Computational Astrophysics 3
Hyperbolic Systems of Conservation Laws

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Outline

- The Euler equations
- Systems of conservation laws
- The advection equation
- Linear systems and hyperbolic systems
- The Bürger's equation
- Riemann invariants
- Shock relations
- The Riemann problem

The Euler equations in conservative form

A system of 3 conservation laws

$$\partial_t \rho + \nabla \cdot \mathbf{m} = 0$$

$$\partial_t \mathbf{m} + \nabla \cdot (\rho \mathbf{u} \times \mathbf{u}) + \partial_x P = 0$$

$$\partial_t E + \nabla \cdot \mathbf{u}(E + P) = 0$$

The vector of ***conservative variables*** $(\rho, \mathbf{m}, E)$

The Euler equations in primitive form

A non-linear system of PDE (quasi-linear form)

$$\partial_t \rho + u \partial_x \rho + \rho \partial_x u = 0$$

$$\partial_t u + u \partial_x u + \frac{1}{\rho} \partial_x P = 0$$

$$\partial_t P + u \partial_x P + \gamma P \partial_x u = 0$$

The vector of ***primitive variables*** $(\rho, \mathbf{u}, P)$

We restrict our analysis to perfect gases $P = (\gamma - 1)\rho\epsilon$

The isothermal Euler equations

Conservative form with conservative variables $\mathbf{U} = (\rho, m)$

$$\partial_t \rho + \partial_x m = 0$$

$$\partial_t m + \partial_x \left(\rho u^2 + \rho a^2\right) = 0$$

Primitive form with primitive variables $\mathbf{W} = (\rho, u)$

$$\partial_t \rho + u \partial_x \rho + \rho \partial_x u = 0$$

$$\partial_t u + u \partial_x u + \frac{a^2}{\rho} \partial_x \rho = 0$$

a is the isothermal sound speed

Systems of conservation laws

General system of conservation laws with ***F*** flux vector.

$$\partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = 0$$

Examples:

1- Isothermal Euler equations
$$\mathbf{U} = (\rho, m)$$
$$\mathbf{F} = \left(u\rho, um + \rho a^2\right)$$

2- Euler equation
$$\mathbf{U} = (\rho, m, E)$$
$$\mathbf{F} = (u\rho, um + P, u(E + P))$$

3- Ideal MHD equations
$$\mathbf{U} = \left(\rho, m_x, m_y, m_z, E, B_x, B_y, B_z\right)$$
$$\mathbf{F} = \left(v_x\rho, v_x m_x + P_{tot} - B_x^2, v_x m_y - B_x B_y, v_x m_z - B_x B_z, \right.$$
$$\left. 0, v_x B_y - v_y B_x, v_x B_z - v_z B_x\right)$$

Primitive variables and quasi-linear form

We define the Jacobian of the flux function as: $\mathbf{J}(\mathbf{U}) = \dfrac{\partial \mathbf{F}}{\partial \mathbf{U}}$

The system writes in the quasi-linear (non-conservative) form

$$\partial_t \mathbf{U} + \mathbf{J}\partial_x \mathbf{U} = 0$$

We define the primitive variables $\mathbf{W}(\mathbf{U})$

and the Jacobian of the transformation $\mathbf{P} = \dfrac{\partial \mathbf{W}}{\partial \mathbf{U}}$

The system writes in the primitive (non-conservative) form

$$\partial_t \mathbf{W} + \mathbf{A}\partial_x \mathbf{W} = 0$$

The matrix A is obtained by $\mathbf{A} = \mathbf{P}\mathbf{J}\mathbf{P}^{-1}$

The system is *hyperbolic* if ***A*** or ***J*** have positive eigenvalues.

The advection equation

Scalar (one variable) linear (u=constant) partial differential equation (PDE)

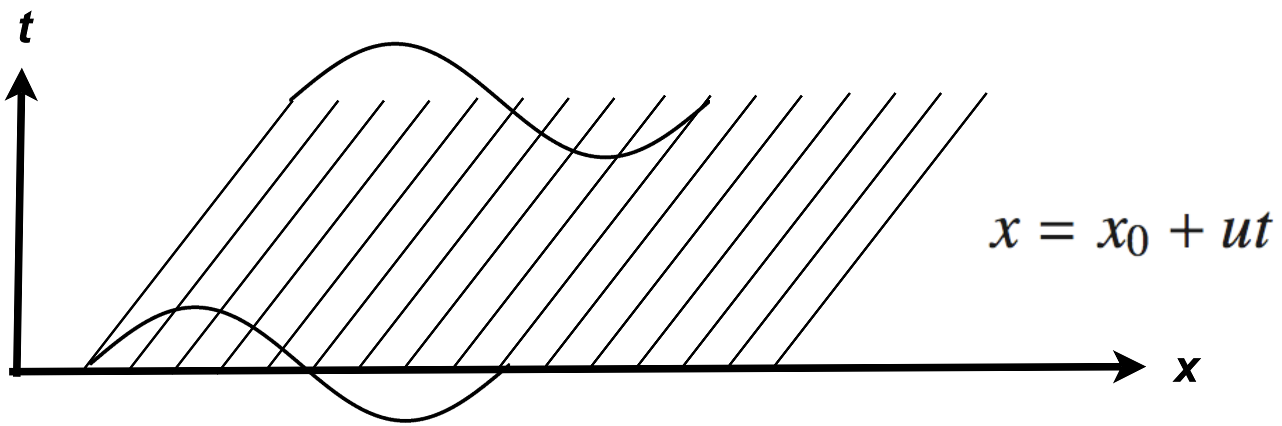
$$\partial_t \rho + u \partial_x \rho = 0$$

Initial conditions: $\rho(x, t = 0) = \rho_0(x)$

Define the function: $\mathcal{I}(t) = \rho(x_0 + ut, t)$

Using the *chain rule*, we have: $\partial_t \mathcal{I} = u \partial_x \rho + \partial_t \rho = 0$

ρ is a ***Riemann Invariant*** along the ***characteristic curves*** defined by u



The isothermal wave equation

We linearize the isothermal Euler equation around some equilibrium state.

$$\mathbf{W} = \mathbf{W_0} + \Delta\mathbf{W}$$

Using the system in primitive form, we get the ***linear*** system:

$$\partial_t \Delta\mathbf{W} + \mathbf{A_0}\partial_x \Delta\mathbf{W} = 0$$

where the constant matrix has 2 real eigenvalues and 2 eigenvectors

$$\mathbf{A_0} = \begin{Bmatrix} u & \rho \\ \dfrac{a^2}{\rho} & u \end{Bmatrix} \qquad \lambda^+ = u + a \qquad \lambda^- = u - a \qquad \Delta\alpha^+ = \frac{1}{2}\left(\Delta\rho + \rho\frac{\Delta u}{a}\right) \qquad \Delta\alpha^- = \frac{1}{2}\left(\Delta\rho - \rho\frac{\Delta u}{a}\right)$$

The previous system is equivalent to 2 independent *scalar linear* PDEs.

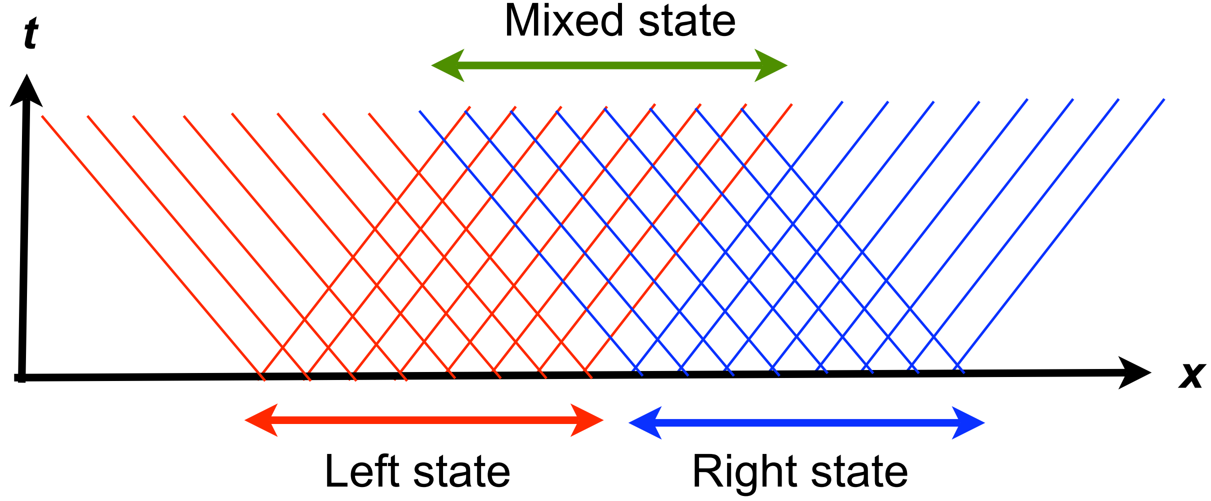
$$\partial_t \Delta\alpha^+ + (u+a)\partial_x \Delta\alpha^+ = 0$$

$$\partial_t \Delta\alpha^- + (u-a)\partial_x \Delta\alpha^- = 0$$

$\Delta\alpha^+$ ($\Delta\alpha^-$) is a Riemann invariant along characteristic curves moving with velocity $u+a$ $(u-a)$

Riemann problem for isothermal waves

Initial conditions are defined by 2 semi-infinite regions with pieceweise constant initial states $(\Delta\rho_R, \Delta u_R)$ and $(\Delta\rho_L, \Delta u_L)$



"Star" state is obtained using the 2 Riemann invariants.

$$u - a < \frac{x}{t} < u + a$$

$$\Delta\rho^* = \Delta\alpha_L^+ + \Delta\alpha_R^-$$

$$\Delta u^* = \frac{a}{\rho}(\Delta\alpha_L^+ - \Delta\alpha_R^-)$$

The adiabatic wave equation

$$\mathbf{A_0} = \begin{Bmatrix} u & \rho & 0 \\ 0 & u & \frac{1}{\rho} \\ 0 & \gamma P & u \end{Bmatrix}$$

$$\lambda^+ = u + a$$
$$\lambda^0 = u$$
$$\lambda^- = u - a$$

$$\Delta\alpha^+ = \frac{1}{2}\left(\frac{\Delta P}{a^2} + \rho\frac{\Delta u}{a}\right)$$
$$\Delta\alpha^0 = \Delta\rho - \frac{\Delta P}{a^2}$$
$$\Delta\alpha^- = \frac{1}{2}\left(\frac{\Delta P}{a^2} - \rho\frac{\Delta u}{a}\right)$$

We define the adiabatic sound speed: $a^2 = \gamma\frac{P}{\rho}$

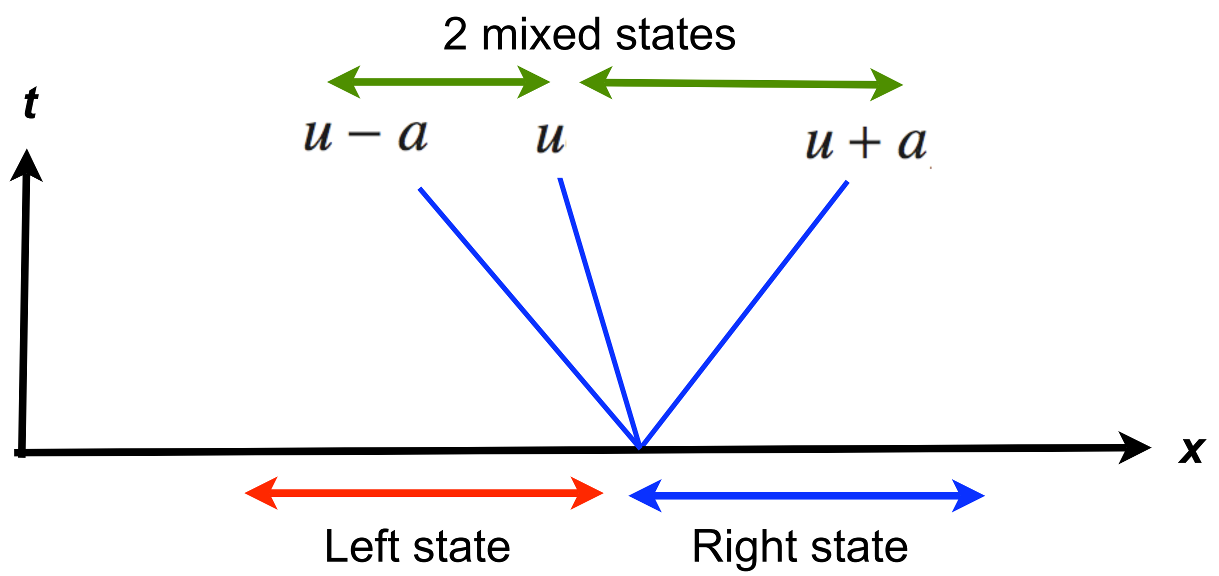
The system is equivalent to the 3 independent scalar PDEs:

$$\partial_t\Delta\alpha^+ + (u+a)\partial_x\Delta\alpha^+ = 0$$
$$\partial_t\Delta\alpha^0 + u\partial_x\Delta\alpha^0 = 0$$
$$\partial_t\Delta\alpha^- + (u-a)\partial_x\Delta\alpha^- = 0$$

$\Delta\alpha^+$, $\Delta\alpha^-$ and $\Delta\alpha^0$ are 3 Riemann invariants along characteristic curves moving with velocity $u+a$, $u-a$ and u.

Riemann problem for adiabatic waves

Initial conditions are defined by 2 semi-infinite regions with pieceweise constant initial states $(\Delta\rho_R, \Delta u_R, \Delta P_R)$ and $(\Delta\rho_L, \Delta u_L, \Delta P_L)$.



Left "star" state: (-,0,+)=(L,R,R) and right "star" state: (-,0,+)=(L,L,R).

$$\Delta u^*_{L,R} = \frac{a}{\rho}(\Delta\alpha^+_L - \Delta\alpha^-_R) \qquad \Delta\rho^*_R = \Delta\alpha^+_L + \Delta\alpha^0_R + \Delta\alpha^-_R$$

$$\Delta P^*_{L,R} = \frac{a}{\rho}(\Delta\alpha^+_L + \Delta\alpha^-_R) \qquad \Delta\rho^*_L = \Delta\alpha^+_L + \Delta\alpha^0_L + \Delta\alpha^-_R$$

The Bürger's equation

Scalar non-linear PDE $\partial_t u + u\partial_x u = 0$ with initial data $u(x, t = 0) = u_0(x)$

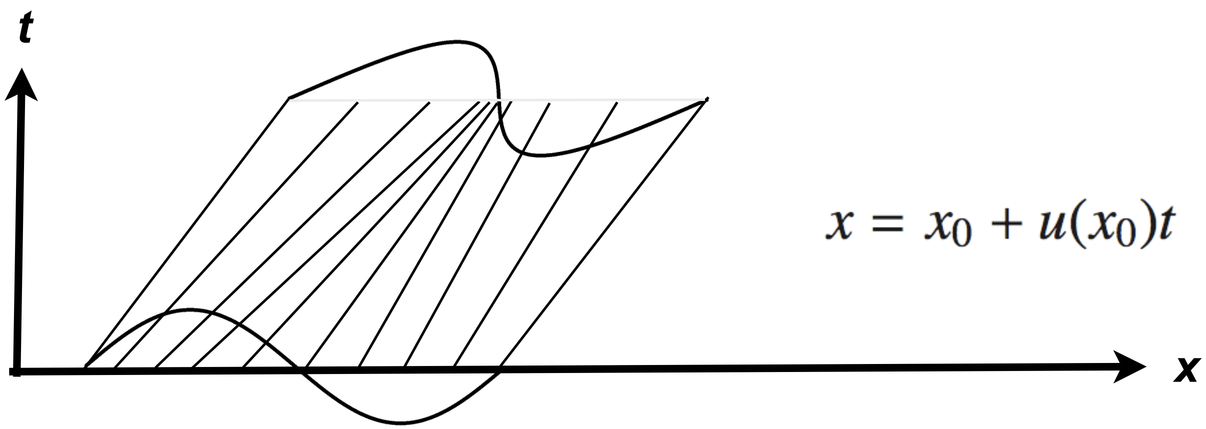
Isothermal Euler equation with vanishing sound speed (*a*=0)

Bürger's equation in conservative form $\partial_t u + \partial_x \frac{u^2}{2} = 0$

Characteristic curve $x(t)$ defined by $x'(t) = u(x(t), t)$

Defining $\mathcal{I}(t) = u(x(t), t)$, we have $\mathcal{I}'(t) = x'(t)\partial_x u + \partial_t u = 0$

$\mathcal{I}(t) = u_0(x)$ is a ***Riemann Invariant*** along ***characteristic lines.***



$$x = x_0 + u(x_0)t$$

Shock formation

Implicit solution: $u(x,t) = u_0(x - u(x,t)t)$

$\partial_t u = u_0'(x_0)\,(-u(x,t) - t\partial_t u)$ gives $\partial_t u = \dfrac{-u(x,t)u_0'(x_0)}{1 + tu_0'}$

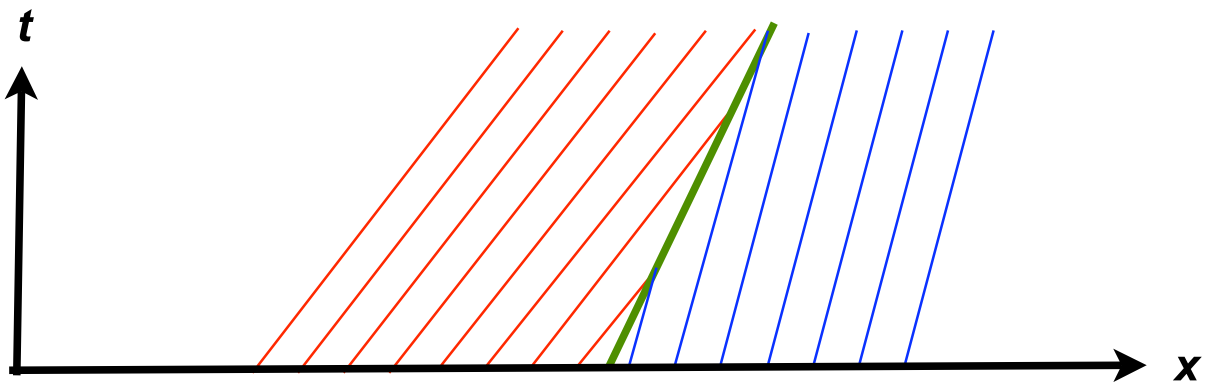
Solution diverges at finite time $T = -\dfrac{1}{\min u_0'}$

Discontinuities appear in the flow; uniqueness of the solution is lost.

Search for *weak solutions* of the flow with an *entropy condition*.

Riemann problem for Bürger's equation

Initial conditions are defined by 2 semi-infinite regions with pieceweise constant initial states u_R and u_L.

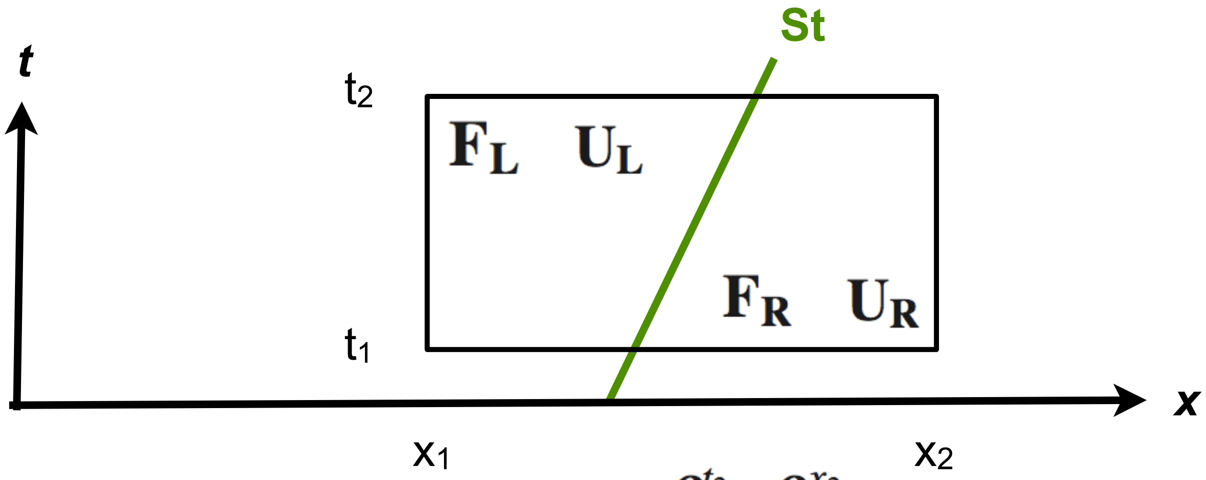


Case 1: $u_L > u_R$

Formation of a shock with velocity $S = \frac{u_L + u_R}{2}$

Solution: If $x < S\,t$ then $u(x,t) = u_L$ else $u(x,t) = u_R$

Shock speed and the Rankine-Hugoniot relation



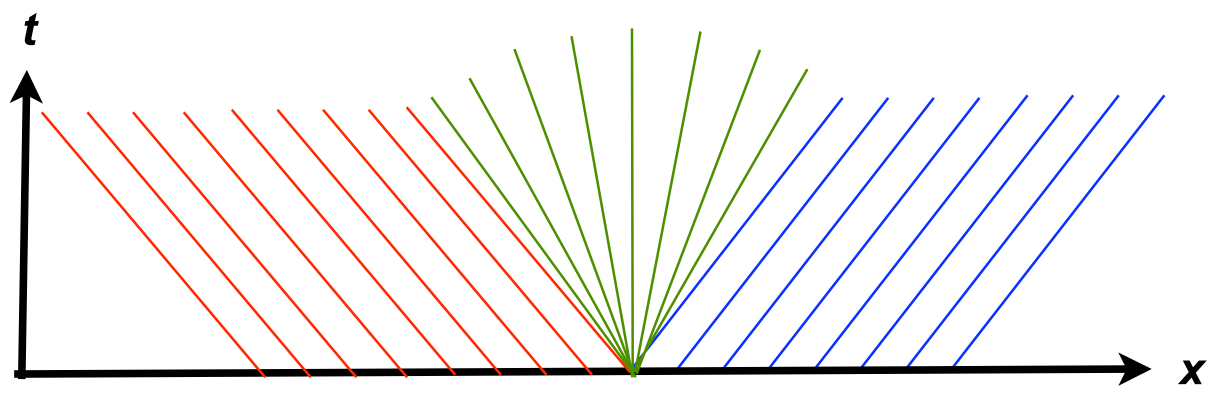
Integral form of the conservation law $$\int_{t_1}^{t_2}\int_{x_1}^{x_2}\left(\partial_t \mathbf{U} + \partial_x \mathbf{F}\right) dx dt = 0$$

$$\int_{x_1}^{x_2} \mathbf{U}(t_2)dx - \int_{x_1}^{x_2} \mathbf{U}(t_1)dx + \int_{t_1}^{t_2} \mathbf{F}(x_2)dt - \int_{t_1}^{t_2} \mathbf{F}(x_1)dt = 0$$

Shock relation: $\mathbf{F_R} - \mathbf{F_L} = S\ (\mathbf{U_R} - \mathbf{U_L})$

Bürger's equation: $\dfrac{u_R^2}{2} - \dfrac{u_L^2}{2} = S\ (u_R - u_L)$ gives $S = \dfrac{u_R + u_L}{2}$

Rarefaction wave



Case 2: $u_L < u_R$

Characteristics are diverging: a rarefaction wave fills the gap.

Solution: If $x < u_L t$ then $u(x,t) = u_L$

If $u_L t < x < u_R t$ then $u(x,t) = \dfrac{x}{t}$

If $x > u_R t$ then $u(x,t) = u_R$

Another solution: a rarefaction shock ?

Vanishing viscosity solution

We know from kinetic theory that the Euler equations are derived under the LTE approximation: viscosity and conductivity are first-order non-LTE effects.

Goal: solve Bürger's equation with viscosity source term.

The entropy solution is the solution with $\nu \rightarrow 0$

$$\partial_t u + u \partial_x u = \nu \partial_x^2 u$$

Hopf-Cole transform: $u = -2\nu \dfrac{\partial_x \Phi}{\Phi}$ we get $\partial_t \Phi = \nu \partial_x^2 \Phi$

Solution of the heat transfer equation $\Phi(x,t) = \displaystyle\int_{-\infty}^{+\infty} \Phi_0(y) \exp\left[-\frac{(x-y)^2}{4\nu t}\right] dy$

with initial condition $\Phi_0(y) = \exp\left[-\dfrac{1}{2\nu}\displaystyle\int_0^y u_0(z) dz\right]$

We finally get the viscosity solution $u(x,t) = \dfrac{\int_{-\infty}^{+\infty} \frac{x-y}{t} \Phi_0(y) \exp\left[-\frac{(x-y)^2}{4\nu t}\right] dy}{\int_{-\infty}^{+\infty} \Phi_0(y) \exp\left[-\frac{(x-y)^2}{4\nu t}\right] dy}$

Riemann problem for vanishing viscosity

Initial conditions are defined by 2 semi-infinite regions with pieceweise constant initial states u_R and u_L.

For y<0: $\Phi_0(y) = \exp\left[-\frac{1}{2\nu}u_L y\right]$ and for y>0: $\Phi_0(y) = \exp\left[-\frac{1}{2\nu}u_R y\right]$

When $\nu \to 0$, the Gaussian converges towards a delta-function and the viscosity solution converges towards $\frac{x - y_{max}}{t}$

where y_{max} is the position of the minimum of the function defined by:

For y<0 $\frac{(y-x)^2}{2t} + u_L y$ and for y>0 $\frac{(y-x)^2}{2t} + u_R y$

Exercise: show that the vanishing-viscosity solution is unique, and is either a compression shock or a rarefaction wave.

Riemann invariants for propagating waves

Define the 3 differential forms:

$$\mathrm{d}\mathcal{I}^+ = \frac{1}{2}\left(\frac{\mathrm{d}P}{a^2} + \rho\frac{\mathrm{d}u}{a}\right) \quad \mathrm{d}\mathcal{I}^- = \frac{1}{2}\left(\frac{\mathrm{d}P}{a^2} - \rho\frac{\mathrm{d}u}{a}\right) \quad \mathrm{d}\mathcal{I}^0 = \mathrm{d}\rho - \frac{\mathrm{d}P}{a^2}$$

These are Riemann invariants along the characteristic curves (u+a, u-a, u)

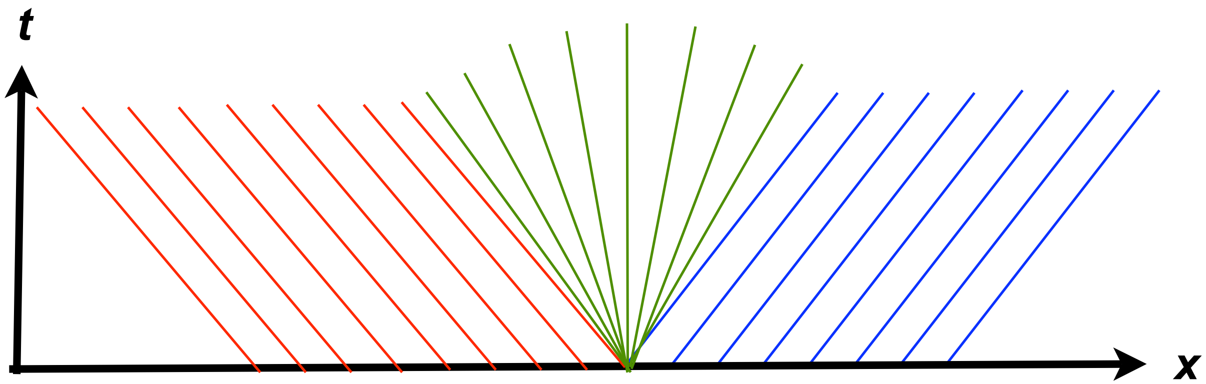
Exercise: using $\mathrm{d}P = \partial_t P + (u+a)\partial_x P$ and the Euler system in primitive form, show that the previous forms are invariants along their characteristic curve.

Right-going waves satisfy $\quad \mathrm{d}\mathcal{I}^- = \mathrm{d}\mathcal{I}^0 = 0$

Left-going waves satisfy $\quad \mathrm{d}\mathcal{I}^+ = \mathrm{d}\mathcal{I}^0 = 0$

Entropy waves satisfy $\quad \mathrm{d}\mathcal{I}^+ = \mathrm{d}\mathcal{I}^- = 0$

Left-going rarefaction wave



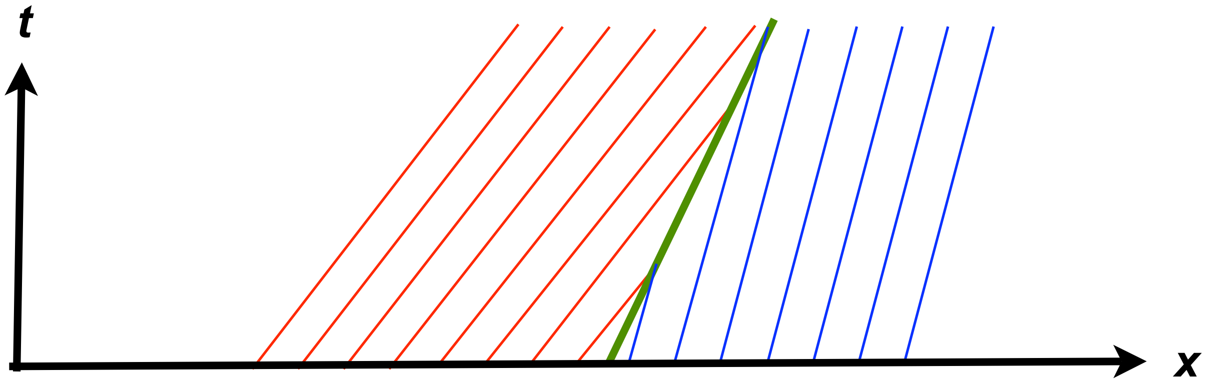
The entropy is conserved across the fan

$$P(x,t) = P_L\left(\frac{\rho}{\rho_L}\right)^{\gamma} \quad a(x,t) = a_L\left(\frac{\rho}{\rho_L}\right)^{\frac{\gamma-1}{2}}$$

$\mathrm{d}\mathcal{I}^+ =$ 0 across the fan, which gives $u + \dfrac{2a}{\gamma - 1} = \text{constant}$

Writing $x = (u-a)t$ we get $u(x,t) = \dfrac{2}{\gamma+1}\left(\dfrac{x}{t} + \dfrac{\gamma-1}{2}u_L + a_L\right)$

Right-going shock wave



Because we have a discontinuity, Riemann invariants are not valid anymore: we use Rankine-Hugoniot shock relations

$$\rho_L u_L - \rho_R u_R = S(\rho_L - \rho_R)$$

$$\rho_L u_L^2 + P_L - \rho_R u_R^2 - P_R = S(\rho_L u_L - \rho_R u_R)$$

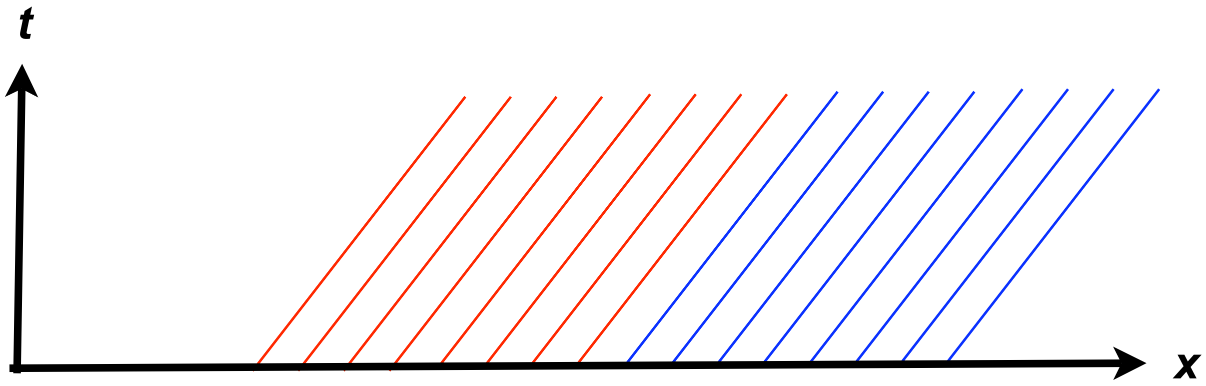
$$(E_L + P_L)u_L - (E_R + P_R)u_R = S(E_L - E_R)$$

One parameter (shock speed) family, fully specified by the right-state.

Exercise: show that for a stationary shock, we get $m = \rho_R u_R$ = constant

and $m(u_L - u_R) + (P_L - P_R) = 0$

Contact discontinuity



Both Riemann invariants and Rankine-Hugoniot relations gives:

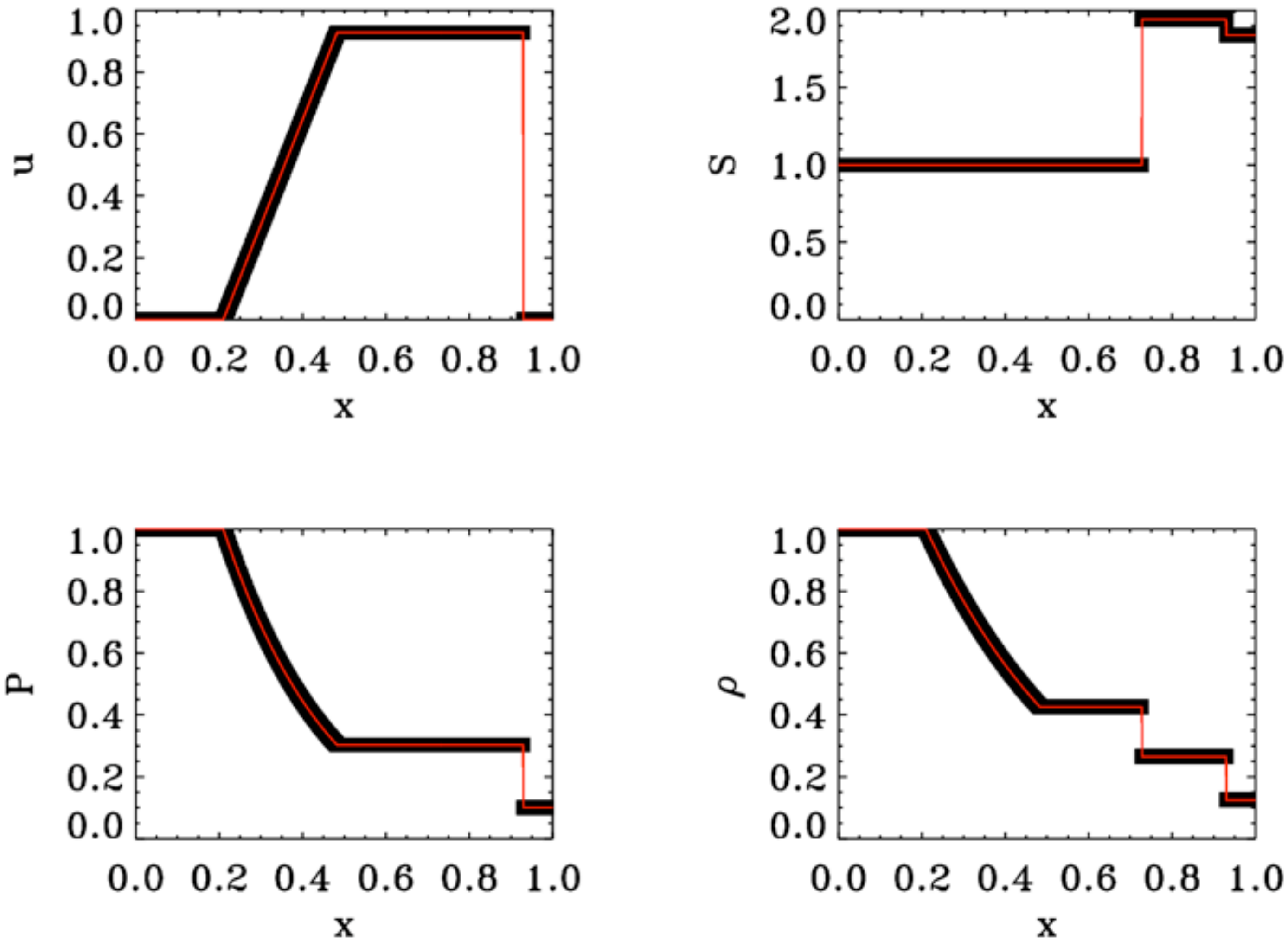
$$P_L = P_R \quad u_L = u_R \quad \text{but} \quad \rho_L \neq \rho_R$$

Characteristic are moving parallel to each other.

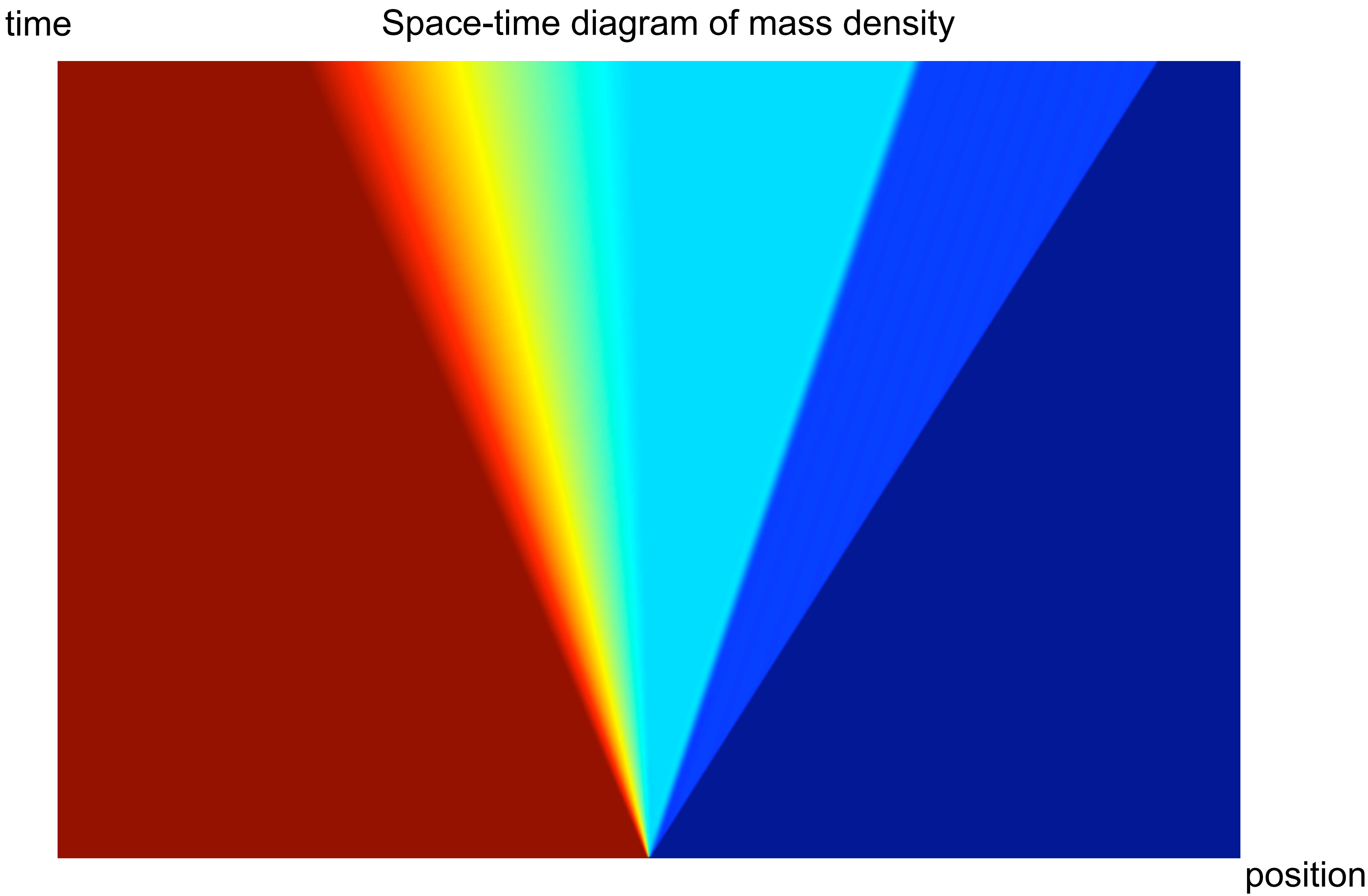
The Sod shock tube

Analytical solution: we match the pressure and the velocity at the tip of the rarefaction wave with the pressure and velocity after the shock.

$$(P_*, u_*)$$



The Sod shock tube



Conclusion

- Hyperbolic systems of conservation laws
- Propagation of waves and formation of shocks
- Vanishing-viscosity solution and weak solutions
- Exact solutions to various Riemann problems

Next lecture: Hydrodynamics 3

Exact solution to the Riemann problem are used to design numerical schemes.

Fundamental property: self-similarity with respect to variable x/t

Toro, E.F., “Riemann Solvers and Numerical Methods for Fluid Dynamics: A Practical Introduction ”, 2nd Edition, Springer