AST 554 Spring 2025 **Homework #5: Langevin'ing**

Due Apr 7, 2025

1. **Langevin dynamics in a magnetic field** (after Krommes Ex. 6.8). Consider random, sub-Debye-scale, electrostatic fluctuations $\delta\boldsymbol{E}(t)$ in the presence of a weak ($\Omega/\omega_{\rm p} \ll 1$), straight magnetic field $\boldsymbol{B}_0 = B_0\hat{\boldsymbol{z}}$, where B_0 is a constant. The classical Langevin equations for a test particle with charge q and mass m moving in these fields are

$$\frac{{\rm d}\boldsymbol{r}}{{\rm d}t} = \boldsymbol{v}, \tag{1a}$$

$$\frac{{\rm d}\boldsymbol{v}}{{\rm d}t} = -\gamma\boldsymbol{v} + \Omega\boldsymbol{v}\times\hat{\boldsymbol{z}} + \boldsymbol{a}(t), \tag{1b}$$

where γ is the drag coefficient, $\Omega \equiv qB_0/mc$ is the cyclotron frequency, and $\boldsymbol{a}(t) = (q/m)\delta\boldsymbol{E}(t)$ is the random acceleration due to the electrostatic fluctuations. In the spirit of Langevin's approach (in which the time axis is coarse grained into time intervals much larger than $\omega_{\rm p}^{-1}$, the transit time through a Debye cloud), one may assume that $\boldsymbol{a}$ is Gaussian white noise:

$$\langle\boldsymbol{a}(t)\rangle = \boldsymbol{0}, \tag{2a}$$

$$\langle\boldsymbol{a}(t)\boldsymbol{a}^{\mathsf{T}}(t')\rangle = \varepsilon\delta(t-t')\mathbf{I}, \tag{2b}$$

where T denotes the transpose and $\mathbf{I}$ is the unit dyadic. Remember that the brackets indicate an *ensemble average* over many individually deterministic realizations of the system. Thus, the position $\boldsymbol{r}$ and velocity $\boldsymbol{v}$ of the test particle are random variables.

(a) One expects that at long times ($\gamma t \gg 1$) the particle will diffuse in space, although it may diffuse at different rates in the directions parallel ($\|$) and perpendicular ($\perp$) to $\boldsymbol{B}_0$. Argue (no mathematics) that the parallel spatial diffusion coefficient is the same one that was calculated in class for the one-dimensional Langevin equation. Then use the simplest random-walk formula you can think of to predict how the cross-field diffusion coefficient ought to scale with γ and B_0 in the physically relevant regime $\gamma/\Omega \ll 1$. Hopefully this scaling will agree with the detailed formula you derive in this problem.

(b) Solve for the random velocity of the particle $\boldsymbol{v}(t)$ in terms of $\boldsymbol{a}(t)$. There are several ways to do this, but my preferred method is to first solve

$$\frac{\partial\mathbf{G}}{\partial t} + \gamma\mathbf{G} - \Omega\mathbf{G}\times\hat{\boldsymbol{z}} = \delta(t-t')\mathbf{I} \tag{3}$$

for the (matrix) Green's function $\mathbf{G}(t;t')$ and then write

$$\boldsymbol{v}(t) = \mathbf{G}(t;0)\boldsymbol{\cdot}\boldsymbol{v}(0) + \int_0^t {\rm d}t'\,\mathbf{G}(t;t')\boldsymbol{\cdot}\boldsymbol{a}(t'). \tag{4}$$

You should find that

$$\mathbf{G}(t;t') = {\rm e}^{-\gamma(t-t')}\begin{pmatrix} \cos[\Omega(t-t')] & \sin[\Omega(t-t')] & 0 \\ -\sin[\Omega(t-t')] & \cos[\Omega(t-t')] & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5}$$

Compute the correlations $\langle v_\parallel(t)\rangle$, $\langle \boldsymbol{v}_\perp(t)\rangle$, $\langle \delta v_\parallel^2(t)\rangle$, and $\langle \delta\boldsymbol{v}_\perp(t)\delta\boldsymbol{v}_\perp^\mathsf{T}(t)\rangle$, where $\delta\boldsymbol{v} \equiv \boldsymbol{v} - \langle\boldsymbol{v}\rangle$ is the departure in the velocity from its mean value.

The rotation matrix $\mathsf{R}(\theta)$ has the following properties:

$$\mathsf{R}^\mathsf{T}(\theta) = \mathsf{R}^{-1}(\theta) = \mathsf{R}(-\theta) \qquad \text{and} \qquad \mathsf{R}(\theta_1)\mathsf{R}(\theta_2) = \mathsf{R}(\theta_1+\theta_2).$$

I just thought I'd say that.[1]

(c) At this point, it should be clear that the solution in the direction parallel to the magnetic field is the same as the solution to the standard one-dimensional Langevin problem. So let's concentrate on the perpendicular dynamics. Integrate $\delta\boldsymbol{v}_\perp(t)$ to find an expression for the perpendicular displacement

$$\delta\boldsymbol{r}_\perp(t) \equiv \int_0^t \mathrm{d}t'\,\delta\boldsymbol{v}_\perp(t') = \int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t''\,\mathsf{G}(t';t'')\boldsymbol{\cdot}\boldsymbol{a}_\perp(t'') = \,?\,. \tag{6}$$

Using your answer, compute

$$\langle\delta\boldsymbol{r}_\perp(t)\delta\boldsymbol{r}_\perp^\mathsf{T}(t)\rangle = \int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' \int_0^t \mathrm{d}s' \int_0^{s'} \mathrm{d}s''\,\mathsf{G}(t';t'')\boldsymbol{\cdot}\langle\boldsymbol{a}_\perp(t'')\boldsymbol{a}_\perp^\mathsf{T}(s'')\rangle\boldsymbol{\cdot}\mathsf{G}^\mathsf{T}(s';s''). \tag{7}$$

Because I don't want to you spend a huge amount of time chugging through the math, here's a generous hint. Be *very* careful that your integration limits ensure causality. This means that, at some point in the calculation, you'll want to interchange integrals so that the integral over t'' in equation (7) runs from 0 to t (instead of from 0 to t') and the accompanying integral over t' then runs from t'' to t (instead of from 0 to t'). Ditto for the integrals over s' and s''. If you take this bit of information for granted and use it, you should at least justify this interchange via some illustrative drawing. After this, to perform the new t' and s' integrals, you'll want to switch to integration variables similar to those used at the bottom of pg. 99 of the lecture notes: $\tau \equiv t' - s'$ and $T \equiv (t'+s')/2$. Again, you should at least justify this change of variables via some illustrative drawing (e.g., like the one at the top of pg. 100 of the notes).

Here's the final answer:

$$\begin{aligned}\langle\delta\boldsymbol{r}_\perp(t)\delta\boldsymbol{r}_\perp^\mathsf{T}(t)\rangle = \frac{\varepsilon}{\gamma^2+\Omega^2}\biggl[t + \frac{1-\mathrm{e}^{-2\gamma t}}{2\gamma} \\ - \frac{2\gamma}{\gamma^2+\Omega^2}\biggl(1 - \mathrm{e}^{-\gamma t}\cos\Omega t + \frac{\Omega}{\gamma}\,\mathrm{e}^{-\gamma t}\sin\Omega t\biggr)\biggr]\mathsf{I}_\perp,\end{aligned} \tag{8}$$

where $\mathsf{I}_\perp \doteq \mathsf{I} - \hat{\boldsymbol{z}}\hat{\boldsymbol{z}}$.

(d) Define the running diffusion tensor by

$$\mathsf{B}(t) \doteq \frac{\mathrm{d}\langle\delta\boldsymbol{r}(t)\,\delta\boldsymbol{r}^\mathsf{T}(t)\rangle}{\mathrm{d}t}. \tag{9}$$

[1] You may be wondering... cross a tensor into a vector? How do I do that?! I found the following website extremely useful on this topic: `https://en.wikiversity.org/wiki/Introduction_to_Elasticity/Tensors`. It writes out a lot of these tensor-tensor and tensor-vector combinations in component form. For example, the ij-component of a tensor G crossed into a vector $\boldsymbol{v}$ is $\epsilon_{kmj}G_{ik}v_m$, where summation over repeated indices is implied. So, $(\mathsf{G}\times\hat{\boldsymbol{z}})_{ij} = G_{ik}\,\epsilon_{kzj}$.

Use (8) to obtain an explicit, simple formula for $\mathbf{B}$ in the *long-time limit*. (Don't do more calculation than is necessary! My solution is just two lines.) It should be of the form

$$v_{\mathrm{th}}\lambda_{\mathrm{mfp}}\Big[\text{something}\,\hat{\boldsymbol{z}}\hat{\boldsymbol{z}} + \text{something else}\,\mathbf{I}_\perp\Big], \tag{10}$$

where $v_{\mathrm{th}}^2 = \varepsilon/\gamma$ from the fluctuation-dissipation theorem and $\lambda_{\mathrm{mfp}} = v_{\mathrm{th}}/\gamma$ is the mean free path. Is your answer compatible with the heuristic scalings you predicted in the first part? If not, start over.

2. **Vlasov Langevin equation in Hermite space**. It is now time to revisit HW02 #2, named "Landau damping via Hermite polynomials". There you derived the following set of linear equations coupling various Hermite moments $m = 0, 1, 2, \ldots$ of the perturbed distribution function $g(t, z, v)$:

$$\frac{\partial g_0}{\partial t} + \frac{\partial}{\partial z}\frac{g_1}{\sqrt{2}} = 0, \tag{11a}$$

$$\frac{\partial g_1}{\partial t} + \frac{\partial}{\partial z}\left(g_2 + \frac{1}{\sqrt{2}}g_0\right) + \frac{\partial}{\partial z}\frac{\varphi(t,z)}{\sqrt{2}} = 0, \tag{11b}$$

$$\frac{\partial g_m}{\partial t} + \frac{\partial}{\partial z}\left(\sqrt{\frac{m+1}{2}}g_{m+1} + \sqrt{\frac{m}{2}}g_{m-1}\right) = -\nu m g_m, \quad m \geq 2. \tag{11c}$$

To remind you of the physical content of these equations: (11a) is the continuity equation, representing the transport of density fluctuations by momentum fluctuations; (11b) is the momentum equation, showing that density fluctuations create potential fluctuations ($\varphi = \alpha g_0$), which accelerate particles (the final term in (11b)) and thus lead to Landau damping; then small-scale structure in velocity space is generated via phase mixing, which manifests as a conservative cascade to higher m in Hermite space (eqns 11b,11c); eventually, small enough scales (i.e., large enough m) are produced that collisions become important (the right-hand side of (11c)). Irreversibility occurs. During this velocity-space cascade of free energy from the low ("fluid") moments of g to the high ("kinetic") moments, the following quadratic quantity is conserved for each wavenumber k:

$$W_k = \frac{1+\alpha}{2}|g_{0,k}|^2 + \frac{1}{2}\sum_{m=1}^{\infty}|g_{m,k}|^2 \tag{12}$$

(Recall that each k is independent of the others because the equations are linear.) Once sufficiently small structure in velocity-space is produced, W_k decays due to collisions:

$$\frac{\mathrm{d}W_k}{\mathrm{d}t} = -\nu\sum_{m=2}^{\infty} m|g_{m,k}|^2. \tag{13}$$

Hopefully you remembered all of that, because now we're going to add a stochastic forcing to the right-hand side of (11a) representing the injection of energy into density fluctuations:

$$\frac{\partial g_0}{\partial t} + \frac{\partial}{\partial z}\frac{g_1}{\sqrt{2}} = \chi(t,z) = \sum_k \chi_k(t)\exp(\mathrm{i}kz). \tag{14}$$

In the spirit of Langevin, take the forcing to be Gaussian white noise, whose ensemble average $\langle \chi_k(t)\chi_k^*(t')\rangle = \varepsilon_k\, \delta(t-t')$. Now that you know a few things about Langevin equations, let's have some fun.

(a) Integrate (14) over time to show that $\langle g_{0,k}(t)\chi_k^*(t)\rangle = \varepsilon_k/2$. Use this to show that, in the presence of this forcing, equation (13) is replaced by

$$\frac{\mathrm{d}W_k}{\mathrm{d}t} = \frac{1+\alpha}{2}\varepsilon_k - \nu \sum_{m=2}^{\infty} m\langle |g_{m,k}|^2\rangle. \tag{15}$$

(b) Equations (11a)–(11c) were obtained from Hermite-transforming the following equations (now including the stochastic forcing of density fluctuations):

$$\frac{\partial g}{\partial t} + v\frac{\partial g}{\partial z} + vF_0\frac{\partial \varphi}{\partial z} = \chi(t,z)F_0 + C[g], \tag{16a}$$

$$\varphi = \alpha \int_{-\infty}^{\infty} \mathrm{d}v\, g, \tag{16b}$$

where $F_0(v) = \exp(-v^2)/\sqrt{\pi}$. Ignoring collisions and Fourier-transforming in space and time, show that the steady-state mean-square fluctuation level in the plasma is

$$\langle |\varphi_k|^2\rangle = \alpha^2\langle |g_{0,k}|^2\rangle = \frac{\varepsilon_k}{2\pi|k|}\int_{-\infty}^{\infty}\mathrm{d}\zeta \left|\frac{Z(\zeta)}{\mathcal{D}_\alpha(\zeta)}\right|^2, \tag{17}$$

where $\zeta = \omega/|k|$, $Z(\zeta) = \int_{-\infty}^{\infty}\mathrm{d}v\, F_0/(v-\zeta)$ is the plasma dispersion function, and

$$\mathcal{D}_\alpha(\zeta) = 1 + \frac{1}{\alpha} + \zeta Z(\zeta) \tag{18}$$

is the dielectric function. Equation (17) is the fluctuation-dissipation relation for the kinetic system (16); given an amount of energy injection ε and a channel of dissipation controlled by α, it predicts the long-time behavior of the electrostatic fluctuations. Again, note that, because of the linearity of (16), there is no coupling between different wavenumbers.

(c) Let $\alpha + 1 \ll 1$, so that the dispersion relation $\mathcal{D}_\alpha(\zeta) = 0$ implies $\zeta \ll 1$. Expand the Z function in its small argument to show that the mode is aperiodic, i.e., it is a purely damped mode:

$$\omega \approx -\mathrm{i}\gamma_{\mathrm{L}}, \quad \text{where} \quad \gamma_{\mathrm{L}} = \frac{1+\alpha}{\sqrt{\pi}}|k| \tag{19}$$

is the rate of collisionless (Landau) damping. A physical example of damping in this regime is the Barnes (1966) damping of compressive fluctuations in high-beta plasmas, for which $1+\alpha \approx 1/\beta_{\mathrm{i}} \ll 1$ (Schekochihin et al. 2009, their equation (190); see also the discussion in the problem setup for HW02 #2). From this, use (17) to show that

$$\langle |\varphi_k|^2\rangle \approx \frac{\varepsilon}{2\gamma_{\mathrm{L}}}. \tag{20}$$

This should look quite familiar from the lectures on the standard Langevin equation, which (in the notation of this problem) reads $\partial\varphi/\partial t + \gamma\varphi = \chi(t)$.

This is a brief educational aside about the opposite limit, $\alpha \gg 1$, which corresponds physically to Landau damping of ion acoustic waves (for $\beta_{\rm i} \ll 1$ and $T_{\rm i}/T_{\rm e} \ll 1$, so that $\alpha \simeq T_{\rm e}/T_{\rm i}$) and of long-wavelength Langmuir waves (for which $\alpha \simeq 2/k^2\lambda_{\rm D}^2$). In this case, one may expand the Z function in its large argument to show that the mode is rapidly oscillating and weakly damped:

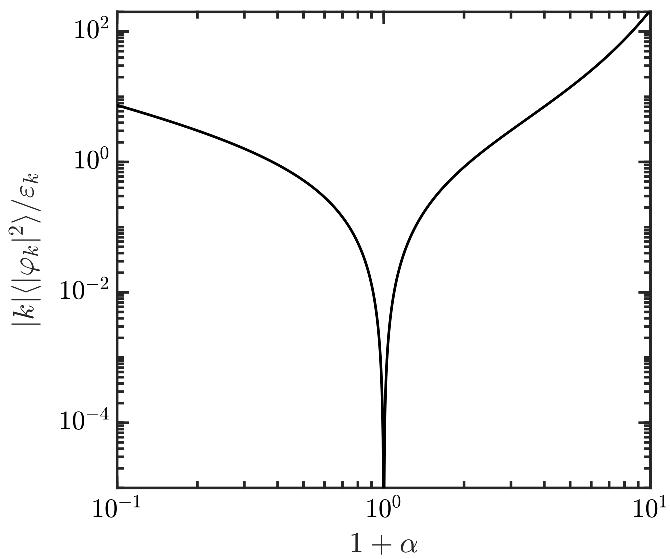
$$\omega \approx \pm\sqrt{\frac{\alpha}{2}}|k| - {\rm i}\gamma_{\rm L}, \quad \text{where} \quad \gamma_{\rm L} = \sqrt{\pi}\frac{\alpha^2}{4}{\rm e}^{-\alpha/2}|k|. \tag{21}$$

From this, one may use (17) to show that

$$\langle|\varphi_k|^2\rangle \approx \frac{\alpha^2\varepsilon_k}{4\gamma_{\rm L}}. \tag{22}$$

This system being only weakly damped, the steady-state fluctuation level is larger than found in part (c). This is actually the same fluctuation-dissipation theorem as that of a Langevin harmonic oscillator governed by the stochastic differential equation $\ddot{\varphi} + \gamma\dot{\varphi} + \omega_0^2\varphi = \dot{\chi}(t)$. This formed the basis for an Irreversibles question on 2023's General Examination, in which this Langevin harmonic oscillator was used as a model to determine the steady-state electrostatic fluctuation level in a weakly coupled plasma in thermal equilibrium.

Here's the fluctuation level versus $1+\alpha$, obtained by performing the integral in (17) numerically:



(d) My guess is that, up to this point, you haven't actually derived an explicit solution for $g_k(t)$. Now do so. You should be able to coax it into the following form:

$$g_k(t) = \int_{-\infty}^{\infty}\frac{{\rm d}\omega}{2\pi}\,g_k(\omega){\rm e}^{-{\rm i}\omega t} = \int_{-\infty}^{\infty}\frac{{\rm d}\omega}{2\pi}\,{\rm e}^{-{\rm i}\omega t}\,\frac{{\rm i}\chi_{k\omega}}{|k|}\left[Z(\zeta) - \frac{1+\alpha}{\alpha}\frac{{\rm sgn}\,k}{v-\omega/k}\right]\frac{F_0}{D_\alpha(\zeta)}. \tag{23}$$

Hermite transform this equation to find

$$g_{m,k}(t) = -\int_{-\infty}^{\infty}\frac{{\rm d}\omega}{2\pi}\,{\rm e}^{-{\rm i}\omega t}\,\frac{{\rm i}\chi_{k\omega}}{|k|}\frac{1+\alpha}{\alpha}\frac{(-{\rm sgn}\,k)^m}{\sqrt{2^m m!}}\frac{Z^{(m)}(\zeta)}{D_\alpha(\zeta)}, \quad m \geq 1, \tag{24}$$

where

$$Z^{(m)}(\zeta) \equiv \frac{{\rm d}^m Z}{{\rm d}\zeta^m} = (-1)^m\int{\rm d}v\,\frac{H_m(v)F_0(v)}{v-\zeta}. \tag{25}$$

(You must be very careful about k vs. $|k|$.) Now, use (24) to show that the mean square fluctuation level in the statistical steady state of each Hermite moment m is given by

$$C_{m,k} \equiv \langle |g_{m,k}|^2 \rangle = \frac{\varepsilon_k}{2\pi|k|} \left(\frac{1+\alpha}{\alpha}\right)^2 \frac{1}{2^m m!} \int_{-\infty}^{\infty} \mathrm{d}\zeta \left| \frac{Z^{(m)}(\zeta)}{\mathcal{D}_\alpha(\zeta)} \right|^2, \quad m \geq 1. \tag{26}$$

The Hermite spectrum (26) characterizes the distribution of free energy in phase space. It is the $m \geq 1$ version of (17).

Equation (26) in its general form is difficult to parse. It's clear that having $1 + \alpha \ll 1$ results in a lower fluctuation level than does $\alpha \gg 1$. This makes sense from (15), since sending $\alpha \to -1$ reduces the effective injection rate of free energy into the plasma by the driving. But let's see if we can't clean (26) up to make it more physically instructive. For that, let us wade into the "inertial range" of the velocity-space cascade, at m's far enough away from both the driving scale and the collisional scale. You will see below that this corresponds to $1 \ll m \ll (k\lambda_{\mathrm{mfp}})^{2/3}$, where λ_{mfp} is the mean free path.

(e) **This part (e) is optional!** A quick glance online reveals that, for $m \gg 1$,

$$\mathrm{e}^{-v^2/2} H_m(v) \approx \left(\frac{2m}{\mathrm{e}}\right)^{m/2} \sqrt{2} \cos\left(v\sqrt{2m} - \pi m/2\right). \tag{27}$$

In this limit, the Hermite transform is somewhat analogous to a Fourier transform in velocity space with "frequency" $\sqrt{2m}/v_{\mathrm{th}}$ (restoring velocity-space units).
Show that, assuming $m \gg 1$ and $\zeta \ll \sqrt{2m}$,

$$Z^{(m)}(\zeta) \approx \sqrt{2\pi}\, \mathrm{i}^{m+1} \left(\frac{2m}{e}\right)^{m/2} \exp(\mathrm{i}\zeta\sqrt{2m} - \zeta^2/2) \tag{28}$$

and, therefore, that

$$C_{m,k} \approx \left[\frac{\varepsilon}{\sqrt{2\pi}\,|k|} \left(\frac{1+\alpha}{\alpha}\right)^2 \int_{-\infty}^{\infty} \frac{\mathrm{d}\zeta\, \mathrm{e}^{-\zeta^2}}{|\mathcal{D}_\alpha(\zeta)|^2} \right] \frac{1}{\sqrt{m}} = \frac{\varepsilon(1+\alpha)}{\sqrt{2}|k|} \frac{1}{\sqrt{m}}. \tag{29}$$

This requires use of Stirling's formula and some manipulation of contours in the complex plane. To evaluate the integral in (29), use can the same technique as was used in class to calculate the electrostatic field fluctuation spectrum in an equilibrium plasma; namely, exploit the fact that $\mathcal{D}_\alpha(\zeta)$ has no poles in the upper-half plane and do a small clockwise loop above $\zeta = 0$ and a large counter-clockwise loop to close the contour. (This is equivalent to using the Kramers-Kronig relations.)

(f) **(This part is required!)** Equation (29) reveals that the *Hermite spectrum* of the free energy $C_{m,k} \propto 1/\sqrt{m}$ is shallow and, in particular, that the total free energy diverges. This is an indication that it must be regulated via collisions, no matter how small is ν. (Remember that larger m corresponds to finer structure in velocity space.) To that end, it is possible to show (though you need not) that, for $m \gg 1$, equation (11c) implies

$$\frac{\partial C_{m,k}}{\partial t} + |k| \frac{\partial}{\partial m} \sqrt{2m} C_{m,k} = -2\nu m C_{m,k}. \tag{30}$$

Thus, the $C_{m,k} \propto 1/\sqrt{m}$ spectrum represents a *constant flux* of free energy in velocity space; this is directly analogous to the constant flux of energy in wavenumber space that goes into the Kolmogorov turbulence spectrum $\propto k^{-5/3}$.

By integrating (30) in steady state ($\partial/\partial t = 0$) and using the steady-state relation

$$\frac{1+\alpha}{2}\varepsilon_k = \nu \sum_{m=2}^{\infty} m C_{m,k} \tag{31}$$

obtained from (15), show that the steady-state Hermite spectrum in the range $m \gg 1$ is given by

$$C_{m,k} = \frac{\varepsilon_k(1+\alpha)}{\sqrt{2m}|k|} \exp\left(-\frac{2\sqrt{2}}{3}\frac{\nu}{|k|}m^{3/2}\right). \tag{32}$$

Thus, show that the cascade of free energy to small scales in velocity space is exponentially cut off at (restoring dimensions)

$$m_{\rm c} \sim (|k|v_{\rm th}/\nu)^{2/3} \tag{33}$$

and, furthermore, that the free-energy dissipation rate $\nu \int^{\infty} {\rm d}m\, m C_{m,k}$ is finite, no matter how small is the collision frequency(!) In other words, *entropy is produced at a rate that is independent of the collision frequency!*

This is the velocity-space analogue of Kolmogorov turbulence: there is a power-law spectrum in the inertial range ($k^{-5/3}$); the accompanying cascade of free energy to small scales in real space is exponentially cut off at the viscous scale; and entropy is produced at a rate that is independent of the viscosity, no matter how small its value.

You can find some code named `landau_langevin.f90` at

`https://www.astro.princeton.edu/~kunz/Site/AST554`

which is based on the code you played with in HW02. This one solves the same set of implicit equations as the other one but with a white-noise forcing χ_k in the equation for $g_{0,k}$:

$$\frac{g_{0,k}^{(n+1)} - g_{0,k}^{(n)}}{\Delta t} = -{\rm i}k\,\frac{g_{1,k}^{(n+1)}}{\sqrt{2}} + \chi_k^{(n)},$$

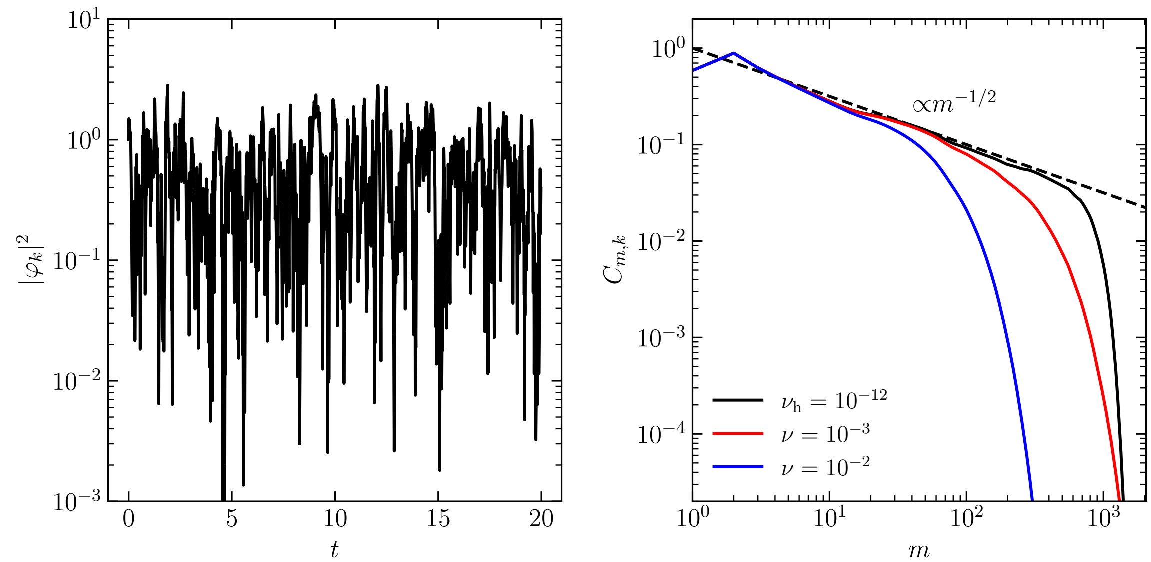
$$\frac{g_{1,k}^{(n+1)} - g_{1,k}^{(n)}}{\Delta t} = -{\rm i}k\left(g_{2,k}^{(n+1)} + \frac{1+\alpha}{\sqrt{2}}g_{0,k}^{(n+1)}\right),$$

$$\frac{g_{m,k}^{(n+1)} - g_{m,k}^{(n)}}{\Delta t} = -{\rm i}k\left(\sqrt{\frac{m+1}{2}}g_{m+1,k}^{(n+1)} + \sqrt{\frac{m}{2}}g_{m-1,k}^{(n+1)}\right) - \left(\nu m + \nu_{\rm h} m^4\right) g_{m,k}^{(n+1)}, \quad 2 \le m \le N_{\rm m}$$

(The forcing scheme starts on line 192.) The default free parameters in this code are as follows: $N_m = 2048$, $\alpha = 1$, $k = 2\pi$, $\nu = 0$, $\nu_{\rm h} = 10^{-12}$, and $g_{0,k}^{(0)} = 1$ (which is arbitrary, because the equations are linear). I've also tried $\nu = 10^{-3}$ and $\nu = 10^{-2}$, both with $\nu_{\rm h} = 0$. You can compile the code using

```
gfortran -o landau_langevin landau_langevin.f90
```

and play with it a bit. Executing the python script given at the end of this problem set produces the plots of $|\varphi_k|^2$ vs time and $C_{m,k}$ vs m shown below. For $\alpha = 1$, the predicted $\langle|\varphi_k|^2\rangle = \varepsilon_k/2\gamma_{\rm eff}$ with $\gamma_{\rm eff} = 0.71|k|$. In the code, the default driving uses $\varepsilon_k = 4$, so that the predicted $\langle|\varphi_k|^2\rangle \simeq 0.45$ for $k = 2\pi$. Examining the plot on the left, this seems about right! The plot on the right shows the $m^{-1/2}$ spectrum, with an exponential cutoff that seems to scale as $\nu^{-2/3}$, just as predicted. Nice!



```
import matplotlib.pyplot as plt
import matplotlib as mpl
import numpy as np

font = 13 ; mpl.rc('text', usetex=True)
mpl.rcParams['text.latex.preamble']=r"\usepackage{amsmath}"
mpl.rc('font', family = 'serif', size = font)

fname = 'landau_langevin.phik2'
data = np.genfromtxt(fname,autostrip=True)
t= data[:,0] ; phik2 = data[:,1]

fig1=plt.figure(figsize=(4,4))
axes = fig1.add_axes([0.18,0.13,0.79,0.84])
axes.semilogy(t,phik2,'k')
axes.tick_params(which='both',direction='in',top=True,right=True)
axes.tick_params(which='major',length=5)
axes.tick_params(which='minor',length=3)
plt.xlabel(r"$t$",fontsize=font)
plt.ylabel(r"$|\varphi_k|^2 $",fontsize=font)
plt.ylim(1e-3,1e1)
plt.show()

fname = 'landau_langevin.gkm'
data = np.genfromtxt(fname,autostrip=True)
time = data[:,0] ; mlab = data[:,1]
gkmr = data[:,2] ; gkmi = data[:,3]

nm = 2048 ; nt = int(time.size/nm) ; spec = np.zeros((nm,nt))
mm = np.arange(1,nm+1) ; indx = np.arange(0,nt)*nm
for m in range(nm):
    spec[m,:]  = (gkmr[indx+m])**2 + (gkmi[indx+m])**2

fig2=plt.figure(figsize=(4,4))
ax = fig2.add_axes([0.18,0.13,0.79,0.84])
ax.loglog(mm,np.mean(spec[:,100:nt-1],1),'k',label=r'$\nu_h=10^{-12}$')
ax.loglog(mm,mm**(-0.5),'--k')
ax.tick_params(which='both',direction='in',top=True,right=True)
ax.tick_params(which='major',length=5)
ax.tick_params(which='minor',length=3)
plt.xlim(1,2048) ; plt.ylim(2e-5,2e0)
plt.text(40,0.22,'$\propto m^{-1/2}$')
plt.xlabel(r"$m$",fontsize=font)
plt.ylabel(r"$C_{m,k}$",fontsize=font)
plt.legend(frameon=False,loc='lower left',prop="size":12)
plt.show()
```