Merton College A2 Hilary Term 2011: Homework 2

1. Griffiths Problem 7.1

2. Griffiths Problem 7.38

   (*Hint*: Recall that the potential of a metal sphere of radius a placed in a uniform electric field E_0 is given by $V(r,\theta) = -E_0 r\cos\theta\left(1-\frac{a^3}{r^3}\right)$. Say, isn't the electric field in a capacitor uniform? Hmmm. *Another hint*: notice that $V = 0$ at $\theta = \pm\pi/2$, which is just what you need in this problem.)

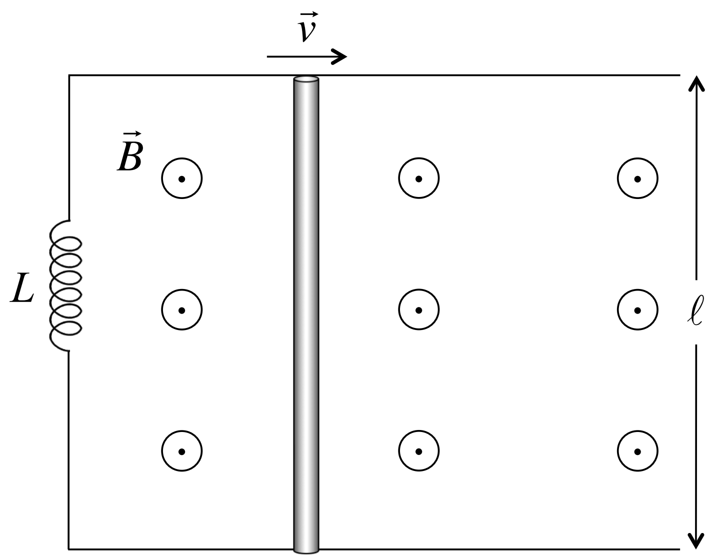
3. Griffiths Problem 9.19

4. See attached.

5. See attached.

In the next two problems, we're going to take a relatively simple induction problem and, both figuratively and literally, "ramp it up". The final problem is neat, because it will involve some induction, some circuits, some Newtonian mechanics, and some ordinary differential equations. The math may or may not be a little tough for you, but I'll take you through it slowly and give you hints where you need them.

4. As shown in the figure below, a bent wire has been placed in the midst of a uniform magnetic field (oriented out of the page). Someone has put an inductor L in the wire and a metal rod (of electrical resistance R) on top of the wire, which completes the circuit. There is no friction between the wire and the rod, and so when the rod moves it does not roll (i.e. you may neglect its moment of inertia in what follows).



(a) At $t = 0$, the rod is pulled to the right with a *constant* velocity v. Which way does the induced current start to flow, clockwise or anticlockwise?

(b) Using the induction equation, $EMF = -d\Phi/dt$, and Kirchoff's voltage law[1], write down an equation for the evolution of the current $I(t)$ in the circuit. Solve this equation, subject to the initial condition $I(0) = 0$.[2] Plot the current versus time.

(c) In order to maintain a constant velocity, a force is being applied to the rod. What is it?

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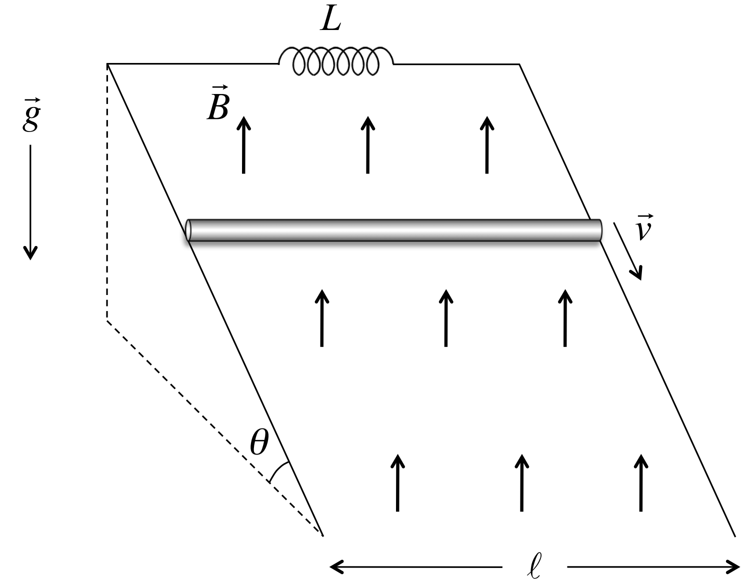
[1] In case you've forgotten Kirchoff's voltage law, it states that voltage sources minus voltage sinks around a closed circuit loop must be zero. Recall that for a resistor, you add $-IR$, whereas for an inductor, you add $-L(dI/dt)$; here, I is the current. Note: the *EMF* counts as a voltage source.

[2] If you don't know how to solve this equation, try a solution of the form $I(t) = \alpha\exp(-t/\tau) + \beta$, where α, β, and τ are constants.

5. Now we're going to investigate a case where the rod doesn't move at a constant velocity, but rather is subject to a constant external force. Take the same setup as above, but place it in a gravitational field and tilt the wire so that it forms an angle θ with the horizontal (see the figure below). The magnetic field makes an angle θ with the plane of the wire. Again, ignore friction. Take the initial velocity of the rod $v(0) = 0$. I suppose the rod needs a mass; call it m.



(a) Write down the two differential equations governing the time evolution of this system: one for the current $I(t)$ in the wire (using Kirchoff's voltage law) and one for the velocity $v(t)$ of the rod (using Newton's second law). These equations should be coupled: v should appear in your current equation; I should appear in your velocity equation.

(b) Here's how to solve them. Take d/dt of your current equation, and then substitute dv/dt from your velocity equation into it. Now you should have one second-order differential equation for the current. Write it down. Define $\varsigma \equiv R/2L$ and $\omega \equiv B\ell\cos\theta/\sqrt{mL}$. Rewrite your equation.

(c) First, let $R = 0$ (i.e. the rod is a perfect conductor). Solve this equation for $I(t)$, subject to the initial condition $I(0) = 0$.[3] Using your current equation from (a), you can also solve for the velocity $v(t)$. Do so. Impose the initial condition $v(0) = 0$, which will determine the final unspecified coefficient in your solution. You should now be able to write down $I(t)$ and $v(t)$. They should be simple harmonic oscillators! Check your answer by calculating the velocity in the limit $t << 1/\omega$; you should obtain something sensible.

(d) Now let R be finite. Solve this equation for $I(t)$.[4] Use this solution to determine $v(t)$. This is a damped driven oscillator! I've plotted solutions on the next page for you for some representative values of ς/ω.

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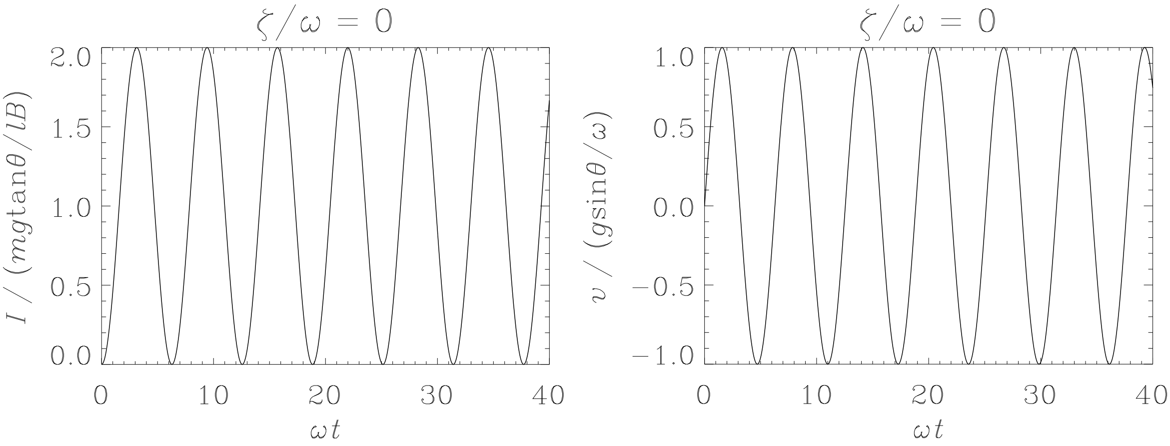
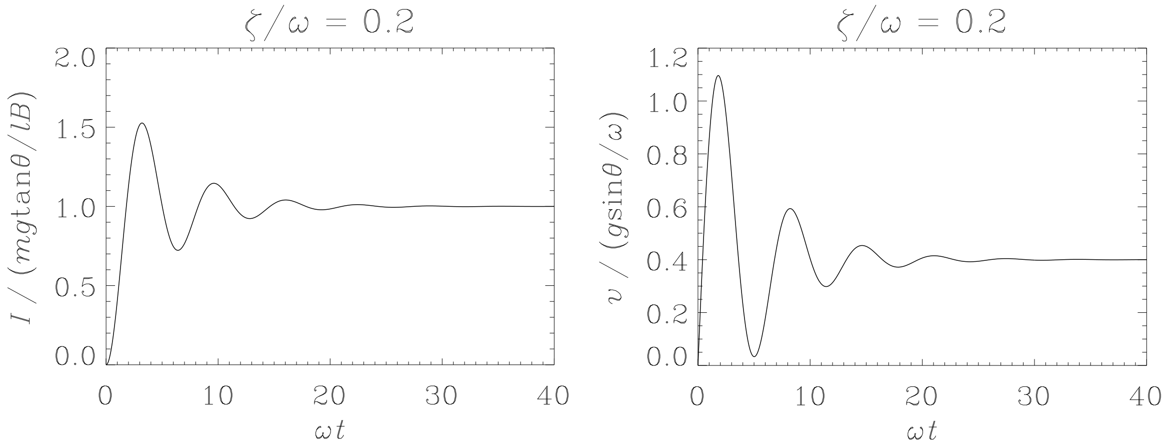
[3] Try a solution of the form $I(t) = \alpha + \beta\cos(\varpi t) + \gamma\sin(\varpi t)$, where α, β, γ and ϖ are constants.

[4] Try a solution of the form $I(t) = \alpha + \beta\exp(-\varsigma t)\cos(\varpi t) + \gamma\exp(-\varsigma t)\sin(\varpi t)$, where α, β, γ and ϖ are constants.

Undamped oscillator:



Under-damped driven oscillator:



Critically-damped driven oscillator: