We all know by now what the electric potential is at a distance d away from a point charge $q > 0$. But what if, at that distance d away, there were a grounded conductor? How would our answer change? Our point charge would certainly induce some negative charge distribution nearby, so our answer would have to account for the contribution from this induced charge. But how do we even know how much induced charge there is?

There is a rather crafty technique used to answer problems just like this, called "the method of images". We'll start with a simple problem to illustrate the idea, and then make the situation more and more complex … just to have some fun.

Let's first make that grounded conductor an infinite grounded conducting plane, which we will place at $z = 0$. (The point charge is at $z = d$.) We need to find an electric potential that satisfies the following boundary conditions:

1. $V = 0$ when $z = 0$ (since the plane is grounded);
2. $V \rightarrow 0$ far from the charge (i.e. for $x^2 + y^2 + z^2 >> d^2$); and
3. $V \rightarrow q/(4\pi\varepsilon_0 r)$ near the point charge, as it should.

Sounds tough. But something's odd here … all of these boundary conditions are satisfied in a completely different setup: our point charge q at $z = d$ and *another* point charge $-q$ at $z = -d$. The potential in this situation is trivial:

$$V(x,y,z) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right].$$

This certainly has $V = 0$ when $z = 0$, has $V \rightarrow 0$ far from the charge, and $V \rightarrow q/(4\pi\varepsilon_0 r)$ near our original point charge. And we know that answers to Poisson's equation are unique if the charges and potentials are specified everywhere, so what's going on? Is this our answer? It turns out that we just solved our original grounded-conductor problem by finding a much simpler problem that satisfied all the same boundary conditions! In this case, that extra charge $-q$ is called our "*image charge*". [The image charge *must* be placed somewhere where we aren't looking for the potential (otherwise we can throw uniqueness out the window)! Indeed, our image charge was placed below the plane (where we didn't care what the potential is).]

Good. Let's do some similar, but tougher, problems:

(a) A point charge q is situated a distance a from the centre of a grounded conducting sphere of radius R. Find the potential outside the sphere. [Hint: the image charge should have a charge $q' = -(R/a)\, q$. Where should it be placed to ensure $V = 0$ at $r = R$? This is sometimes called the "method of inversion", though I still like to think of it as an image problem.]

(b) If you add yet *another* image charge to your solution for (a), you can generalise to the case of a sphere held at any potential V_0. What should this image charge be and where should it be placed?

(c) Let's return to the original planar problem described in the introduction, but replace the grounded plane with an infinite uniform linear dielectric material of susceptibility χ_e. What is the analogous method-of-images problem? What is the image charge now? Where should it be placed? What is the potential for $z > 0$? [Careful! Your boundary conditions require you to know the potential for $z < 0$ *and* $z > 0$ and so you will actually need two image charges: one to obtain V above the plane and one to obtain V below the plane.] Check your answer by taking $\chi_e \to \infty$; you should recover $q' = -q$.

(d) Again, let's return to the original planar problem described in the introduction (with the *grounded* plane), but replace the point charge with a dipole $\vec{p}$. Show that its energy is

$$U = -\frac{p_{\parallel}^2 + 2p_{\perp}^2}{64\pi\varepsilon_0 d^3},$$

where $p_{\parallel}$ and $p_{\perp}$ are the components of $\vec{p}$ parallel and perpendicular to the plane, respectively. How much energy is required to rotate it from a perpendicular orientation to a parallel one? [First hint: what is the image dipole? Think by analogy with the setup described in the introduction. Be careful which direction the image dipole is pointing! Second hint: you probably found a 32 in the denominator of U instead of a 64 … why should you divide by a factor of 2?]

(e) Now combine the setups in (c) and (d): put a dipole $\vec{p}$ above an infinite uniform dielectric medium. How does your answer to (d) change?