Aperture Photometry in Band Limited Data

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1 Band Limited Data

Band limited data, that is, data that has non-zero Fourier components in only a restricted part of k-space $|k| < \Delta k$, has some remarkable properties when properly sampled. In particular, if the signal is sampled at the integers i, and points are at least as closely spaced as $\pi/\Delta k$ (the *Nyquist rate*), the exact value of the signal f at some general point ξ is given by

$$f(\xi) = \sum_{i=-\infty}^{\infty} f(i) \frac{\sin((i-\xi)\pi)}{(i-\xi)\pi}. \tag{1}$$

This result is referred to as *sinc interpolation*. If ξ is an integer, j say, this correctly reduces to $f(\xi) = f(j)$.

Note that setting $f(x) = 1$, this reduces to

$$\sum_{i=-\infty}^{\infty} \frac{\sin((i-\xi)\pi)}{(i-\xi)\pi} = 1$$

so the sinc interpolation coefficients are properly normalised.

The generalisation to higher dimensions is immediate.

2 Photometry

If our data is band limited, we can use equation 1 to shift the image so that its centre falls at some desired point, say $(0, 0)$; after this operation let us write the intensity at pixel i, j as $f_{ij} \equiv f(i, j)$.

10	0.004	0.003	-0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000
9	-0.007	-0.004	0.004	0.000	-0.001	0.000	0.000	-0.001	0.000	0.000
8	0.012	0.007	-0.007	0.000	0.001	-0.001	0.001	0.000	-0.001	0.000
7	-0.019	-0.011	0.013	0.001	-0.001	0.000	-0.001	0.001	0.000	-0.001
6	0.032	0.018	-0.020	-0.001	0.002	-0.001	0.000	-0.001	0.001	0.000
5	-0.063	-0.035	0.032	0.001	-0.004	0.002	-0.001	0.000	-0.001	0.000
4	0.260	0.142	-0.060	-0.005	0.005	-0.004	0.002	-0.001	0.001	-0.001
3	1.042	1.011	0.637	0.017	-0.005	0.001	-0.001	0.001	0.000	0.000
2	1.009	0.947	1.146	0.637	-0.060	0.032	-0.020	0.013	-0.007	0.004
1	1.019	1.016	0.947	1.011	0.142	-0.035	0.018	-0.011	0.007	-0.004
0	0.956	1.019	1.009	1.042	0.260	-0.063	0.032	-0.019	0.012	-0.007
	0	1	2	3	4	5	6	7	8	9

Table 1: $R = 3.75$, $\sum C_{ij}/(\pi R^2) = 1.0002$.

To reiterate, this process is exact. We can then evaluate the total flux F within an aperture of radius R as

$$
\begin{aligned}
F &= \int_{x^2+y^2<R^2} f(x,y)dxdy && (2)\\
&= \int_{x^2+y^2<R^2} \sum_{ij} f_{ij} \frac{\sin((i-x)\pi)}{(i-x)\pi} \times \frac{\sin((j-y)\pi)}{(j-y)\pi} dxdy \\
&= \sum_{ij} f_{ij} \int_{x^2+y^2<R^2} \frac{\sin((i-x)\pi)\sin((j-y)\pi)}{(i-x)\pi(j-y)\pi} dxdy \\
&\equiv \sum_{ij} f_{ij} C_{ij} && (3)
\end{aligned}
$$

where

$$
C_{ij} \equiv \int_{x^2+y^2<R^2} \frac{\sin((i-x)\pi)\sin((j-y)\pi)}{(i-x)\pi(j-y)\pi} dxdy. \qquad (4)
$$

The integral defining C_{ij} cannot be evaluated analytically, but it is independent of the image being measured, so it can be evaluated numerically and then tabulated. Tables 1 and 2 give the values of C_{ij} in the first quadrant for $R = 3.75$ (the SDSS 3” aperture) and $R = 7$.

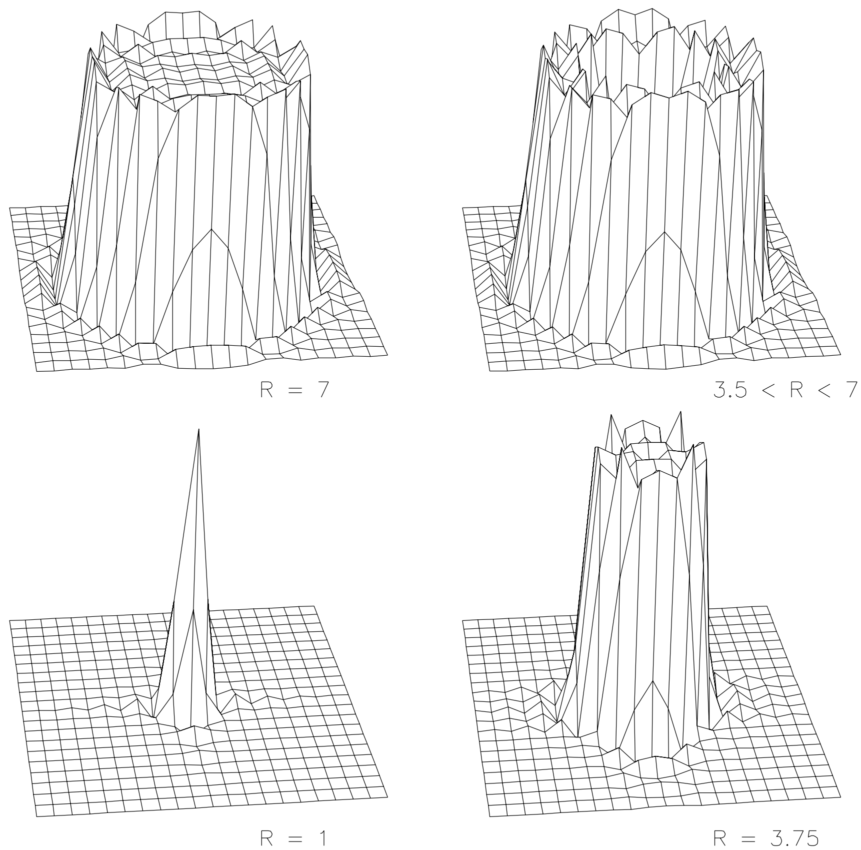

Figure 1: The coefficients for exact integration over a set of four circular apertures, with radii $R = 1$, $R = 3.75$ (see Table 1), $R = 7$ (see Table 2), and an annulus $3.5 < R < 7$; all radii are in pixels.

	0	1	2	3	4	5	6	7	8	9	10
9	0.04	0.04	0.03	-0.02	-0.01	0.01	0.00	0.00	0.00	0.00	0.00
8	-0.09	-0.08	-0.06	0.02	0.02	-0.01	0.01	0.00	0.00	0.00	0.00
7	0.49	0.42	0.22	-0.03	-0.05	0.02	-0.01	0.00	0.00	0.00	0.00
6	1.09	1.10	1.03	0.81	0.23	-0.06	0.02	-0.01	0.01	0.00	0.00
5	0.96	0.93	0.99	1.03	1.08	0.37	-0.06	0.02	-0.01	0.01	-0.01
4	1.03	1.05	1.01	0.98	0.96	1.08	0.23	-0.05	0.02	-0.01	0.01
3	0.94	0.99	0.95	1.06	0.98	1.03	0.81	-0.03	0.02	-0.02	0.01
2	1.05	1.01	1.04	0.95	1.01	0.99	1.03	0.22	-0.06	0.03	-0.02
1	0.99	0.95	1.01	0.99	1.05	0.93	1.10	0.42	-0.08	0.04	-0.02
0	1.06	0.99	1.05	0.94	1.03	0.96	1.09	0.50	-0.09	0.04	-0.03

Table 2: $R = 7.0$, $\sum C_{ij}/(\pi R^2) = 1.0012$.

3 Is SDSS Data Band Limited?

Let us take the PSF to be a Gaussian:

$$e^{-r^2/(2\alpha^2)};$$

as usual its Fourier Transform is

$$e^{-k^2\alpha^2/2},$$

and we may take the sample points to be at the integers. If we assume that the data is Nyquist sampled for some bandlimit Δk, we must have $\Delta k = \pi$; the PSF thus has an amplitude of

$$e^{-\Delta k^2\alpha^2/2}$$

at the band limit, i.e.

$$= e^{-\pi^2\alpha^2/2}. \quad (5)$$

Our pixels are 0.4" , so for 1" FWHM seeing , $\alpha = 2.5/(2\sqrt{(2\ln 2)}) = 1.06$, so the amplitude at the band limit is only 0.4%; for a FWHM of 0.8" (the best images that the optics can deliver, for which the PSF is admittedly non-Gaussian), the band limit amplitude is still only 2.8%; if the seeing is worse than an arcsecond, the situation only improves.

One might worry about noise; after all, noise is uncorrelated from pixel to pixel, and thus can have features that are sharper than the PSF — i.e. it breaks the band limit. Remarkably, if the noise per pixel is constant (as is the case for objects with surface brightness appreciably lower than the sky), the noise in the sinc-interpolated image has the same variance as in the input image, and the noise is uncorrelated from pixel to pixel; for Gaussian noise, this implies that the noise in each pixel is independent (see appendix 7).

4 Application to Pixellated Data

The above discussion refers to point sampling of a signal, whereas in fact CCDs have pixels. Measuring the intensity within a pixel corresponds to convolving the incoming data with a top-hat function the width of a pixel.

This implies that the function $f(x)$ is not the true intensity distribution in the focal plane, but rather that distribution convolved with the size of the pixels; when we sinc-shifted the data to be centred on a pixel, it was this pixel-smoothed image that we shifted.

In the Fourier domain, this integration over a pixel is equivalent to multiplying by a sinc function, namely

$$\frac{\sin(k/2)}{k/2}.$$

If the pixels critically sample the data, at the band limit this has the value $2/\pi = 0.637$; note that there are no zeros within the band limit.

When we perform the integration of equation 2, we are interested in the true, not pixel-smoothed, value of the intensity. If we revisit the derivation of the sinc interpolation formula in appendix 6, it is clear that we can correct for this by using instead of the regular sinc formula 1, the modified result

$$f(\xi) = \sum_{i=-\infty}^{\infty} f(i) D_i(\xi)$$

where

$$\begin{aligned} D_j(\xi) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{k/2}{\sin(k/2)} e^{ik(j-\xi)} dk \\ &= \frac{1}{\pi} \int_{0}^{\pi} \frac{k/2}{\sin(k/2)} \cos(k(j-\xi)) dk \end{aligned} \tag{6}$$

10	0.006	0.004	-0.003	0.000	0.000	0.000	-0.001	0.000	0.000	0.000
9	-0.010	-0.006	0.007	0.000	0.000	0.000	0.000	0.000	0.001	0.000
8	0.017	0.009	-0.013	0.002	0.001	0.000	0.000	0.000	0.000	0.001
7	-0.026	-0.014	0.021	-0.004	-0.002	0.001	-0.001	0.000	0.000	0.000
6	0.044	0.024	-0.034	0.003	0.003	-0.002	0.001	0.000	0.000	0.000
5	-0.081	-0.042	0.057	-0.004	-0.002	0.003	-0.002	0.001	0.000	0.000
4	0.242	0.099	-0.124	0.009	0.003	-0.002	0.003	-0.002	0.001	0.000
3	1.087	1.095	0.664	-0.058	0.009	-0.004	0.003	-0.004	0.002	0.000
2	1.010	0.898	1.227	0.664	-0.124	0.057	-0.034	0.021	-0.013	0.007
1	1.018	1.039	0.898	1.095	0.100	-0.042	0.024	-0.014	0.009	-0.006
0	0.931	1.019	1.010	1.089	0.242	-0.083	0.043	-0.026	0.017	-0.010
	0	1	2	3	4	5	6	7	8	9

Table 3: $R = 3.75$, $\sum C_{ij}/(\pi R^2) = 1.0002$. Interpolation formula with pixel correction.

Unfortunately, this integral does not seem to be analytically tractable. To make progress let us expand $(k/2)/\sin(k/2)$ in a Taylor series:

$$D_j(\xi) \approx \frac{1}{\pi}\int_0^\pi \left(1+\frac{k^2}{24}\right)\cos(k(j-\xi))dk.$$

Writing $\lambda \equiv \pi(j-\xi)$ we have

$$\begin{aligned} &= \frac{\sin(\lambda)}{\lambda} + \frac{\pi^2}{24\lambda^3}\Big((\lambda^2-2)\sin(\lambda) + 2\lambda\cos(\lambda)\Big) \qquad (7)\\ &\sim 1 + \frac{\pi^2}{72} - \frac{1}{6}\left(1+\frac{\pi^2}{40}\right)\lambda^2 + O(\lambda^4). \end{aligned}$$

We can use this result to calculate coefficients for aperture magnitudes corrected for pixellation; such tables are given, for $R = 3.75$ and $R = 7$, in tables 3 and 4. [1]

[1] Note that the numerical scheme used to calculate the tables is not exact; in particular the quoted coefficients are not symmetrical about the line $x = y$. They are quoted for illustration only.

10	-0.04	-0.03	-0.03	0.02	0.01	-0.01	0.01	0.00	0.00	0.00	0.00
9	0.07	0.05	0.04	-0.03	-0.02	0.02	-0.01	0.00	0.00	0.00	0.00
8	-0.13	-0.11	-0.07	0.05	0.03	-0.03	0.01	-0.01	0.00	0.00	0.00
7	0.49	0.41	0.19	-0.11	-0.06	0.04	-0.02	0.01	-0.01	0.00	0.00
6	1.13	1.16	1.09	0.88	0.17	-0.10	0.04	-0.02	0.01	-0.01	0.01
5	0.95	0.89	0.97	1.03	1.20	0.33	-0.10	0.04	-0.03	0.02	-0.01
4	1.04	1.08	1.02	0.97	0.91	1.20	0.17	-0.06	0.03	-0.02	0.01
3	0.91	0.99	0.92	1.09	0.98	1.03	0.88	-0.11	0.05	-0.03	0.02
2	1.08	1.00	1.07	0.92	1.02	0.98	1.09	0.19	-0.07	0.04	-0.03
1	0.99	0.93	1.00	0.99	1.08	0.89	1.16	0.41	-0.11	0.05	-0.03
0	1.09	0.99	1.08	0.91	1.04	0.95	1.12	0.51	-0.13	0.07	-0.04
	0	1	2	3	4	5	6	7	8	9	10

Table 4: $R = 7.0$, $\sum C_{ij}/(\pi R^2) = 0.9987$. Interpolation formula with pixel correction.

5 Correction of Derived PSF quantities for Pixellation

As an alternative to the correction procedure of the previous section, which is after all a deconvolution with all its attendent worries, we can decide to live with the pixellation, and to measure the properties of the seeing-and-pixel convolved image.

One question that immediately arises is, "what is the effect of the pixellation on the PSF?". If we take the PSF to be a sum of Gaussians, we can readily provide an answer.

The pixellation multiplies the PSF's Fourier transform by

$$\frac{\sin(k/2)}{k/2}.$$

so for Gaussian seeing, the resulting image that we measure has a PSF given by the inverse Fourier transform of

$$\frac{\sin(k/2)}{k/2} exp(-k^2\alpha^2/2).$$

Expanding the first term as before, this integral may be evaluated:

$$P \equiv \exp\left(-r^2/(2\alpha^2)\right)\left(1+\frac{1}{24\alpha^2}\left(x^2/\alpha^2-1\right)\right)\left(1+\frac{1}{24\alpha^2}\left(y^2/\alpha^2-1\right)\right).$$

α	f	f_n	f/f_n
0.25	2.383	1.528	1.560
0.50	1.204	1.155	1.042
0.75	1.081	1.072	1.009
1.00	1.044	1.041	1.003
1.25	1.028	1.026	1.001
1.50	1.019	1.018	1.001
1.75	1.014	1.014	1.000
2.00	1.011	1.010	1.000

Table 5: The relationship between a pixel-convolved Gaussian and the original Gaussian. Column one gives the with parameter of the original Gaussian; column two the width of the Gaussian which best fits the pixel-convolved original; column three gives the width that one would arrive at by adding in quadrature the pixel's variance to the original Gaussian; and column four is the ratio of columns two and three.

Because the value of $\sin(k/2)/(k/2)$ at $k = 0$ is unity, the total flux in this PSF is the same as in the original Gaussian, namely $2\pi\alpha^2$.

This functional form is well represented by a circularly symmetrical Gaussian with a suitably chosen width parameter, $f\alpha$. Table 5 lists the value of f, and also the value implied by naïvely adding the PSF's variance with that of a square pixel: $\alpha'^2 = \alpha^2 + 1/12$ (note that we add $1/12$ rather than the $1/6$ you might expect for a 2-dimensional pixel, as the value α is itself a 1-dimensional quantity — $\langle r^2 \rangle = 2\alpha^2$). The final column lists the ratio of the "true" correction factor to the "naïve" one, and may be fit (for $\alpha \geq 0.5$) with the function $1 + 0.00286/\alpha^{3.9}$; the maximum error is about 2×10^{-4}.

6 Appendix: Derivation of Sinc Interpolation Formula

This derivation is extrememly well known (see, for example, Bracewell), but I repeat it here as I need to generalise the result in section 4.

Consider a function $f(x)$ that we have sampled at the points id, that is, instead of measuring the function $f(x)$ itself, we have measured

$$g(x) \equiv \sum_i f(id)\delta(x - id).$$

If we Fourier transform g, we get

$$g(k) = f \otimes \sum_j \delta\left(k - \frac{j}{2\pi d}\right)$$

where $f(k)$ is f's Fourier transform, and $\otimes$ indicated a convolution. If you think for a moment about g, you'll see that it consists of multiple copies of $f(k)$ centred at the points $j/2\pi$; in general these copies will overlap and information about f will be irretrievably lost. If, however, $f(k)$ is only non-zero in the range $|k| < \Delta k$, and the δ functions $\delta(k - j/2\pi d)$ are sufficiently widely spaced, there will be no overlap between the copies of $f(k)$; the condition for this to be true is clearly that

$$\frac{1}{\pi d} \geq \Delta k,$$

i.e. that

$$d \leq \frac{\pi}{\Delta k}. \tag{8}$$

This is the famous Nyquist condition.

If we define $\Theta(k)$ as

$$\Theta(k) = \begin{cases} 1, & \text{if } |k| < \Delta k \\ 0, & \text{otherwise.} \end{cases}$$

then, if we have a properly sampled function, we can recover $f(k)$ as

$$f(k) = g(k) \times \Theta(k)$$

i.e.

$$\begin{aligned} f(x) &= \Theta(x) \otimes g(x) \\ &= \frac{\sin(\pi x)}{\pi x} \otimes g(x) \\ &= \frac{\sin(\pi x)}{\pi x} \otimes \sum_i f(id)\delta(x - id) \\ &= \sum_i f(id) \frac{\sin(\pi(x - id))}{\pi(x - id)} \end{aligned}$$

If we now set $d = 1$ we recover equation 1.

7 Appendix: Proofs of Noise Properties of Sinc-Interpolated Data

If we have function $f(x)$ which is band limited at $\pm\pi$, we know that the values f_i, the values of function $f(x)$ sampled at all integers i, completely specify f; in this case equation 1 reduces to

$$f(\xi) = \sum_{i=-\infty}^{\infty} f_i \frac{\sin((i-\xi)\pi)}{(i-\xi)\pi}.$$

We may calculate the expectation value $\langle f(\xi)f(\xi+k)\rangle$ (where k is some integer) as

$$\sum_{i,j=-\infty}^{\infty} \langle f_i f_j \rangle \frac{\sin((i-\xi)\pi)\sin((j-\xi-k)\pi)}{\pi^2(i-\xi)(j-\xi-k)}$$

Let us consider the case that f is a noise signal, i.e. $\langle f_i f_j\rangle = \sigma_i^2 \delta_{ij}$; this expression then becomes

$$\begin{aligned}
\langle f(\xi)f(\xi+k)\rangle &= \sum_{i=-\infty}^{\infty} \sigma_i^2 \frac{\sin((i-\xi)\pi)\sin((i-\xi-k)\pi)}{\pi^2(i-\xi)(i-\xi-k)} \\
&= (-)^k \frac{1}{\pi^2} \sum_{i=-\infty}^{\infty} \sigma_i^2 \frac{\sin^2((i-\xi)\pi)}{(i-\xi)(i-\xi-k)} \\
&= (-)^k \frac{\sin^2(\pi\xi)}{\pi^2} \sum_{i=-\infty}^{\infty} \frac{\sigma_i^2}{(i-\xi)(i-\xi-k)}.
\end{aligned}$$

If the noise is stationary, $\sigma_i^2 = \sigma^2$, this reduces to

$$\begin{aligned}
&= (-)^k \frac{\sin^2(\pi\xi)\sigma^2}{\pi^2} \sum_{i=-\infty}^{\infty} \frac{1}{(i-\xi)(i-\xi-k)} \\
&\equiv (-)^k \frac{\sin^2(\pi\xi)\sigma^2}{\pi^2} S(k,\xi). \qquad (9)
\end{aligned}$$

Let us consider this sum:

$$\begin{aligned}
S(k,\xi) &= \sum_{i=-\infty}^{\infty} \frac{1}{(i-\xi)(i-\xi-k)} \qquad (10) \\
&= \frac{1}{\xi(\xi+k)} + \frac{1}{k}\sum_{i\neq 0}^{\infty}\left[\frac{1}{\xi-i} - \frac{1}{\xi+k-i}\right]
\end{aligned}$$

(if $k \neq 0$).

$$= \frac{1}{\xi(\xi+k)} + \frac{1}{k}\sum_{i=1}^{\infty}\left[\frac{1}{\xi-i} + \frac{1}{\xi+i} - \frac{1}{\xi+k-i} - \frac{1}{\xi+k+i}\right]$$

Inspection of the terms in this sum reveals that all but two cancel, resulting in the simple expression

$$\begin{aligned} &= \frac{1}{\xi(\xi+k)} + \frac{1}{k}\left[\frac{1}{\xi+k} - \frac{1}{\xi}\right] \\ &= 0. \end{aligned} \tag{11}$$

As claimed, we see that sinc-interpolated values separated by an integral (but non-zero, the $k \neq 0$ condition) number of pixels are uncorrelated.

For $k = 0$,

$$\begin{aligned} \langle f(\xi)^2 \rangle &= \frac{\sigma^2 \sin^2(\pi\xi)}{\pi^2} \sum_{i=-\infty}^{\infty} \frac{1}{(i-\xi)^2} \\ &= \sigma^2 \end{aligned} \tag{12}$$

(This last simplification uses the standard result that $\sum(1/(i-d)^2) = \pi^2/\sin^2(\pi d)$; the disbelieving reader may check it in tables, or by using maple or mathematica).

It is not too surprising that the variance is σ^2. The sinc-interpolation weights sum to one, so the eigenvalues of the covariance matrix are unchanged; as we just showed that the off-diagonal elements vanish, it must be true that the diagonal values, the variances, are unchanged.