For a few special configurations of electric charge, the equipotential lines have a simple form, which provides a method to determine the capacity of a set of corresponding surfaces. In this problem we'll explore the case of line charges. For it, you will need an expression for the electric potential a distance r away from an infinitely-long line charge with charge density per unit length λ; here it is:

$$V = -\frac{\lambda}{2\pi\varepsilon_0}\ln r \ .$$

Okay. Off we go…

(a) Two parallel infinitely-long line charges with charge density per unit length $\pm\lambda$ are oriented along the z-axis and located at $x = \pm b$, $y = 0$. Show that the equipotential surfaces (i.e. $V =$ constant) can be described by the equation

$$(x - x_0)^2 + y^2 = R^2$$

for all z; i.e. the equipotentials are circular cylinders with radii R centred at x_0. Find explicit solutions for x_0 and R in terms of b and the parameter

$$\xi \equiv \exp\left(\frac{2\pi\varepsilon_0 V}{\lambda}\right) .$$

Sketch some representative equipotential lines. (Note that $\xi - 1$ may be positive or negative.)

(b) Now consider two infinitely-long circular conducting cylinders located at $x = \pm d$, $y = 0$ with finite radii $a < d$. Prove that the capacitance per unit length C of this system is given by

$$\frac{1}{C} = \frac{1}{\pi\varepsilon_0}\ln\left[\frac{d}{a} + \sqrt{\frac{d^2}{a^2} - 1}\right] \quad \left(= \frac{1}{2\pi\varepsilon_0}\cosh^{-1}\left[\frac{2d^2}{a^2} - 1\right]\right) .$$

Hints: (1) conductors are equipotentials and (2) there's a reason you did part (a) first! Start by finding which equipotential corresponds to each cylinder. Note that this method also works for cylinders of different radii, including the case of one cylinder placed off-centre inside another.

(c) Find the work required to move the two cylinders from $x = \pm d$ to $x = \pm d\,(1+\delta)$, where $\delta << 1$, while keeping their potentials fixed.

(d) Repeat (c) for a fixed charge instead. Are the answers to (c) and (d) different? If they are different, explain physically why this is so.