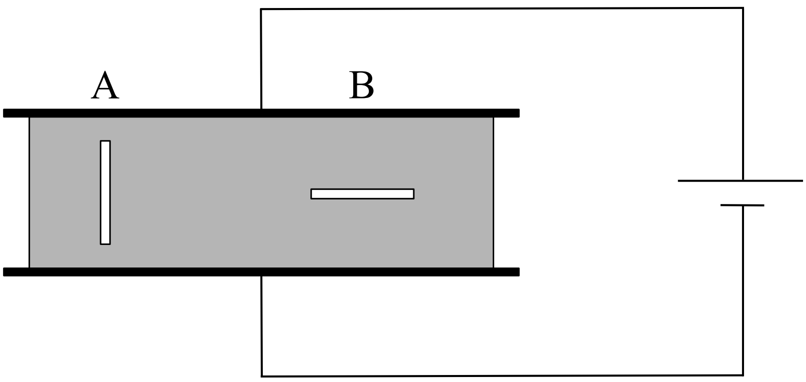
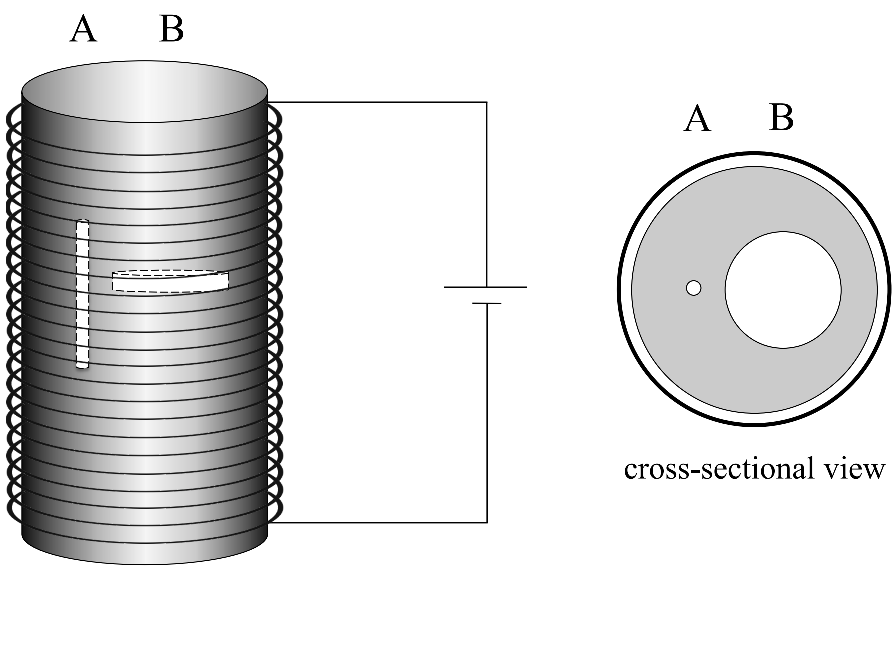
Mansfield College A2 Hilary Term 2011: Homework 1

Part I: Short Answer Questions

1. A capacitor is filled with a linear dielectric material of constant permittivity $\varepsilon > \varepsilon_0$, which has two small cracks in it as shown in the figure. In which of the two cracks (A or B) is the electric field strength greater? Briefly explain your answer.

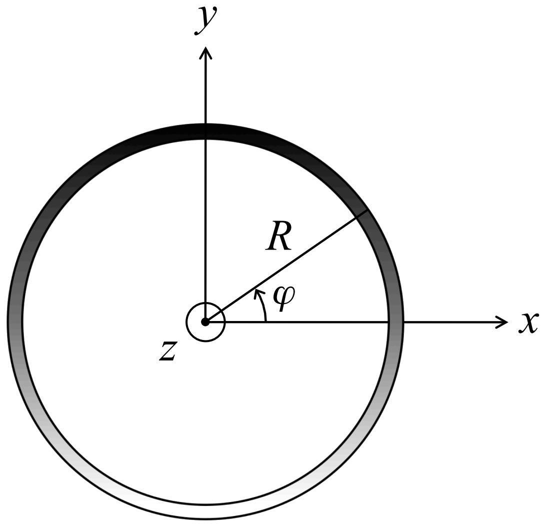


2. A solenoid is filled with a linear magnetic material of constant permeability $\mu > \mu_0$, which has two small cracks in it as shown in the figure. In which of the two cracks (A or B) is the magnetic field strength greater? Briefly explain your answer.



Part II: Long Answer Questions

3. An insulating circular ring of radius R lies in the x-y plane, centred at the origin. It carries a linear charge density $\lambda = \lambda_0 \sin\varphi$, where λ_0 is a constant and φ is the usual azimuthal angle measured from the x axis (see figure). Assume the ring is infinitesimally thin.



a. Using Coulomb's law, calculate the electric field at the point $(x, y, z) = (0, 0, d)$. Discuss the limits $d = 0$ and $d >> R$. There are simpler charge configurations that can reproduce your answer in these limits. What are they?

b. At $t = 0$, the ring is set spinning at a constant angular velocity $\vec{\omega} = \omega\,\hat{z}$. Is this an electric dipole, magnetic dipole, or something else? Justify your answer by explicitly calculating the appropriate moment at $t = 0$.

c. Whilst the ring is spinning, a magnetic field is produced. Using the Biot-Savart law, calculate the magnetic field at the point $(x, y, z) = (0, 0, d)$ at $t = 0$. You may assume $\omega << c/R$ and $\omega << c/d$, where c is the speed of light. The following may be useful: $\hat{\varphi} = -\sin\varphi\,\hat{x} + \cos\varphi\,\hat{y}$.

4. A charge density $\sigma(\theta) = \sigma_0 \cos^2\theta$ is glued over the surface of a spherical shell of radius R.

a. Show that the electric potential V satisfies Laplace's equation $\nabla^2 V = 0$ inside and outside of the spherical shell. Write down the most general solution for the potential in spherical coordinates (r, θ, φ), making use of the Legendre polynomials $P_\ell(\cos\theta)$.

b. Write down the boundary conditions on V at $r = 0$, $r = R$, and $r \to \infty$.

c. Using your general solution from (a) and your boundary conditions from (b), solve for the potential inside (V_{in}) and outside (V_{out}) of the shell.

d. Embed the spherical shell in an infinite linear dielectric medium with dielectric constant $\varepsilon > \varepsilon_0$. Write down the new boundary conditions on V at $r = R$.

5. A conducting spherical shell of radius R is placed in an otherwise uniform electric field of strength E_0.

a. Write down the boundary conditions on V at $r = R$ and $r \to \infty$.

b. Using your general solution from (a) and your boundary conditions from (b), solve for the potential outside of the shell V_{out}.

c. Find the charge density $\sigma(\theta)$ on the surface of the shell. Explain your answer physically.

d. Embed the spherical shell in an infinite linear dielectric medium with dielectric constant $\varepsilon > \varepsilon_0$. Write down the new boundary conditions on V at $r = R$.

Possibly Useless Information:

$$\int_0^{\pi} P_\ell(\cos\theta) P_{\ell'}(\cos\theta)\, \sin\theta\, d\theta = \frac{2}{2\ell + 1}\, \delta_{\ell\ell'}$$

$$P_0(\cos\theta) = 1 \qquad P_1(\cos\theta) = \cos\theta \qquad P_2(\cos\theta) = \frac{3}{2}\cos^2\theta - \frac{1}{2}$$