Due May 1, 2024

Generals prep. Make sure you can provide brief explanations of the following: Spitzer–Härm problem; Chapman–Enskog expansion; parallel, gyro-, and perpendicular viscosity; pressure anisotropy; parallel, diamagnetic, and perpendicular heat flux. Be able to quickly figure out which species (electrons or ions) dominates each transport coefficient and why.

Another thing you ought to know for generals is how to do a linear analysis, obtain a dispersion relation, and analyze it. This problem should be good practice. Let's start with a review of Braginskii transport...

The single-fluid Braginskii-MHD equations for an ion-electron plasma include the continuity equation

$$\frac{\mathrm{d}n}{\mathrm{d}t} = -n\boldsymbol{\nabla}\cdot\boldsymbol{u}, \tag{1}$$

the momentum equation

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = -\frac{1}{mn}\boldsymbol{\nabla}\cdot\left(\mathbf{P} + \frac{B^2}{8\pi}\mathbf{I} - \frac{B^2}{4\pi}\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\right), \tag{2}$$

the induction equation

$$\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t} = \boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{u} - B\boldsymbol{\nabla}\cdot\boldsymbol{u}, \tag{3}$$

and the entropy equation

$$\frac{3}{2}p\frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{p}{n^{5/3}} = -\left(\mathbf{P} - p\mathbf{I}\right):\boldsymbol{\nabla}\boldsymbol{u} - \boldsymbol{\nabla}\cdot\boldsymbol{q}, \tag{4}$$

where $\mathrm{d}/\mathrm{d}t \doteq \partial/\partial t + \boldsymbol{u}\cdot\boldsymbol{\nabla}$ is the comoving (Lagrangian) time derivative, and $p \doteq (1/3)\,\mathrm{tr}\,\mathbf{P}$. The plasma is quasi-neutral, with number densities satisfying $n_{\mathrm{i}} = n_{\mathrm{e}} \doteq n$. Due to the smallness of the electron mass, $m \simeq m_{\mathrm{i}}$ in the momentum equation (2). The other symbols have their usual meanings, but it is important to note that summation over species is implied in the pressure tensor $\mathbf{P}$ and the heat flux $\boldsymbol{q}$:

$$\mathbf{P} = \mathbf{P}_{\mathrm{i}} + \mathbf{P}_{\mathrm{e}} \qquad \text{and} \qquad \boldsymbol{q} = \boldsymbol{q}_{\mathrm{i}} + \boldsymbol{q}_{\mathrm{e}}. \tag{5}$$

In a strongly magnetized plasma (*viz.*, $k\rho_{\mathrm{i}} \ll 1$ and $\omega \ll \Omega_{\mathrm{i}}$ for a disturbance of frequency ω and wavenumber k), the pressure tensor

$$\mathbf{P} = p_\perp\left(\mathbf{I} - \hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\right) + p_\parallel\,\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} = p\,\mathbf{I} - \left(\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} - \frac{\mathbf{I}}{3}\right)(p_\perp - p_\parallel) \tag{6}$$

and the heat flux

$$\boldsymbol{q} = q\hat{\boldsymbol{b}}. \tag{7}$$

That is, the transport of momentum and heat are *anisotropic* with respect to the magnetic-field direction. Cross-field transport is completely stifled by the particles' small gyroradii.

In the collisional limit (*viz.*, $k\rho_i \ll k\lambda_{\mathrm{mfp}} \ll 1$ and $\omega \ll \nu_{ii} \ll \Omega_i$), the pressure anisotropy is dominated by the contribution from the ions; it is given by

$$p_\perp - p_\parallel = \frac{3p_i}{\nu_i}\left(\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} - \frac{\mathsf{I}}{3}\right):\boldsymbol{\nabla u}, \tag{8}$$

where $\nu_i^{-1} \doteq 0.96\tau_{ii}$. The heat flux is dominated by the electrons (provided $T_i \sim T_e$):

$$q = -\frac{5}{2}\frac{p_e}{m_e\nu_e}\,\hat{\boldsymbol{b}} \cdot \boldsymbol{\nabla}T_e, \tag{9}$$

where $\nu_e^{-1} \doteq 1.26\tau_{ee}$. Both results are due to Braginskii (1965). In what follows, take $T_i = T_e$.

(a) Show that (3) implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln B = (\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} - \mathsf{I}):\boldsymbol{\nabla u} \tag{10}$$

and, therefore, that

$$p_\perp - p_\parallel = \frac{p_i}{\nu_i}\frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{B^3}{n^2} = \frac{p_i}{\nu_i}\left(\frac{\mathrm{d}}{\mathrm{d}t}\ln nB - \frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{n^3}{B^2}\right). \tag{11}$$

Thus, in a weakly collisional, magnetized plasma, the pressure anisotropy is set by a balance between adiabatic production ($p_\perp \propto nB$, $p_\parallel \propto n^3/B^3$) and collisional relaxation. I said this in class, but it's important enough to re-iterate it here.

Consider a stationary, spatially uniform plasma with $p_\perp = p_\parallel = nT$, threaded by a uniform magnetic field $\boldsymbol{B}_0$. For simplicity, drop the heat flux $\boldsymbol{q}$ in the entropy equation (4). For such an equilibrium, the linear dispersion relation governing small-amplitude plane-wave disturbances with frequency ω and wavevector $\boldsymbol{k} = k_\parallel\hat{\boldsymbol{b}} + \boldsymbol{k}_\perp$ is separable into Alfvén waves satisfying $\omega = \pm k_\parallel v_A$ and magnetosonic waves satisfying

$$\omega^2 + \mathrm{i}\omega\frac{k_\perp^2}{k^2}k_\parallel^2\mu - k_\parallel^2 v_A^2 = \frac{\left(\omega^2 + \mathrm{i}\omega\frac{2}{3}k_\parallel^2\mu\right)\left[k_\perp^2 v_A^2 + \mathrm{i}\omega\frac{k_\perp^2}{k^2}\left(k_\parallel^2 - \frac{k^2}{3}\right)\mu\right]}{\omega^2 - k^2 a^2 + \mathrm{i}\omega\frac{2}{3}\left(k_\parallel^2 - \frac{k^2}{3}\right)\mu}, \tag{12}$$

where

$$\text{(Alfvén speed)} \quad v_A^2 \doteq \frac{B_0^2}{4\pi mn},$$

$$\text{(ion thermal speed)} \quad v_{\mathrm{thi}}^2 \doteq \frac{2T_i}{m},$$

$$\text{(adiabatic sound speed)} \quad a^2 \doteq \frac{5}{3}v_{\mathrm{thi}}^2,$$

$$\text{(viscous diffusion coefficient)} \quad \mu \doteq \frac{3}{2}\frac{v_{\mathrm{thi}}^2}{\nu_i}.$$

I'm not asking you to derive this, but you should be able to if asked.

(b) Give a physical explanation for why linear Alfvén waves are not viscously damped. (While technically correct, "they satisfy $\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}:\boldsymbol{\nabla u} = 0$" is not a physical explanation.) Then, take the following limits of (12):

- $k^2 a^2 \to \infty$, to show that slow modes are viscously damped at a rate $k_\parallel^2 \mu\,(k_\perp^2/2k^2)$.
- $k_\perp = 0$, to show that undamped pseudo-Alfvén waves are decoupled from viscously damped sound waves.[1]
- $k_\parallel = 0$, to show that fast modes are damped at a rate $k_\perp^2 \mu/18$.

Explain each result physically (e.g., "Fast modes are viscously damped because...").

(c) Wait a second. I only count six roots: (1,2) forward- and backward-propagating Alfvén waves; (3,4) forward- and backward-propagating slow waves; and (5,6) forward- and backward-propagating fast waves. But there are seven time derivatives in equations (1)–(4). (One of the time derivatives in the induction equation is superfluous, being replaced by the solenoidality constraint on $\boldsymbol{B}$.) Where's the missing seventh root? (Hint: it's called the "entropy mode".)

(d) Now re-instate the heat flux in the entropy equation (4) and show that

$$\frac{\delta p}{p} = \frac{5}{3}\frac{\delta n}{n}\left[\frac{(3/5)k_\parallel^2\chi - (3/2)\mathrm{i}\omega}{k_\parallel^2\chi - (3/2)\mathrm{i}\omega}\right], \tag{13}$$

where $\chi \doteq (5/8)(v_{\mathrm{the}}^2/\nu_{\mathrm{e}})$. The resulting dispersion relation for the magnetosonic fluctuations is the same as (12), but with

$$a^2 \to a^2\left[\frac{(3/5)k_\parallel^2\chi - (3/2)\mathrm{i}\omega}{k_\parallel^2\chi - (3/2)\mathrm{i}\omega}\right]. \tag{14}$$

Take the fast-conduction limit, $\omega_{\mathrm{cond}} \doteq (2/5)k_\parallel^2\chi \gg \omega$. Take the slow-conduction limit, $\omega_{\mathrm{cond}} \ll \omega$. Do these limits give results that make sense? (Write more than, "yes".)

(e) Using equation (14) in the magnetosonic dispersion relation (12), we now see clearly that there are seven roots in total. What happened?

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[1] If the terminology "pseudo-Alfvén" is unfamiliar to you, a pseudo-Alfvén wave is a slow mode with $\omega^2 = k_\parallel^2 v_{\mathrm{A}}^2$; the two branches, Alfvén and slow, often (but not always) have the same dispersion relation when $k_\perp = 0$.