Due April 28, 2025

Generals prep. Make sure you can provide brief definitions of the following terms: Krook operator, Lenard–Bernstein (or Dougherty) operator, Lorentz operator, Rosenbluth potentials, runaway electrons, and Dreicer field. Also know $\nu_{\rm ee}, \nu_{\rm ei} : \nu_{\rm ii} : \nu_{\rm ie} = 1 : \sqrt{m_{\rm e}/m_{\rm i}} : m_{\rm e}/m_{\rm i}$.

1. **Fast ions and neutral-beam heating**. In this problem, you will explore an important and practical application of the Landau collision operator: the consequences of injecting an energetic beam of neutrals into a hot, ionizing plasma. In many tokamak experiments, such "neutral-beam injection" (NBI) is used to heat a magnetically confined plasma to thermonuclear temperatures. The basic atomic processes that involve the absorption of energetic neutrals by the plasma and the consequent production of fast ions are:

$$\begin{aligned} \text{charge exchange:} &\quad {\rm H}^0 + {\rm H}^+ \to {\rm H}^+ + {\rm H}^0 \\ \text{electron ionization:} &\quad {\rm H}^0 + e^- \to {\rm H}^+ + 2e^- \\ \text{proton ionization:} &\quad {\rm H}^0 + {\rm H}^+ \to {\rm H}^+ + {\rm H}^+ + e^-. \end{aligned}$$

(The energetic particle is identified in red.) For an injection energy of ∼50 keV per nucleon, these three processes are almost equally likely (cross section $\sigma \sim 10^{-16}\ {\rm cm}^2$). After the absorption of the injected energetic neutral by any of these processes, we are left with a fast ion and an electron. The electron heats up rapidly to the bulk electron thermal speed $v \sim v_{\rm th,e}$ due to its small mass. The fast ion joins other similarly produced fast ions to form a hot ion component (subscript h) with $v_{\rm thi} \ll v_{\rm h} \ll v_{\rm th,e}$. This problem guides you through a calculation of the properties of that hot ion distribution.

Assume the following: that the NBI is into a magnetic field that gyro-tropizes the velocity-space distribution function $f_{\rm h}$ of the hot ions (i.e., $f_{\rm h}$ is independent of gyrophase); that the background cold ions and electrons are unshifted Maxwellians; and that the density of hot ions $n_{\rm h}$ is small compared with the density of the background ions $n_{\rm i}$.[1] In this case, the hot-ion distribution function satisfies the test-particle form of the Fokker–Planck equation:

$$\frac{\partial f_{\rm h}}{\partial t} = C[f_{\rm h}] + S_{\rm h}, \tag{1}$$

where $C[f_{\rm h}]$ is the test-particle collision operator (to be determined by you below) and $S_{\rm h}$ is a source term describing the production of hot ions through NBI. The latter is modeled by

$$S_{\rm h} \doteq \dot{n}_{\rm h}\,\delta(\boldsymbol{v} - \boldsymbol{v}_0)\,H(t) = \frac{\dot{n}_{\rm h}}{2\pi v_0^2}\,\delta(v - v_0)\,\delta(\xi - \xi_0)\,H(t), \tag{2}$$

where $\dot{n}_{\rm h}$ is the production rate of hot ions, $\xi \doteq \cos\theta$ is the (cosine of the) pitch angle, and the Heaviside function $H(t)$ indicates that the source is turned on at $t = 0$.

[1] In truth, the hot ions impart both momentum and energy to the background plasma and thus can give rise to net motion (e.g., toroidal rotation in a tokamak with tangential injection). But let us neglect this here. The assumption that $n_{\rm h} \ll n_{\rm i}$ is a good one: few energetic particles are required to produce the desired heating.

(a) Use the assumptions given above alongside $v_{\rm th,i} \ll v_{\rm h} \ll v_{\rm th,e}$ to simplify terms in the Landau collision operator and thus obtain

$$\tau_{\rm s}\, C[f_{\rm h}] \approx \frac{m_{\rm i}}{m_{\rm h}}\frac{v_{\rm c}^3}{v^3}\,\mathcal{L}(f_{\rm h}) + \frac{1}{v^2}\frac{\partial}{\partial v}\big(v_{\rm c}^3 + v^3\big)\, f_{\rm h}, \tag{3}$$

where

$$\tau_{\rm s} \doteq \frac{m_{\rm i}}{m_{\rm h}}\frac{m_{\rm h}^2 v_{\rm c}^3}{4\pi Z_{\rm h}^2 e^4 n_{\rm i} \ln \lambda_{\rm hi}} \qquad \text{and} \qquad v_{\rm c} \doteq v_{\rm th,e}\left(\frac{3\sqrt{\pi}}{4}\frac{m_{\rm e}}{m_{\rm i}}\right)^{1/3}. \tag{4}$$

Explain your reasoning and all of your steps! What is the physical meaning of each of the three terms on the right-hand side of (3)?

To help guide you, fill out the following table using the ordering $v_{\rm thi} \ll v_{\rm h} \ll v_{\rm th,e}$ in the formulae for the collision rates for test particles scattering off of a Maxwellian background plasma (see pp. 121–122 of the lecture notes):

collision frequencies (with $\nu_0^{\rm he} \doteq 4\pi n_{\rm e} Z^2 e^4 \ln\lambda / m_{\rm h}^2 v_{\rm th,e}^3$)	$\beta = {\rm i}$	$\beta = {\rm e}$
slowing down ($\nu_{\rm s}^{{\rm h}\beta}/\nu_0^{\rm he}$)	???	???
perpendicular diffusion ($\nu_{\perp}^{{\rm h}\beta}/\nu_0^{\rm he}$)	???	???
parallel diffusion ($\nu_{\parallel}^{{\rm h}\beta}/\nu_0^{\rm he}$)	???	???
energy loss ($\nu_{\varepsilon}^{{\rm h}\beta}/\nu_0^{\rm he}$)	???	???

(b) Using (3) for the test-particle collision operator, solve (1) to show that

$$f_{\rm h}(v,\xi,t) = \frac{\dot{n}_{\rm h}\tau_{\rm s}}{2\pi(v^3+v_{\rm c}^3)}\sum_{\ell=0}^{\infty}\left(\ell+\frac{1}{2}\right)P_\ell(\xi)P_\ell(\xi_0)\left(\frac{v^3}{v_0^3}\frac{v_0^3+v_{\rm c}^3}{v^3+v_{\rm c}^3}\right)^{(m_{\rm i}/6m_{\rm h})\,\ell(\ell+1)}$$
$$\times\, H[t-\tau(v)]H(v_0 - v), \tag{5}$$

where $P_\ell(\xi)$ is the ℓth Legendre polynomial satisfying $\mathcal{L}[P_\ell] = -(\ell/2)(\ell+1)P_\ell$, and

$$\tau(v) \doteq \frac{\tau_{\rm s}}{3}\ln\left(\frac{v_0^3+v_{\rm c}^3}{v^3+v_{\rm c}^3}\right) \tag{6}$$

is the time taken for a hot ion to slow from speed v_0 to $0 < v < v_0$. [Hint: Expand $f_{\rm h}$ in Legendre polynomials and use

$$\delta(\xi-\xi_0) = \sum_{\ell=0}^{\infty}\left(\ell+\frac{1}{2}\right)P_\ell(\xi)P_\ell(\xi_0)$$

to expand $\delta(\xi-\xi_0)$ as well. You may further find it useful to define a new function $g_{\rm h} \doteq (v^3+v_{\rm c}^3)f_{\rm h}$ and then refresh your memory on how to solve inhomogeneous linear PDEs with non-constant coefficients using the method of characteristics. You're on the right track if you need to solve a separable ODE that looks like ${\rm d}w/{\rm d}t = -(v_{\rm c}^3+w^3)/(\tau_{\rm s} w^2)$ and if you eventually must use something like $\delta(w(t)-w(0)) = \delta(t)/|w'(0)|$. Please do not hesitate to ask Silvia or me if you need any further assistance!!!]

(c) Use (5) to show that the distribution function at late times for particles satisfying $v_0 > v \gg v_c$ is given by

$$f_{h,eq}(v) = \frac{\dot{n}_h \tau_s}{2\pi(v^3 + v_c^3)} H(v_0 - v)\delta(\xi - \xi_0). \quad (7)$$

This is called the "slowing-down distribution". Sketch it for $\xi = \xi_0$. In addition, use (5) to show that, for particles satisfying $v \ll v_c$, the late-time distribution is approximately isotropic (i.e., $\ell = 0$ represents the largest contribution).

(d) Take the energy moment $m_h v^2/2$ of (1) to obtain the following equation relating the power input by the hot ions Q_h into the background electrons and ions to the injection and evolution of the energy density of the hot ions:

$$\int d\boldsymbol{v} \, \frac{1}{2} m_h v^2 \frac{\partial f_h}{\partial t} \doteq \frac{d}{dt} n_h \varepsilon_h = -Q_{he} - Q_{hi} + \dot{n}_h \varepsilon_{h0} H(t), \quad (8)$$

where $\varepsilon_{h0} \doteq m_h v_0^2/2$ and

$$Q_{he} \doteq -\int d\boldsymbol{v} \, \frac{1}{2} m_h v^2 C[f_h, f_{Me}] = m_h \dot{n}_h \int_{v(t)}^{v_0} dv \, v \frac{v^3}{v^3 + v_c^3}, \quad (9a)$$

$$Q_{hi} \doteq -\int d\boldsymbol{v} \, \frac{1}{2} m_h v^2 C[f_h, f_{Mi}] = m_h \dot{n}_h \int_{v(t)}^{v_0} dv \, v \frac{v_c^3}{v^3 + v_c^3}. \quad (9b)$$

(This includes proving the final equalities in (9).)

(e) The plasma in the Joint European Torus (JET) attains densities $\sim 10^{14}$ cm^{-3} and temperatures ~ 10 keV. Neutral beams on JET are at 110–140 keV. Estimate v_c for JET parameters. Using (9a,b) as a guide, is the power flowing predominantly into the background ions or into the electrons? Use (6) to compute the time required for a hot ion to slow down from an initial velocity v_0 to a final velocity $v_f \doteq \sqrt{3T_i/m_i}$. Take $Z_h = 1$, $m_h = m_i = 3670 m_e$, and $\ln \lambda = 17$.