Zonal Flow Generation: Basic Theoretical Insights and Analytical Tools[a]

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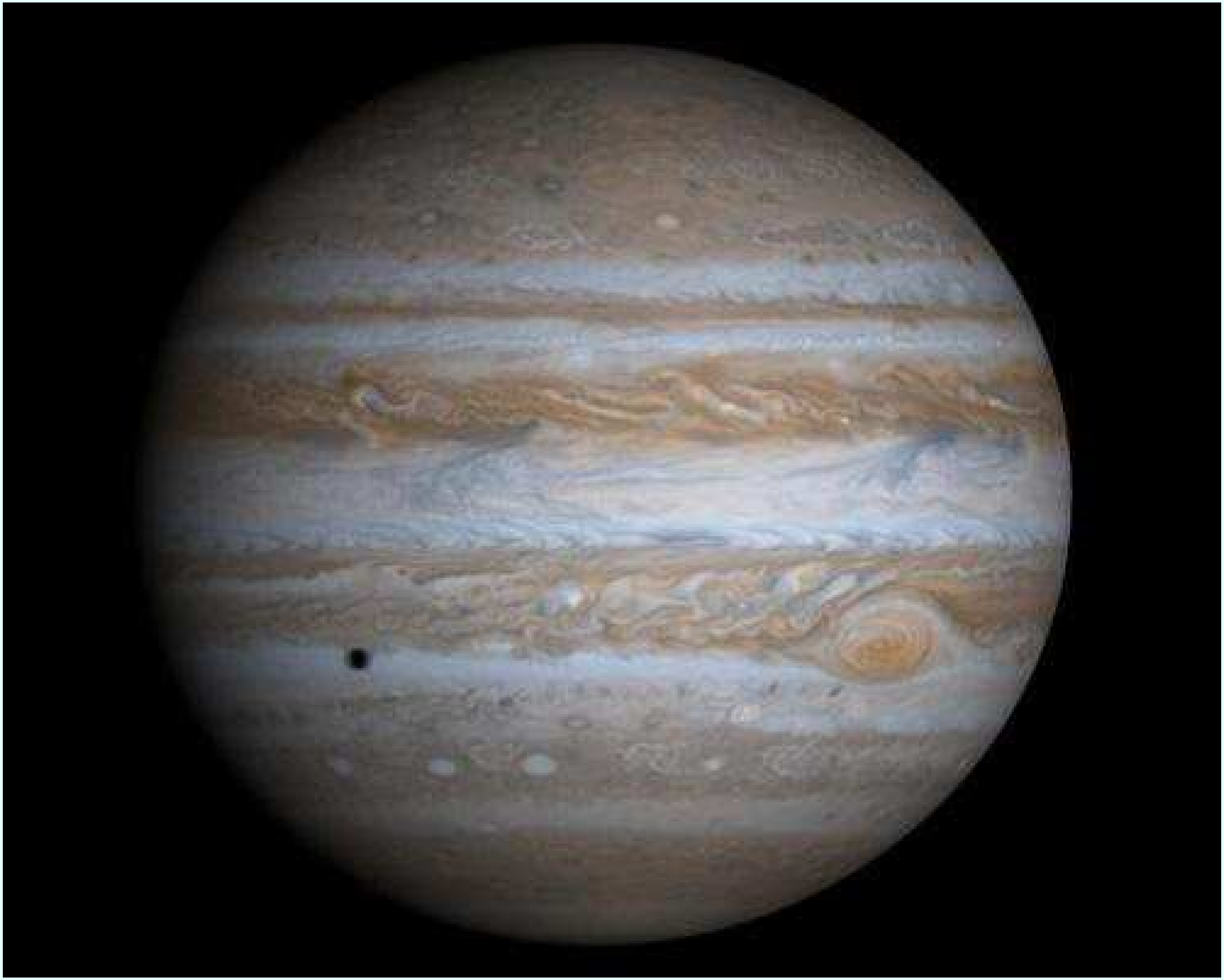


Zonal flows exist in tokamaks; they are believed to help regulate the level of tokamak microturbulence.

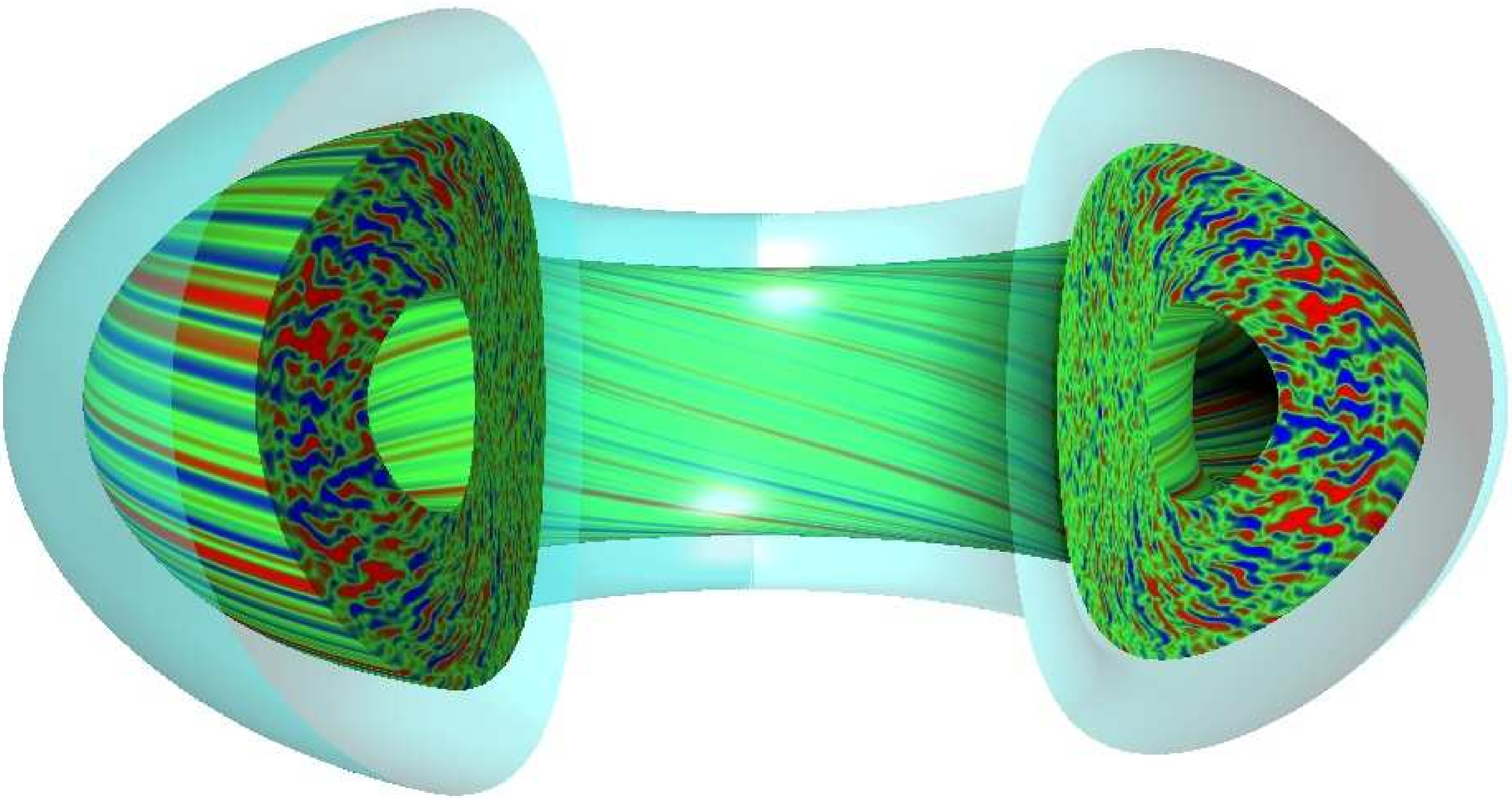


FIG. 1: Microturbulence in a tokamak simulation (Jeff Candy; from the GYRO **web site.).**

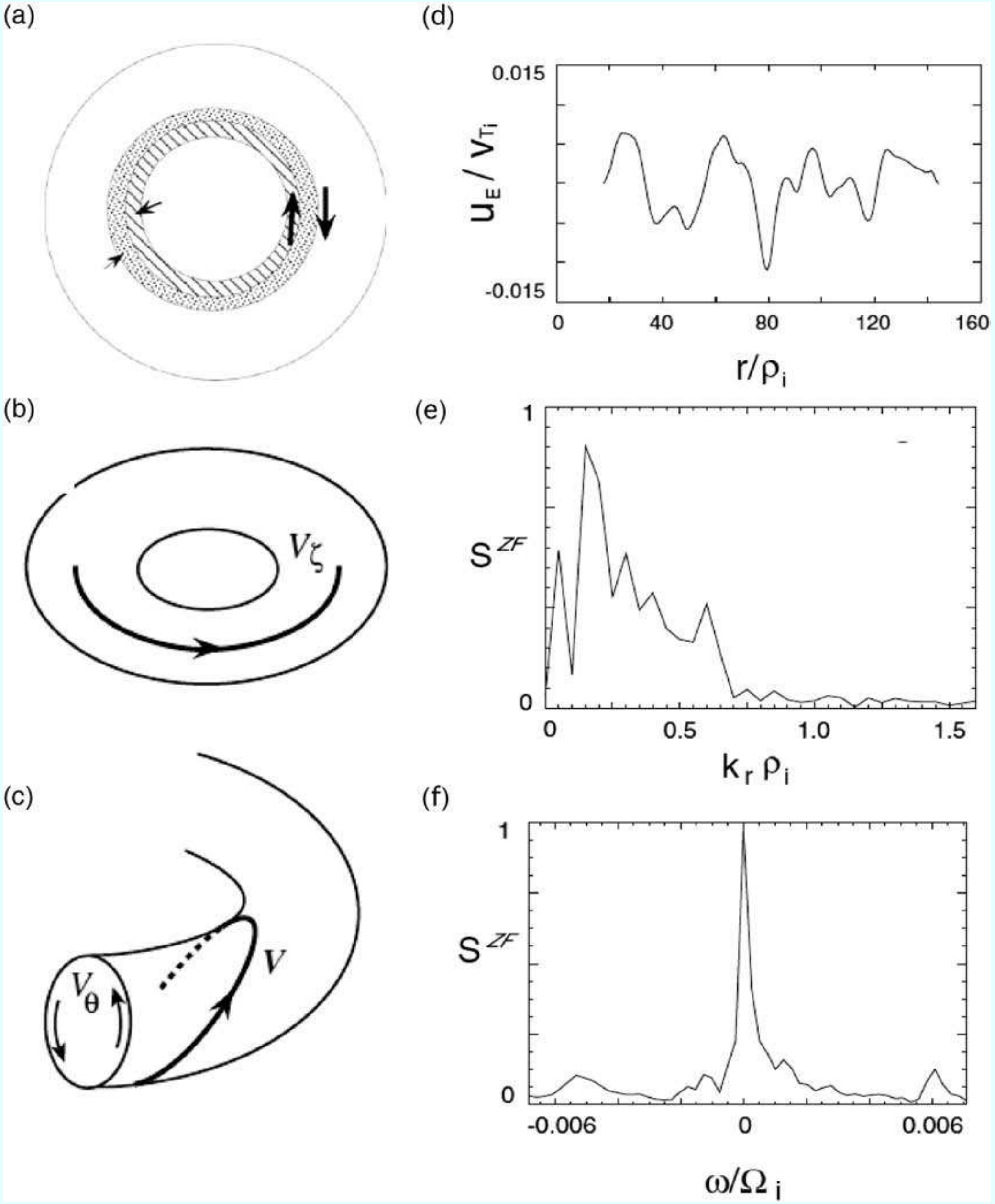


FIG. 2: Illustration of sheared ZFs in tokamaks. From review article of Fujisawa (2009).

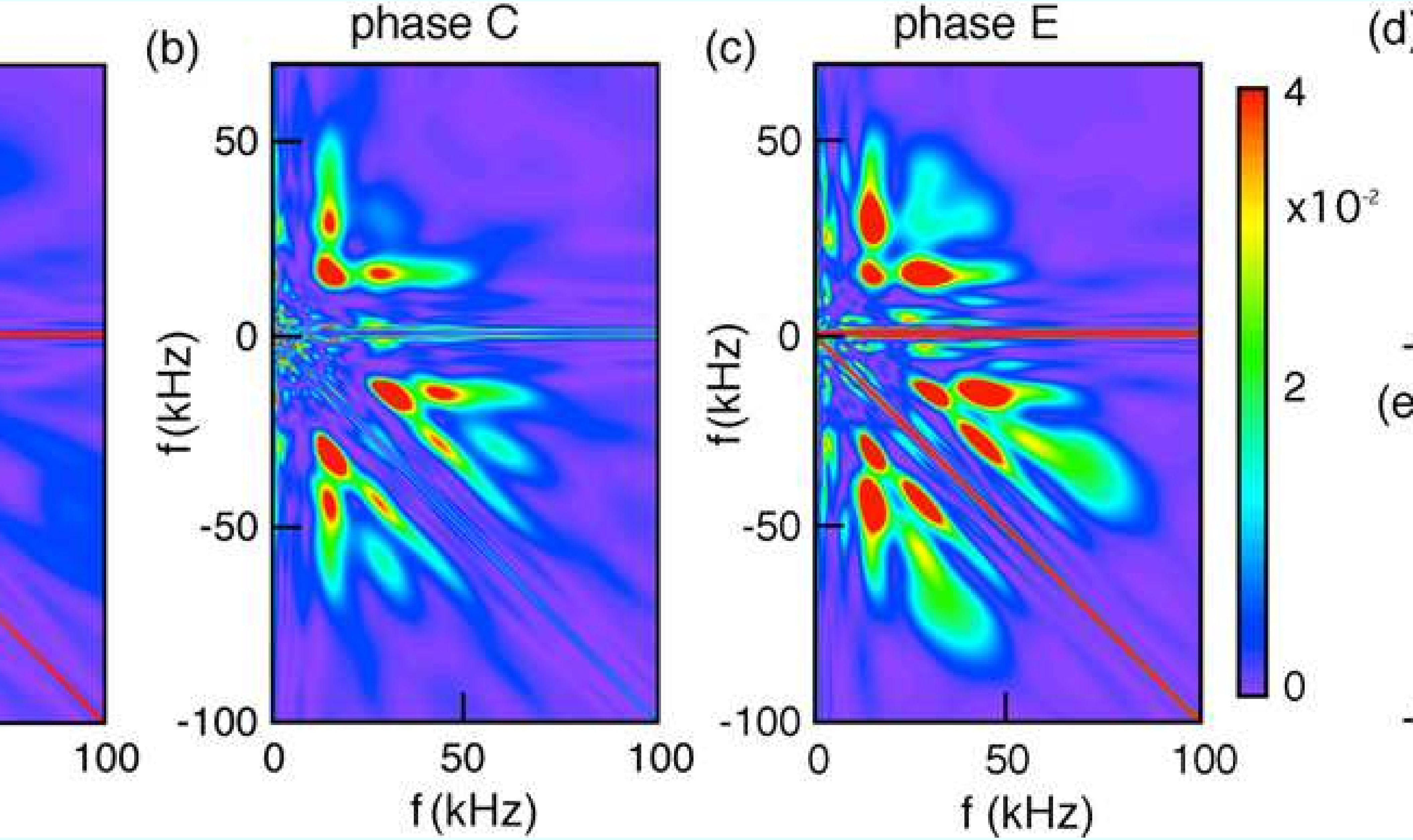


FIG. 3: Bicoherency analysis is useful. From Fujisawa (2009).

It is fruitful to treat self-generated ZFS as a separate component of a saturated state.

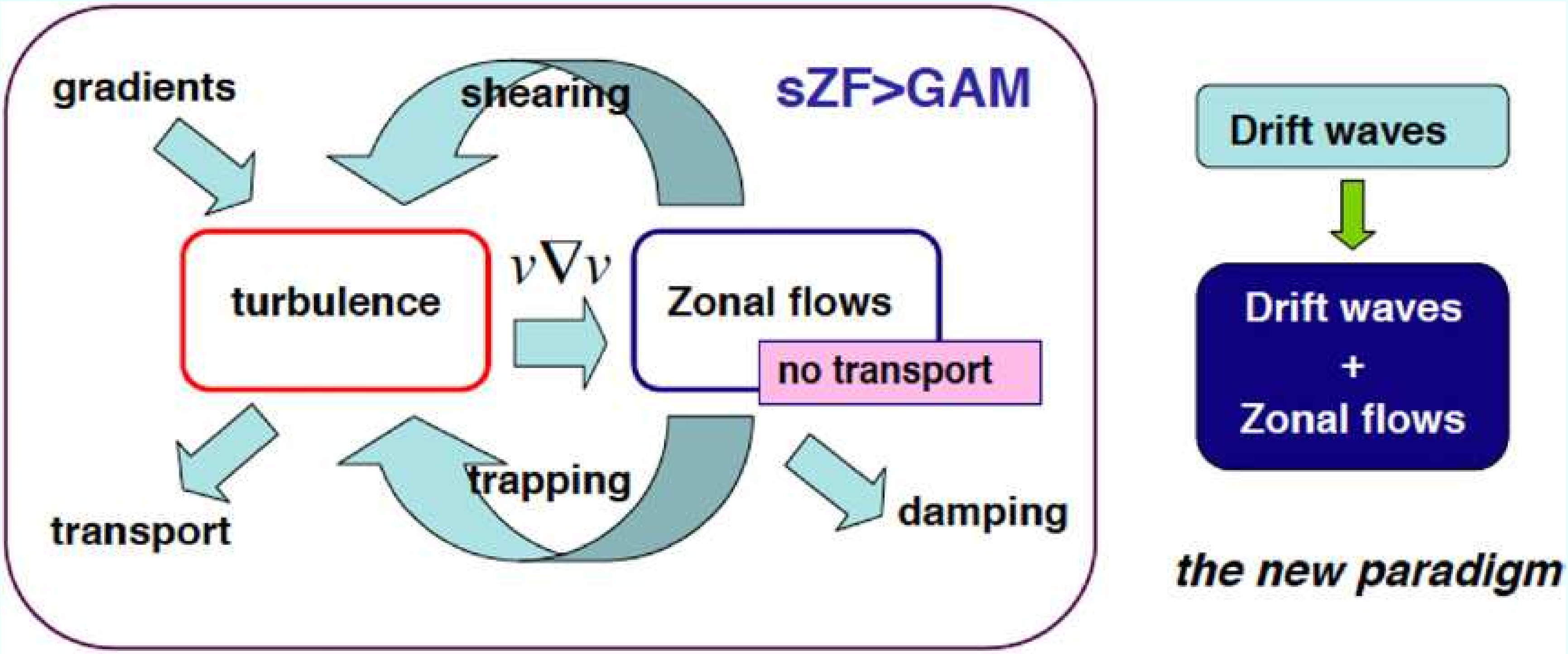
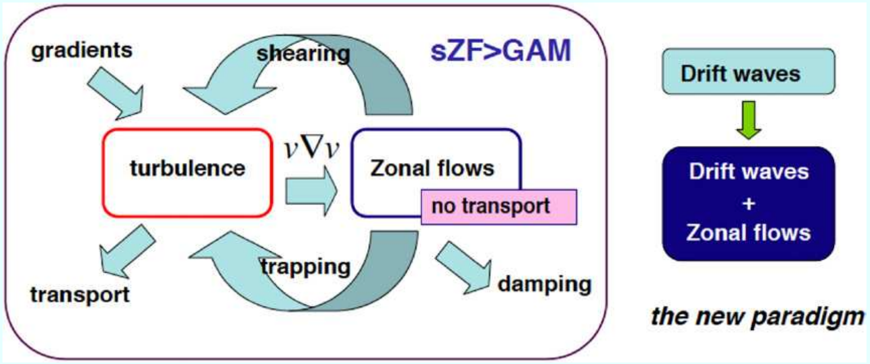


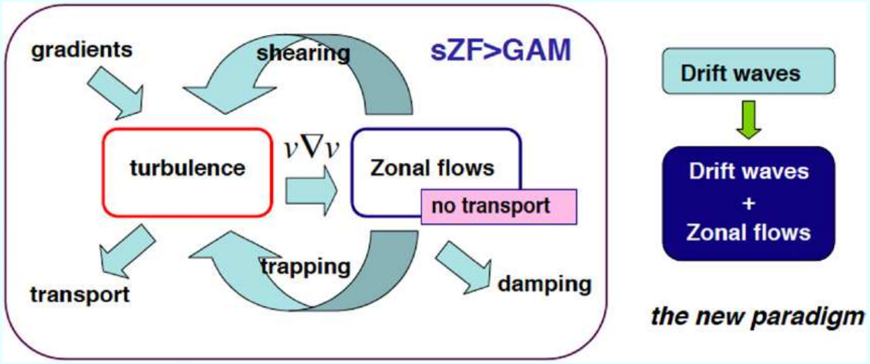
FIG. 4: From review article of Fujisawa (2009).

Goal: **Gain some conceptual and analytical understanding of self-consistent states of zonal flows and turbulence ($\Rightarrow$ implement some sort of *averaging procedure*).**

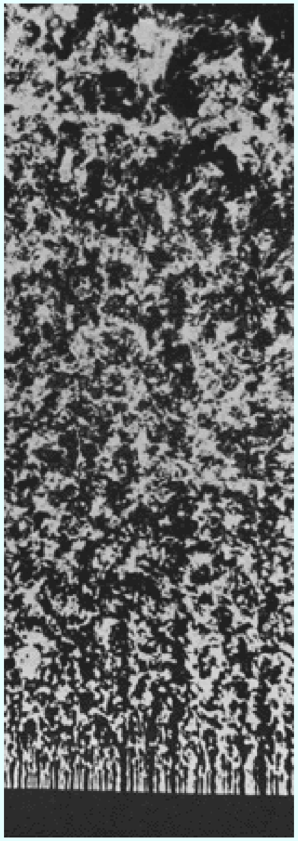
Statistical closures: A blessing and a curse…



Statistical closures: A blessing and a curse. . .



Let's begin with homogeneous turbulence. . .



Use random forcing to model instability drive...

3D Navier–Stokes: replace macroscopic, inhomogeneous boundary conditions,

$$\partial_t \vec{u} + \vec{u} \cdot \vec{\nabla} \vec{u} = \cdots + \text{(b.c.'s)}, \tag{1}$$

with homogeneous, isotropic stirring:

$$\partial_t \vec{u} + \vec{u} \cdot \vec{\nabla} \vec{u} - \cdots = \widetilde{\vec{f}}(\vec{x}, t), \tag{2}$$

where $\widetilde{\vec{f}}$ is Gaussian white noise spectrally concentrated at long wavelengths:

$$\langle \widetilde{\vec{f}} \rangle = \vec{0}, \quad \langle \delta \widetilde{\vec{f}}(\vec{x}, t) \delta \widetilde{\vec{f}}^T(\vec{x}', t') \rangle = \mathsf{F}(\vec{x} - \vec{x}') \delta(t - t'). \tag{3}$$

Make the usual Reynolds decomposition

$$\widetilde{\vec{u}} = \vec{U} + \delta \vec{u} \quad (\text{note } \langle \delta \vec{u} \rangle = \vec{0}). \tag{4}$$

Then the mean velocity $\vec{U}$ obeys

$$\partial_t \vec{U} + \vec{U} \cdot \vec{\nabla} \vec{U} = -\vec{\nabla} \cdot \underbrace{\langle \delta \vec{u}\, \delta \vec{u} \rangle}_{\text{Reynolds stress}} + \cdots . \tag{5}$$

The effect of the Reynolds stress vanishes for homogeneous turbulence ($\partial_x\langle\ldots\rangle = 0$)...

For homogeneous turbulence, one has $\langle\delta\vec{u}(\vec{x},t)\delta\vec{u}(\vec{x},t)\rangle(\cancel{\vec{x}})$, so

$$\vec{\nabla}\cdot\langle\delta\vec{u}\,\delta\vec{u}\rangle = \vec{0}. \tag{6}$$

Also, can take $\vec{U} = \vec{0} \Rightarrow$ traditional focus on wave-number spectrum $E(k)$.

Now reinstate some boundary conditions and/or inhomogeneity:

- let's say it's inhomogeneous in x,
- homogeneous in y and z.

Now Reynolds stresses can drive flows:

$$\partial_t\vec{U}(x) + U_x\partial_x\vec{U}(x) = -\partial_x\langle\delta u_x(\vec{x})\delta\vec{u}(\vec{x})\rangle(x) + \cdots. \tag{7}$$

For example, in magnetized plasmas Reynolds stresses due to microturbulence can drive poloidal *zonal flows*:

$$\partial_t U_y(x) = -\partial_x\langle\delta u_x\,\delta u_y\rangle(x) + \cdots. \tag{8}$$

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Zonal flows can emerge by spontaneous symmetry breaking of a homogeneous turbulent state.

$$\partial_t \vec{U} + \vec{U} \cdot \vec{\nabla} \vec{U} = -\vec{\nabla} \cdot \langle \delta \vec{u}\, \delta \vec{u} \rangle [\mathcal{E}, \vec{U}] - \mathcal{D}[\vec{U}], \tag{9a}$$

$$\partial_t \mathcal{E} = \mathcal{N}[\mathcal{E}, \vec{U}] - \mathcal{D}[\mathcal{E}] + \underbrace{\mathcal{F}}_{\text{homogeneous forcing}}. \tag{9b}$$

Assume that one imposes no macroscopic inhomogeneity (periodic b.c.'s, constant profile gradients, etc.). Then:

- This system has statistically homogeneous solutions with $\vec{U} \equiv \vec{0}$. If one insists that the statistics are homogeneous, then homogeneous solutions are all one gets.
- But if one allows for the possibility of inhomogeneous statistics, then *a bifurcation may occur:*

Inhomogeneous solutions can emerge by spontaneous symmetry breaking of the homogeneous state.

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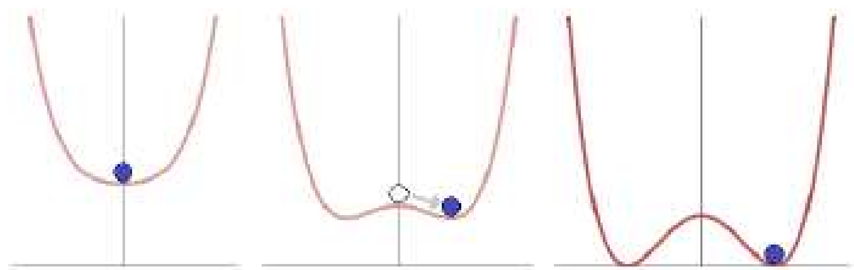
Spontaneous symmetry breaking is well known in physics. . .



Spontaneous symmetry breaking simplified: - At high energy levels *(left)* the ball settles in the center, and the result is symmetrical. At lower energy levels *(right)*, the overall "rules" remain symmetrical, but the "Mexican hat" potential comes into effect: "local" symmetry is inevitably broken since eventually the ball must roll one way (at random) and not another.

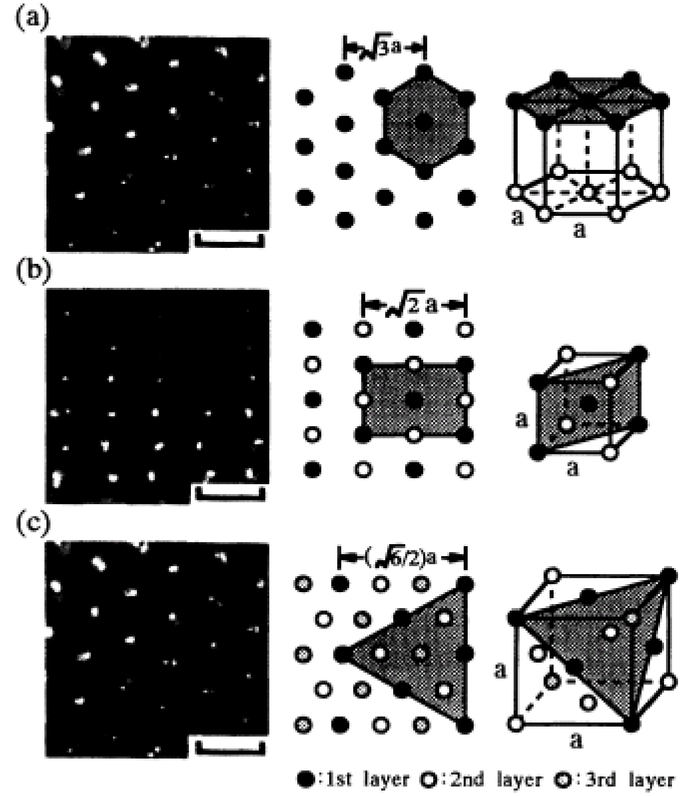


FIG. 3. Micrographs and sketches of the different crystal structures. (a) Hexagonal; (b) bcc; (c) fcc. The center column corresponds to the structures in the micrographs. The graded areas in the sketches are normal to the optical axis. The bars correspond to 200 μm.

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BROKEN SYMMETRIES AND THE MASSES OF GAUGE BOSONS

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(Received 31 August 1964)

In a recent note[1] it was shown that the Goldstone theorem,[2] that Lorentz-covariant field theories in which spontaneous breakdown of symmetry under an internal Lie group occurs contain zero-mass particles, fails if and only if the conserved currents associated with the internal group are coupled to gauge fields. The purpose of the present note is to report that, as a consequence of this coupling, the spin-one quanta of some of the gauge fields acquire mass; the longitudinal degrees of freedom of these particles (which would be absent if their mass were zero) go over into the Goldstone bosons when the coupling tends to zero. This phenomenon is just the relativistic analog of the plasmon phenomenon to which Anderson[3] has drawn attention: that the scalar zero-mass excitations of a superconducting neutral Fermi gas become longitudinal plasmon modes of finite mass when the gas is charged.

about the "vacuum" solution $\varphi_1(x) = 0$, $\varphi_2(x) = \varphi_0$:

$$\partial^\mu\{\partial_\mu(\Delta\varphi_1) - e\varphi_0 A_\mu\} = 0, \tag{2a}$$

$$\{\partial^2 - 4\varphi_0^2 V''(\varphi_0^2)\}(\Delta\varphi_2) = 0, \tag{2b}$$

$$\partial_\nu F^{\mu\nu} = e\varphi_0\{\partial^\mu(\Delta\varphi_1) - e\varphi_0 A^\mu\}. \tag{2c}$$

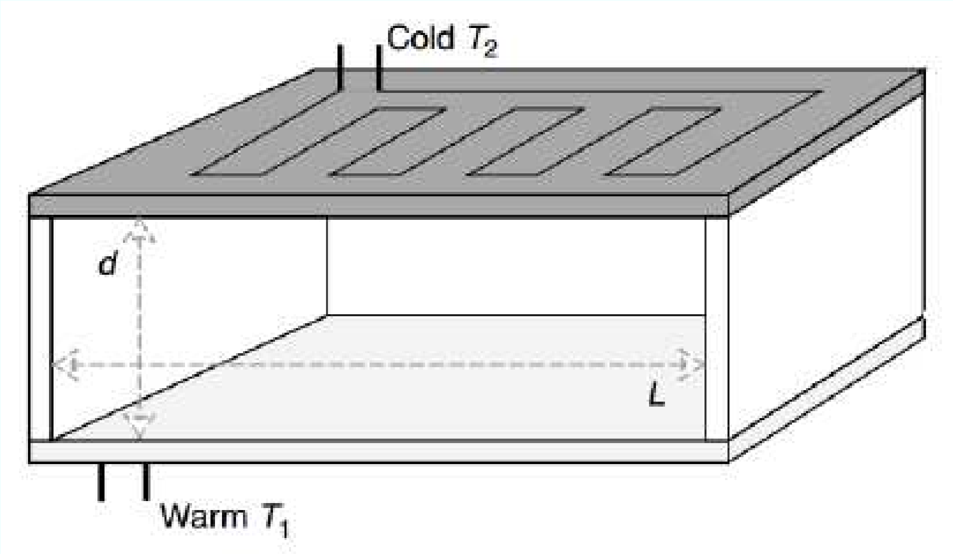
Equation (2b) describes waves whose quanta have (bare) mass $2\varphi_0\{V''(\varphi_0^2)\}^{1/2}$; Eqs. (2a) and (2c) may be transformed, by the introduction of new variables

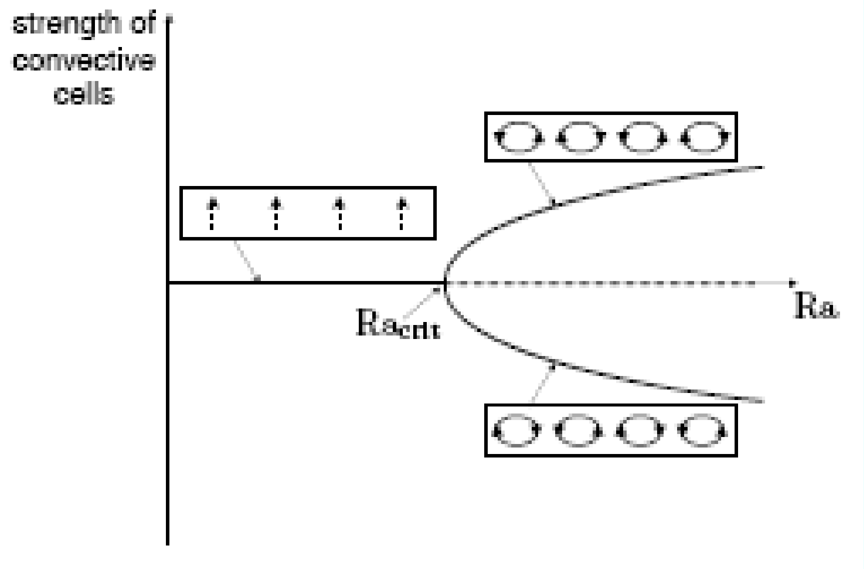
$$B_\mu = A_\mu - (e\varphi_0)^{-1}\partial_\mu(\Delta\varphi_1),$$
$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu = F_{\mu\nu}, \tag{3}$$

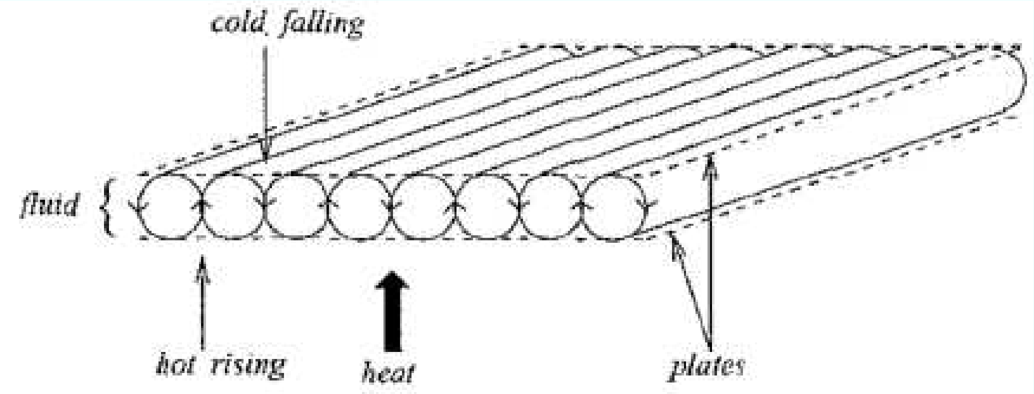
into the form

$$\partial_\mu B^\mu = 0, \quad \partial_\nu G^{\mu\nu} + e^2\varphi_0^2 B^\mu = 0. \tag{4}$$

Symmetry breaking also happens in Rayleigh–Bénard convection.







There is a useful analogy between the convection problem and the problem of zonal-flow generation.

	convection	ZFs
small ΔT:	microscopic Gibbsian fluctuations macroscopically motionless	homogeneous microturbulence no ZFs
	translationally invariant in horizontal direction	
larger ΔT:	steady rolls	steady ZFs
	translational symmetry is broken	

- We will apply methodology used in the theory of convection and, more generally, pattern formation to the problem of zonal-flow generation.
- One possible scenario: Zonal flows emerge via a bifurcation from a state of homogeneous turbulence (to an inhomogeneous state with interacting ZFs and microturbulence).
- We calculate the stability of those ZFs by determining the 'Busse stability balloon.'
- Hence the title of our recent paper: J. Parker & J. Krommes, Zonal flows as pattern formation: Merging jets and the ultimate jet length scale (arXiv 1301.5059).

How does one handle the statistical closure problem for inhomogeneous turbulence?

Sidebar on *realizability*:

- The characteristic function (Fourier transform) of a PDF is its moment generating function.
- The logarithm of the characteristic function is the *cumulant generating function*.
- The fact that $P(x) \geq 0$ implies an infinite number of *realizability inequalities* between the various moments M_n or cumulants C_l.

$$P(x) \to \langle e^{-ikx} \rangle = \sum_{n=0}^{\infty} \frac{(-ik)^n}{n!} M_n, \tag{10a}$$

$$\ln \langle e^{-ikx} \rangle = \sum_{l=1}^{\infty} \frac{(-ik)^l}{l!} C_l. \tag{10b}$$

E.g., $M_2 \geq M_1^2$, $M_4 - M_2^2 \geq (M_3 - M_1 M_2)^2/(M_2 - M_1^2)$.

- A good statistical closure must be realizable.

Presently, second-order cumulant expansion (CE2) is popular. CE2 is realizable.

One has the general cumulant expansion

$$P(x) = \int \frac{dk}{2\pi} \, e^{ikx} \exp\left(\sum_{l=1}^{\infty} \frac{(-ik)^l}{l!} C_l \right). \tag{11}$$

Marcienkiewicz (1939) Theorem: Either

- retain just C_1 and C_2 (Gaussian approximation)

or

- must retain *all* cumulants in order to ensure realizability.

Some possibilities:

- inhomogeneous direct-interaction approximation (realizable, but extremely complicated);
- second-order cumulant expansion (CE2; realizable);
- third-order cumulant expansion [CE3; not realizable, but maybe can be patched up (Marston)].

'Stochastic Structural Stability Theory' (SSST)

$$\begin{gathered}\text{primitive amplitude equation}\\ \Downarrow \\ \text{Reynolds decomposition with zonal average} \\ \Downarrow \\ (\partial_t - L)U = \text{(Reynolds stress)}, \\ (\partial_t - L)\delta\psi = \underbrace{U\delta\psi}_{\text{ZF–eddy}} + \underbrace{\delta\psi\,\delta\psi - \langle\delta\psi\,\delta\psi\rangle}_{\text{eddy–eddy}} + \delta f^{(\mathrm{ext})}.\end{gathered} \tag{12}$$

Two possibilities: Either

- neglect eddy–eddy altogether ($\equiv$ CE2)

or

- parametrize eddy–eddy by internal forcing (maybe plus energy-conserving damping):

$$(\partial_t - L)\delta\psi = U\delta\psi + \underbrace{(-\eta\delta\psi)}_{\text{(neglect??)}} + \underbrace{\delta f^{(\mathrm{int})} + \delta f^{(\mathrm{ext})}}_{\delta f}. \tag{13}$$

Now discuss the stability of the statistical ensemble.

From

$$(\partial_t - L)\delta\psi = U\delta\psi + \delta f, \tag{14}$$

form the equation for the covariance $C \doteq \langle \delta\psi(\vec{x}, t)\delta\psi(\vec{x}', t)\rangle$:

$$\partial_t C(t,t) = \text{(Hermitian part of)}(LC + UC + \langle \delta f(t)\delta\psi(t)\rangle). \tag{15}$$

Assume white-noise forcing: $\langle \delta f \, \delta\psi\rangle = F(\vec{x}, \vec{x}')$. Then we arrive at the forced CE2 equations:

$$\partial_t U(x) = \dots \quad \text{(zonal } \textit{mean flow} \text{ or `}\textit{amplitude}\text{' eq'n)}, \tag{16a}$$

$$\partial_t C_{\vec{k}}(x, x') = \cdots + F_{\vec{k}}(x, x') \quad \text{(eddy } \textit{covariance} \text{ eq'n)}. \tag{16b}$$

Now discuss symmetry breaking in the *statistical ensemble*, which can be 'structurally stable' or 'structurally unstable' to the emergence of stable zonal flows...

'Stochastic ($\equiv$ statistical) Structural Stability Theory' (SSST)

(Farrell & Ioannou).

Neglecting the eddy–eddy interactions precludes inverse cascade and the Rhines mechanism.

The Rhines mechanism:

small-scale forcing

$\downarrow$

inverse cascade

$\downarrow$

stopped by anisotropic linear wave physics

$\downarrow$

preferential excitation of zonal fluctuations

- Even without the Rhines mechanism, zonal flows can still be generated by direct coupling from turbulence to zonal scales.
- For the time being, we will study just this latter mechanism.

The SSST/CE2 equations have been studied for a variety of physical situations or models.

- Farrell & Ioannou (various papers on SSST)
- Farrell & Ioannou (2009) — Hasegawa–Wakatani model for drift waves in plasmas *(initial motivation for our work)*

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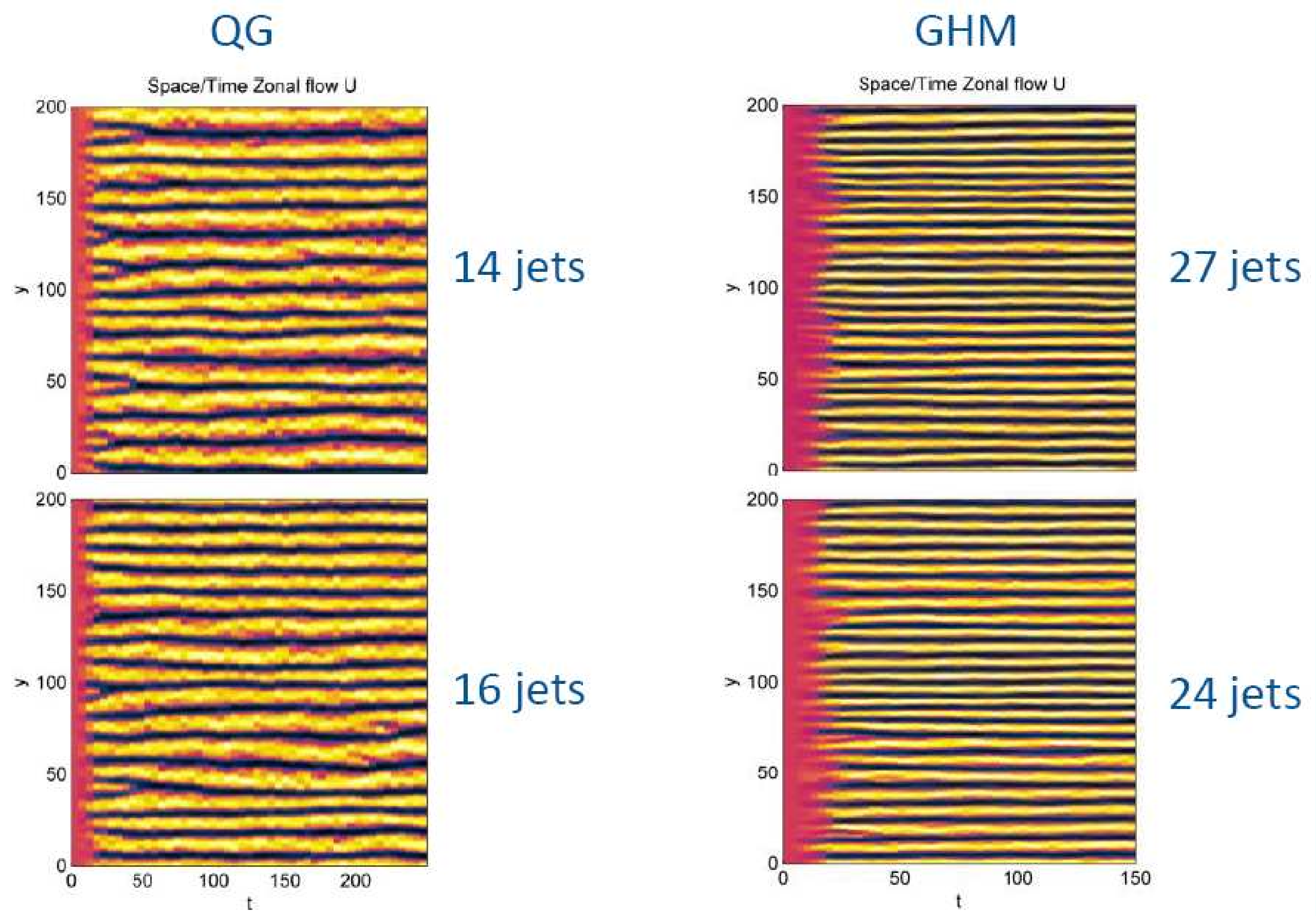
One may ask, Is there anything else left to do? Yes!

- Analytically understand the nature of the stochastically stable states with ZFs that are observed in simulations.
- E.g., predict the characteristic jet wave number, jet spacing, etc.

Surprise: The jet wave number is not unique!

The jet wave number is not unique.



A simplified example is instructive.

As an oversimplified example, suppose the equations to be solved are

$$\partial_t U(x) + \underbrace{\mu}_{\text{damping}} U = \partial_x \underbrace{C(x,x)}_{\text{Reynolds stress}}, \tag{17a}$$

$$\partial_t C(x,x') + 2\mu C = \underbrace{U(x)}_{\text{ZF}}\,\underbrace{C(x,x')}_{\text{turb.}} + C(x,x')U(x') + F(x,x'). \tag{17b}$$

(In detail, there are other gradients, structure in the homogeneous y direction, etc.; the real equations are complicated.) Change variables to

$$\rho \doteq x - x', \quad X \doteq \frac{1}{2}(x + x'). \tag{18}$$

Then $C(x,x') \equiv C(\rho \mid X)$ and

$$\partial_t U(X) + \mu U = \partial_X C(X), \tag{19a}$$

$$\partial_t C(\rho \mid X) + 2\mu C = [U(X + \tfrac{1}{2}\rho) + U(X - \tfrac{1}{2}\rho)]C(\rho \mid X) + F(\rho). \tag{19b}$$

- Note that there is a steady solution with $U = 0$, $C(\rho \mid X) = C(\rho)$.

The characteristic zonal wave number is not unique!

Assume $C \sim e^{iqX}$. Then the structure of the s.s. equations is

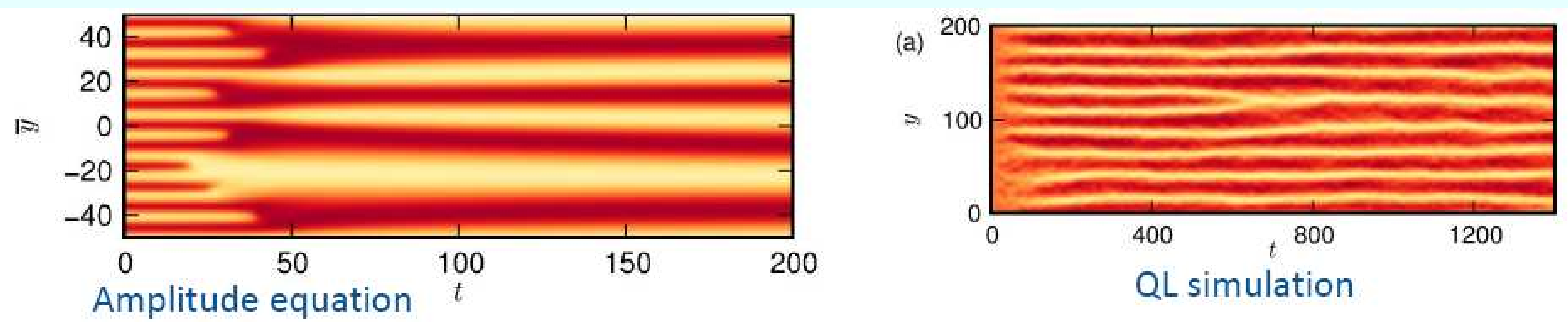
$$\mu U \sim qC, \quad 2\mu C \sim UC + F \tag{20}$$

or

$$\mu^{-1} qC^2 + 2\mu C - F = 0. \tag{21}$$

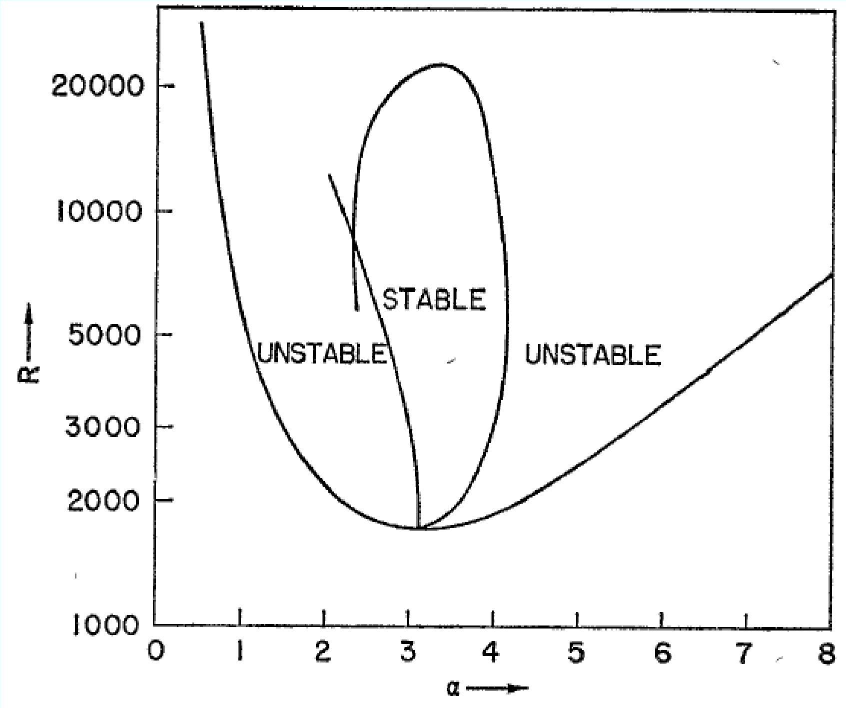
There is a solution for $q = 0$ (homogeneous turbulence) and *also* a continuous range of solutions for nonvanishing q. *(Cf. the real Ginzburg–Landau equation.)*

- Allows for the possibility of *merging jets*.



- The problem is completely analogous to the calculation of the *Busse stability balloon* for the Rayleigh–Bénard convection problem.

The Busse stability balloon delimits the stable modes inside the neutral curve.



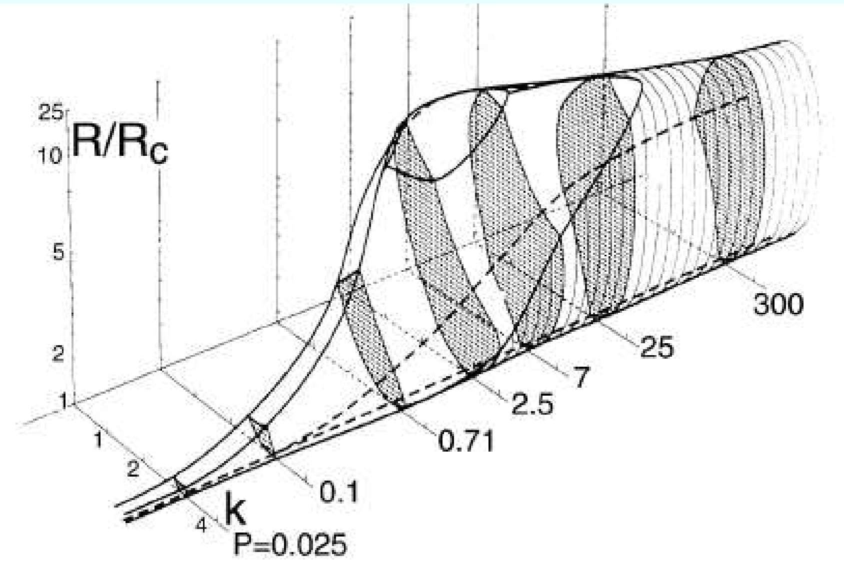


FIG. 5: Left: There is a continuous band of steady roll solutions inside the neutral curve, but only a subset is stable (delimited by the 'Busse stability balloon'). Right: The stability balloon is a volume in parameter space.

- We would like to do the analogous calculation for a relevant plasma model. Hasegawa–Wakatani? No; too complicated.

Calculation of the stability balloon for the generalized Hasegawa–Mima equation.

A prototypical vorticity equation (with forcing and dissipation) is

$$\partial_t \zeta + \vec{u} \cdot \vec{\nabla} \zeta - \kappa \partial_y \phi = \underbrace{\widetilde{f}}_{\text{random forcing}} - \underbrace{(\mu \zeta - \nu \nabla_\perp^2 \zeta)}_{\text{dissipation}}. \tag{22}$$

Various important physical models arise by specifying the relation between vorticity ζ and stream function (or potential) ϕ. In general,

$$\zeta = (\nabla_\perp^2 - \widehat{\alpha})\phi. \tag{23}$$

Particular cases (all with eddy shearing, but no damped eigenmodes):

- *barotropic vorticity equation* (infinite deformation radius): $\widehat{\alpha} = 0$.
- *quasigeostrophic equation* (finite deformation radius): $\widehat{\alpha} = L_d^{-2}$.
- *generalized Hasegawa–Mima equation (2D):*

$$\widehat{\alpha} = \begin{cases} 0 & \text{ZF mode } (k_y = 0) \\ 1 & \text{DW mode (otherwise).} \end{cases} \tag{24}$$

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The CE2 equations are nontrivial, even for just generalized Hasegawa–Mima.

$$\partial_t U(X,t) + (\mu - \nu\partial_X^2)U = -\frac{\partial}{\partial X}[\underbrace{\partial_x\partial_y C(0,0,X,t)}_{\text{Reynolds stress}}, \tag{25a}$$

$$\partial_t W(x,y \mid X,t) + (U_1 - U_2)\partial_y W - (U_1'' - U_2'')\left(\nabla_\perp^2 + \frac{1}{4}\partial_X^2\right)\partial_y C$$
$$- [2\kappa - (U_1'' + U_2'')]\partial_X\partial_y\partial_x C = \underbrace{F - \mathcal{D}[W]}_{\text{balance for homogeneous turbulence}}\,. \tag{25b}$$

Here
$C(\vec{x}_1, \vec{x}_2, t) \doteq \langle\delta\phi(\vec{x}_1,t)\delta\phi(\vec{x}_2,t)\rangle \equiv C(\underbrace{\vec{x}_1 - \vec{x}_2}_{\vec{x}} \mid \underbrace{\tfrac{1}{2}(\vec{x}_1 + \vec{x}_2)}_{\vec{X}})$

and $W(\vec{x}_1, \vec{x}_2, t) \doteq \langle\delta\zeta(\vec{x}_1,t)\delta\zeta(\vec{x}_2,t)\rangle = \nabla_{\perp,1}^2\nabla_{\perp,2}^2 C(\vec{x}_1,\vec{x}_2,t)$.

- First, calculate the neutral curve by extending the calculations of Srinivasan & Young (2012) for the barotropic vorticity equation.

Now calculate ZF equilibria for wave numbers lying inside the neutral curve.

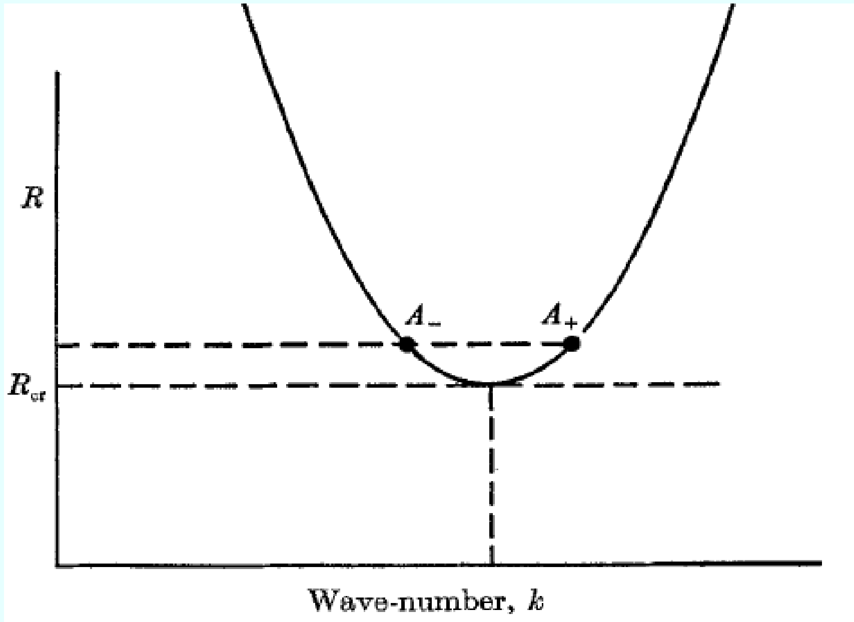


FIG. 6: Above criticality, a continuous band of equilibria lies inside the neutral curve.

- **Fourier expand the equations in all variables.**
- **Galerkin truncation.**
- **Solve numerically for the equilibria.** *(This is nontrivial.)*

Finally, calculate the stability of the equilibria.

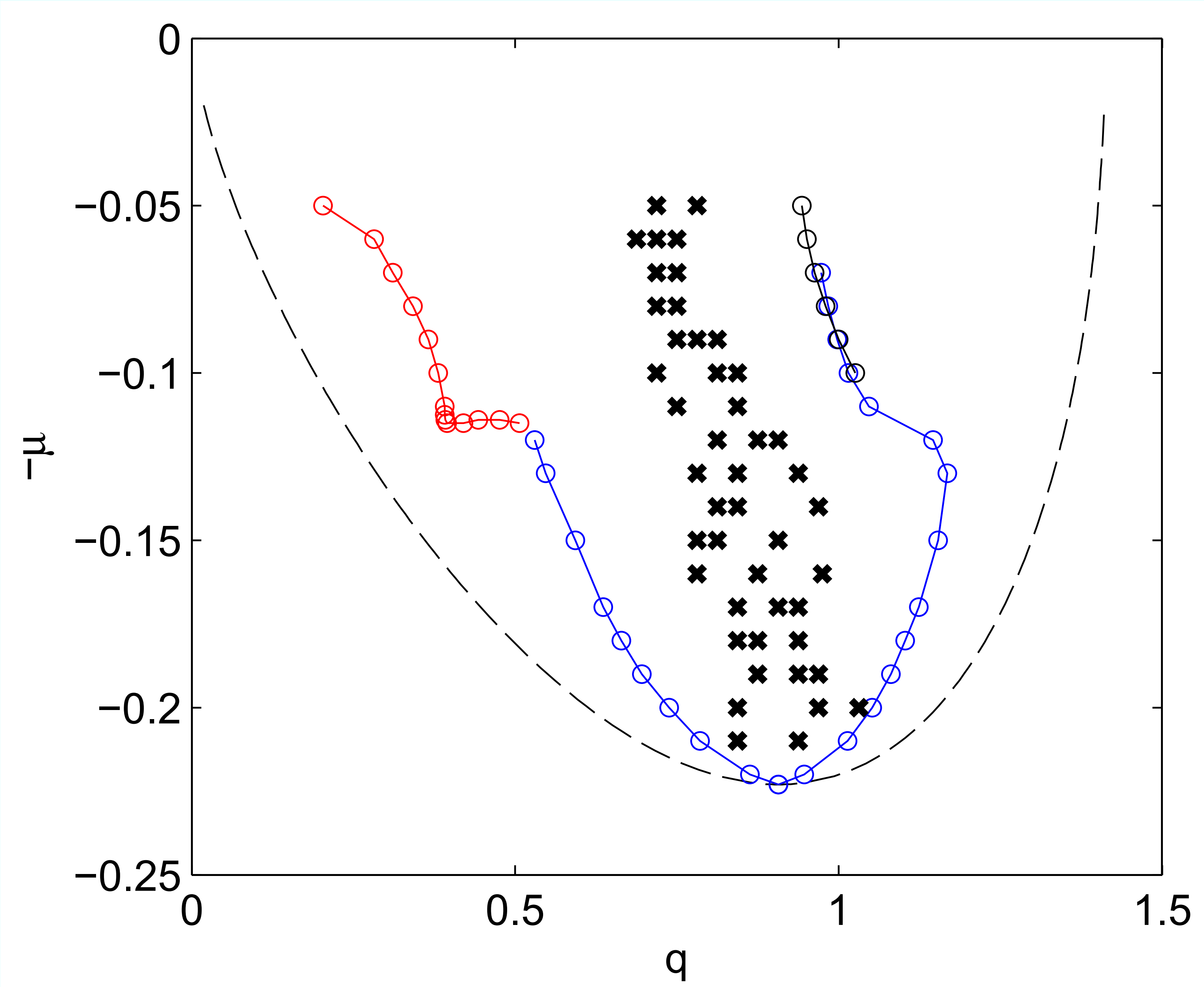
- Linearize around the ZF equilibrium; study perturbations.
- The equilibrium is periodic in X, so perturbations can be expanded as a Bloch state:

$$\Delta U(X,t) = e^{\sigma t} e^{iQX} \sum_p \Delta U_p e^{ipqX}, \tag{26a}$$

$$\Delta W(x,y \mid X,t) = e^{\sigma t} e^{iQX} \sum_{mnp} \Delta W_{mnp} e^{imax} e^{inby} e^{ipqX}. \tag{26b}$$

Look for eigenvalues $\sigma < 0$ $(\forall Q)$.

The stability balloon for the generalized HME is interesting and has not been completely analyzed.



Summary...

- Begin with primitive forced, dissipative nonlinear equation (e.g., generalized Hasegawa–Mima equation).
- Study from the point of view of 'stochastic structural stability'.
- I.e., study stability of homogeneous turbulent state; spontaneous symmetry breaking $\Rightarrow$ emergence of inhomogeneous ZFs.
- Model with inhomogeneous cumulant expansion (simplest: CE2).
- Examine stability of steady ZFs to secondary instabilities; calculate the Busse stability balloon.

The Busse stability balloon constrains the wavelength of the steady zonal flows that can be generated from turbulence.

- Implication: Analogous calculations may be useful in related problems (e.g., MRI turbulence in accretion discs).

Loose ends and future research directions. . .

- Need better physical understanding of the boundaries of the stability balloon.
- CE2 completely ignores eddy–eddy interactions.
- CE3 is better (Marston), although since it is not realizable one must be tricky.
- Anisotropic forcing may change the picture (Srinivasan).
- White-noise forcing is artificial. What about linear instability?
  - Nontrivial to deal with: With linear instability, need some kind of eddy–eddy term in order to permit a homogeneous steady state of turbulence.
  - Explore a closure model (simpler than inhomogeneous DIA!).

Future research (continued)...

- (For the fusion physicists:) How does the Dimits shift fit into the picture? We don't understand yet...
- Relation to critical balance arguments?
- Study a more complicated model in which eddy shearing and coupling to damped eigenmodes compete $\Rightarrow$ better understanding of physical systems in which zonal flows are self-consistently coupled to turbulence.

Future research (continued)...

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Our understanding of zonal flow generation is quite incomplete. It is a fertile area for further research in various fields (fusion, astrophysics, geophysics, etc.), but it's a difficult topic. Patience is a virtue.