Mansfield College A2 Hilary Term 2011: Homework 4

1. Griffiths Problem 9.16

2. Griffiths Problem 9.30

3. Griffiths Problem 9.36

4. Lecturer's Problem Set 5, Number 6

5. Magnetic fields play an important role in a wide variety of astrophysical systems. While cosmic magnetism is of great theoretical interest in its own right, one quickly acquires a strong desire to get a telescope and try to actually *measure* the strength of these fields. Unfortunately, there are few available techniques, some being more accurate than others and most of which being quite difficult to employ. In this problem we'll explore one of the more straightforward techniques for measuring astrophysical magnetic fields, called "Faraday rotation". The idea is rather simple. There are different indices of refraction for left- and right-handed circularly polarised waves that propagate parallel to an external magnetic field. Therefore the left- and right-handed components of a linearly polarized wave accumulate a phase difference as they traverse the medium, so that the direction of linear polarization changes with time/distance. The magnetic field strength can be then deduced from this phase difference.

   Consider a polarisable medium threaded by a uniform magnetic field $\vec{B}_0 = B_0\,\hat{z}$. We approximate the medium as a cloud of electrons with charge $-e$, mass m, and volume number density n_0, which are bound to their locations by a spring constant $K = m\omega_0^2$. Send in an electromagnetic wave of frequency $\omega = 2\pi/\lambda$ with (transverse) electric field $\vec{E} = E_x\hat{x} + E_y\hat{y}$ and magnetic field strength $B << B_0$.

   a. First, show that the equation governing the electromagnetic wave is given by

   $$\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial z^2}\right)\vec{E} = \frac{n_0 e}{\varepsilon_0}\frac{d^2\vec{r}}{dt^2},$$

   where $\vec{r} = x\hat{x} + y\hat{y}$ is the position of the electrons. Consider solutions of the form $\exp(ikz - i\omega t)$ and write E_x and E_y in terms of x and y.

   b. Next, use Newton's second law to write an equation of motion for $\vec{r}$. Treat the electrons as simple harmonic oscillators with natural frequency ω_0, which are being driven by the Lorentz force. Assuming solutions of the form $\exp(ikz - i\omega t)$, write x and y of the electrons in terms of E_x and E_y. At some point it'll be convenient to introduce the cyclotron frequency $\omega_c \equiv eB_0/m$.

   c. Combine your results from parts (a) and (b) to show that an electromagnetic wave with left-handed (right-handed) polarization has an index of refraction n_+ (n_-),

   $$n_\pm^2 = 1 + \frac{\Omega_p^2}{\omega_0^2 - \omega^2 \mp \omega\omega_c},$$

where $\Omega_p^2 = n_0 e^2 / m\varepsilon_0$ is the square of the plasma frequency. This means that right- and left-circularly polarised electromagnetic waves will propagate at different speeds. Which one travels faster?

d. The answer from part (c) implies that, if each polarisation of wave propagates through the same thickness of the electron cloud, then they will emerge having different phases. Thus the total plane of polarisation has rotated. Show that the plane of polarization is rotated through an angle (in radians)

$$\Delta\chi \cong \frac{e^3}{2\varepsilon_0 m^2 c\omega^2} \int_0^d n_e B \, dl \equiv \mathrm{RM}\ \lambda^2$$

after traversing a cloud of size d, where we have defined the Rotation Measure (RM), which has units of rad m^{-2}. In your derivation, you may make the following assumptions: ω_c^2, $\omega_0^2 << \omega^2$ and $n \approx \frac{1}{2}\left(n_+ + n_-\right) \approx 1$. (Note: in general, the field strength B in the integral should be $B_{\parallel}$, the strength of the magnetic field parallel to the line of sight.)

e. Faraday rotation is a useful tool that is often used to determine the magnetic field strength and structure in both our Galaxy and external galaxies.[1] Suppose one observed 21 cm (1.4 GHz) emission and measured an RM of 300 rad m^{-2}. Assuming typical distances and average electron densities characteristic of our Galaxy's interstellar medium (5 kpc and 0.05 cm^{-3}, respectively), estimate the average strength of our Galaxy's magnetic field in μG.[2] You may assume that the magnetic field is oriented along the line-of-sight.

[1] See, e.g., http://adsabs.harvard.edu/abs/2011ApJ...728...97V
[2] Useful unit conversions: 1 pc = 3.0857 x 10^{16} m; 1 cm^{-3} = 10^6 m^{-3}; 1 μG = 10^{-10} T.