant condition: $\max(C_x, C_y) < 1$

Cost: 2 Riemann solves per time step.

Second order based on corresponding 1D higher order method.

Unsplit schemes

Godunov scheme

No predictor step.

Flux functions computed using 1D Riemann problem at time t^n in each normal direction.

2 Riemann solves per step.

Courant condition: $C_x + C_y < 1$

Runge-Kutta scheme

Predictor step using the Godunov scheme and $\Delta t/2$.

Flux functions computed using 1D Riemann problem at time $t^{n+1/2}$ in each normal direction.

4 Riemann solves per step.

Courant condition: $C_x + C_y < 1$

Corner Transport Upwind

Predictor step in transverse direction only using the 1D Godunov scheme.

Flux functions computed using 1D Riemann problem at time $t^{n+1/2}$ in each normal direction.

4 Riemann solves per step.

Courant condition: $\max(C_x, C_y) < 1$

The Godunov scheme for 2D advection

Solve 1D Riemann problem at each face

$$a > 0 \quad u_{i+1/2,j}^{n+1/2} = u_{i,j}^{n} \qquad b > 0 \quad u_{i,j+1/2}^{n+1/2} = u_{i,j}^{n}$$

Perform a 2D unsplit conservative update

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} + a\frac{u_{i+1/2,j}^{n+1/2} - u_{i-1/2,j}^{n+1/2}}{\Delta x} + b\frac{u_{i+1/2,j}^{n+1/2} - u_{i-1/2,j}^{n+1/2}}{\Delta y} = 0$$

We get the following first-order linear scheme

$$u_{i,j}^{n+1} = u_{i,j}^{n}\left(1 - C_x - C_y\right) + u_{i-1,j}^{n} C_x + u_{i,j-1}^{n} C_y$$

Modified equation for 2D advection equation (exercise):

$$\partial_t u + a\partial_x u + b\partial_y u = a\frac{\Delta x}{2}(1 - C_x)\partial_x^2 u + b\frac{\Delta y}{2}(1 - C_y)\partial_y^2 u - ab\Delta t\partial_x\partial_y u$$

Differential form has 2 positive eigenvalues if:

$C_x > 0 \qquad C_y > 0 \qquad$ and $\qquad C_x + C_y < 1$

CTU scheme for 2D advection

Solve 1D Riemann problem at each face using transverse predicted states

$$a > 0 \quad u^{n+1/2}_{i+1/2,j} = u^{n+1/2,y}_{i,j} \qquad b > 0 \quad u^{n+1/2}_{i,j+1/2} = u^{n+1/2,x}_{i,j}$$

Predicted states are obtained in each direction by a 1D Godunov scheme.

$$u^{n+1/2,y}_{i,j} = u^n_{i,j}\left(1 - C_y/2\right) + u^n_{i,j-1} C_y/2$$

$$u^{n+1/2,x}_{i,j} = u^n_{i,j}(1 - C_x/2) + u^n_{i-1,j} C_x/2$$

during $\Delta t/2$

We get the following first-order linear scheme

$$u^{n+1}_{i,j} = u^n_{i,j}(1 - C_x)\left(1 - C_y\right) + u^n_{i-1,j} C_x \left(1 - C_y\right) + u^n_{i,j-1}(1 - C_x) C_y + u^n_{i-1,j-1} C_x C_y$$

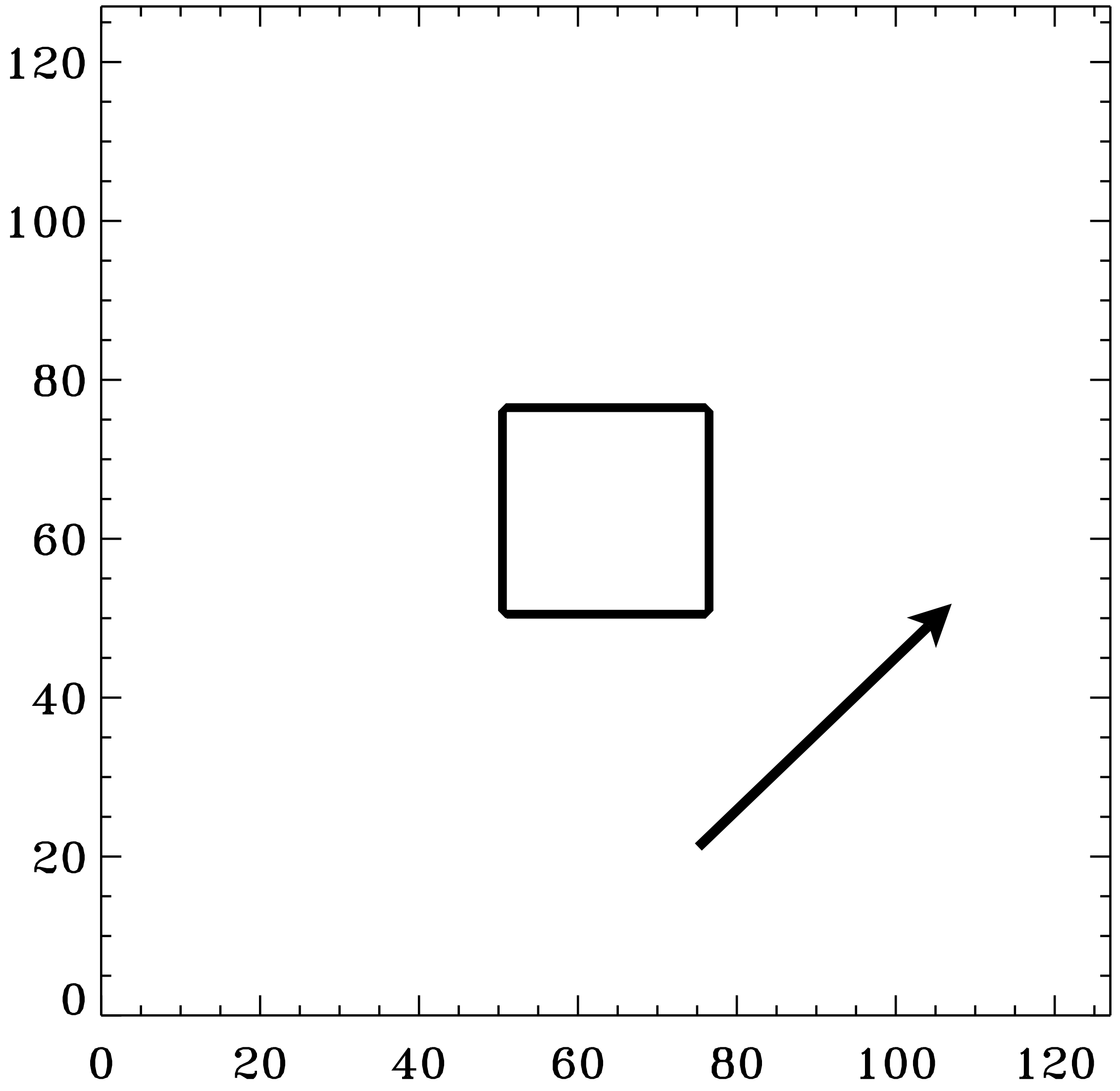
Diffusion term in the modified equation is now (exercise):

$$a\frac{\Delta x}{2}(1 - C_x)\partial^2_x u + b\frac{\Delta y}{2}(1 - C_y)\partial^2_y u$$

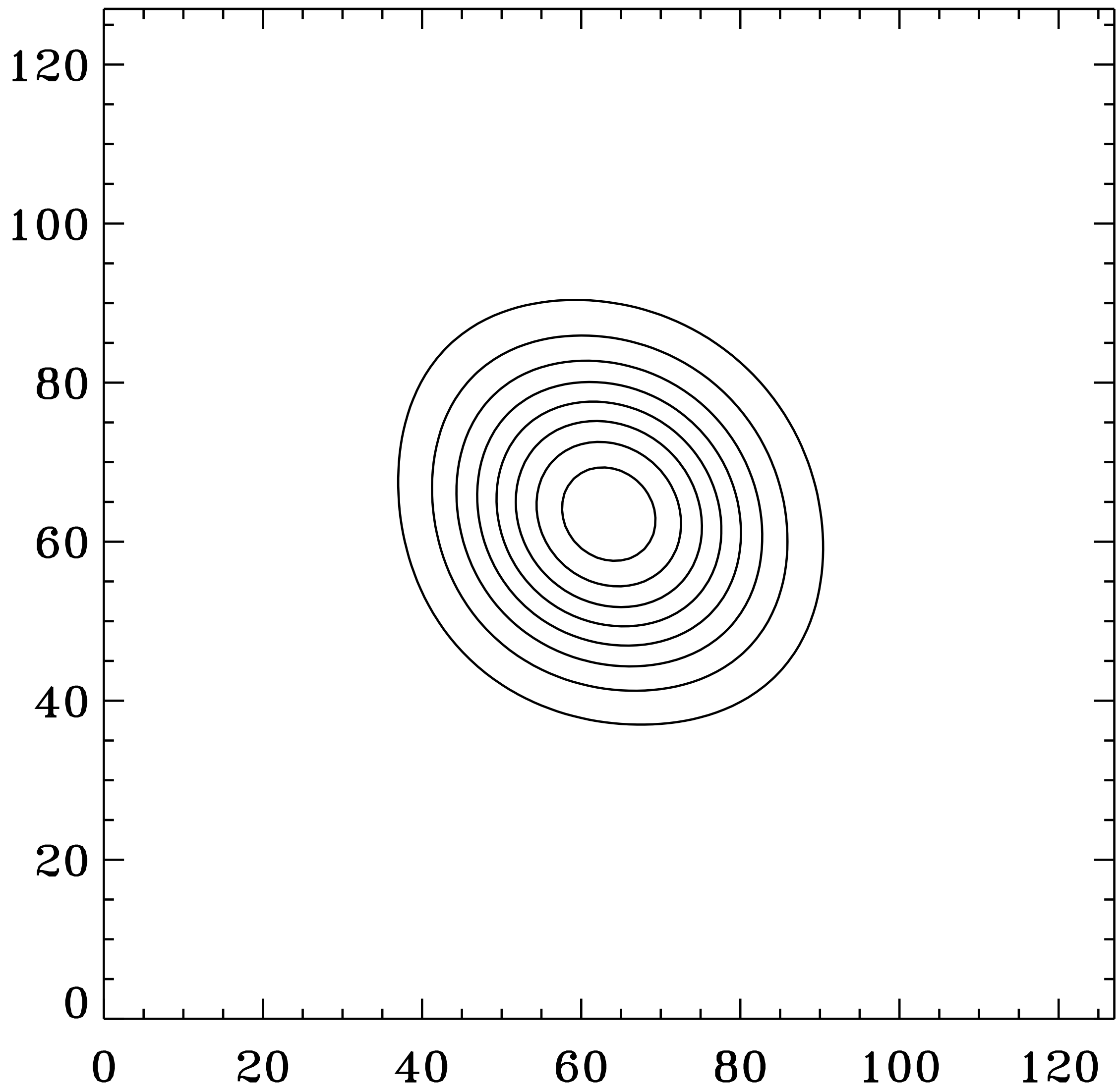
$$0 < C_y < 1$$

$$0 < C_x < 1$$

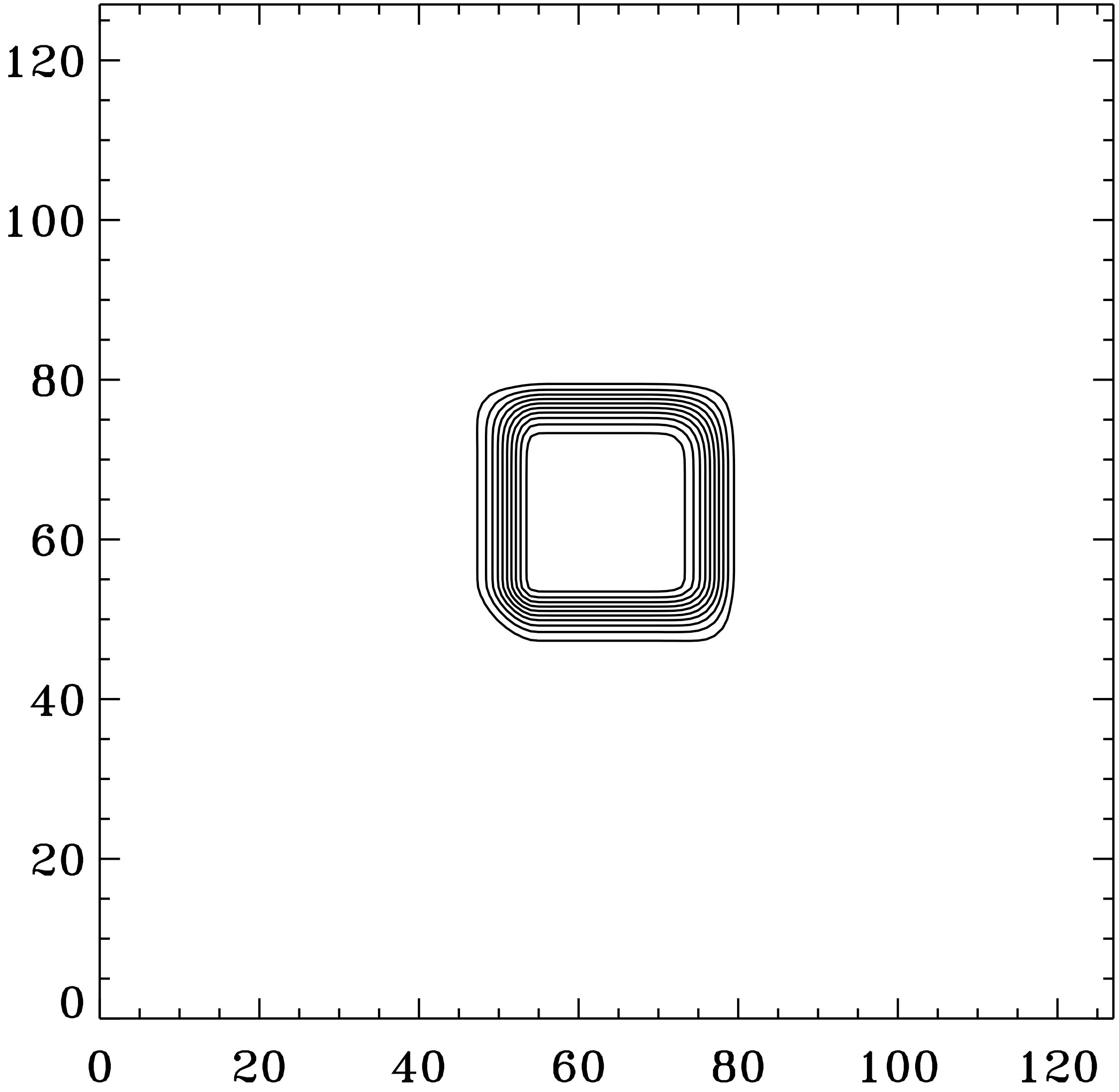
Advection of a square with Godunov scheme



Advection of a square with Godunov scheme



Second-order Godunov scheme



Conclusion

- Upwind scheme for stability
- Modified equation analysis and numerical diffusion
- Godunov scheme: self-similarity of the Riemann solution
- Riemann solver: wave-by-wave upwinding
- Multiple dimensions: predictor-corrector scheme
- Need for higher-order schemes

Next lecture: Hydrodynamics 4