Due March 24, 2025

Generals prep. Make sure you can provide brief definitions of the following terms: test-particle superposition principle, polarization drag, Bremsstrahlung. Also, know 1 eV $\sim 10^4$ K

1. **Cerenkov wakes from moving test charges**. In class, we showed that a moving test charge provokes a polarization response from the background plasma through which it travels. This response dresses the test particle, so that the steady-state potential

$$\varphi(t,\boldsymbol{r}) = \frac{q_\mathrm{T}}{2\pi^2}\int \mathrm{d}\boldsymbol{k}\,\frac{\exp[\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{R}_0-\boldsymbol{V}_0 t)]}{k^2\,\mathcal{D}(\boldsymbol{k}\cdot\boldsymbol{V}_0,\boldsymbol{k})} \tag{1}$$

for a particle of charge q_T with initial position $\boldsymbol{R}_0$ and velocity $\boldsymbol{V}_0$, where

$$\mathcal{D}(\omega,\boldsymbol{k}) = 1 + \sum_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{k^2}\frac{1}{n_\alpha}\int \mathrm{d}\boldsymbol{v}\,\frac{\boldsymbol{k}\cdot\partial f_\alpha/\partial\boldsymbol{v}}{\omega-\boldsymbol{k}\cdot\boldsymbol{v}+\mathrm{i}0} \tag{2}$$

is the dielectric function. The other symbols have their usual meanings: $\boldsymbol{k}$ is the wavevector, $\omega_{\mathrm{p}\alpha}^2 \equiv 4\pi q_\alpha^2 n_\alpha/m_\alpha$ is the square of the plasma frequency, and $f_\alpha(\boldsymbol{v})$ and n_α are the (spatially uniform) distribution function and number density of the background (i.e., undisturbed) plasma, respectively. In class, we discussed the physically reasonable result that

$$\varphi(t,\boldsymbol{r}) \approx \frac{q_\mathrm{T}}{|\boldsymbol{r}-\boldsymbol{R}_0-\boldsymbol{V}_0 t|}\,\exp\bigl(-K_\mathrm{D}|\boldsymbol{r}-\boldsymbol{R}_0-\boldsymbol{V}_0 t|\bigr) \quad \text{for} \quad V_0 \ll v_{\mathrm{th}\alpha} \tag{3}$$

and

$$\varphi(t,\boldsymbol{r}) \approx \frac{q_\mathrm{T}}{|\boldsymbol{r}-\boldsymbol{R}_0-\boldsymbol{V}_0 t|} \quad \text{for} \quad V_0 \gg v_{\mathrm{th}\alpha}, \tag{4}$$

where $v_{\mathrm{th}\alpha}$ is the thermal speed of species α; i.e., a sufficiently slow particle has efficient Debye shielding, whereas a sufficiently fast particle has inefficient Debye shielding.[1] In this problem, you will analytically compute the first-order correction to (3) and then examine the exact solution, which was numerically obtained and presented in the lecture notes. You will also examine the corresponding density response of the plasma through which the test charge travels (so-called "Cerenkov wakes").

(a) For simplicity, take the background plasma to be composed of Maxwellian electrons and cold, immobile ions. Show that the dielectric function may be written as

$$\mathcal{D}(\boldsymbol{k}\cdot\boldsymbol{V}_0,\boldsymbol{k}) = 1 + \frac{k_\mathrm{De}^2}{k^2}\bigl[1+\zeta_\mathrm{e}Z(\zeta_\mathrm{e})\bigr], \tag{5}$$

where $k_\mathrm{De}^2 \equiv 4\pi e^2 n_\mathrm{e}/T_\mathrm{e}$ is the square of the inverse Debye length, $Z(\zeta)$ denotes the plasma dispersion function, and $\zeta_\mathrm{e} \equiv \hat{\boldsymbol{k}}\cdot\boldsymbol{V}_0/v_\mathrm{the}$.

[1] In (3), we have used $K_\mathrm{D}^2 \equiv -\sum_\gamma \omega_{\mathrm{p}\gamma}^2 \int \mathrm{d}u\,(1/u)(\partial F_\gamma/\partial u)$, where $F_\gamma(u) = (1/n_\gamma)\int \mathrm{d}\boldsymbol{v}\,f_\gamma(\boldsymbol{v})\,\delta(u-\hat{\boldsymbol{k}}\cdot\boldsymbol{v})$, as the square of the Debye wavenumber for an arbitrary distribution function. For the usual Maxwellian distribution, $K_\mathrm{D}^2 = \sum_\gamma 4\pi q_\gamma^2 n_\gamma/T_\gamma \equiv k_\mathrm{D}^2$.

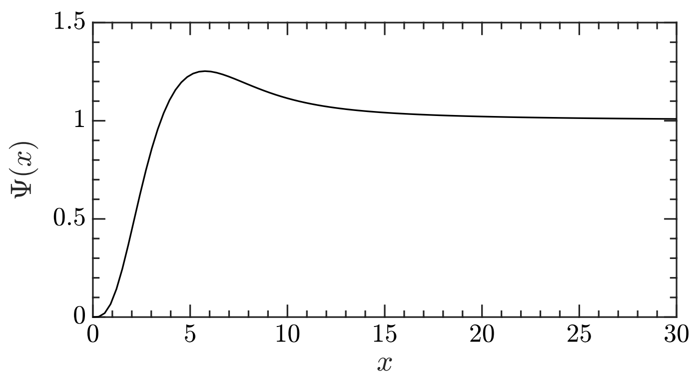
(b) Suppose $V_0/v_{\rm the} \ll 1$. Expand the Z function in its small argument to show that the first-order correction to (3) is given by

$$\delta\varphi(\boldsymbol{\mathscr{r}}) = \frac{q_{\rm T} k_{\rm De}}{(k_{\rm De}\mathscr{r})^3} \frac{4\Psi(k_{\rm De}\mathscr{r})}{\sqrt{\pi}}\, \hat{\boldsymbol{\mathscr{r}}} \boldsymbol{\cdot} \frac{\boldsymbol{V}_0}{v_{\rm the}}, \tag{6}$$

where $\boldsymbol{\mathscr{r}} \equiv \boldsymbol{r} - \boldsymbol{R}_0 - \boldsymbol{V}_0 t$ and the function

$$\Psi(x) \equiv \frac{1}{2}\int_0^\infty {\rm d}y\, \frac{\sin y - y\cos y}{(y^2/x^2+1)^2} \tag{7}$$

is shown in the figure below:



At large distances from the test charge, it's clear from the figure that $\Psi(k_{\rm De}\mathscr{r}) \to 1$.[2] The potential $\delta\varphi(\boldsymbol{\mathscr{r}})$ thus falls off as $1/\mathscr{r}^3$ at large distances. Note further the asymmetry revealed by (6): observers in front of the moving test charge ($\hat{\boldsymbol{\mathscr{r}}} \boldsymbol{\cdot} \boldsymbol{V}_0 > 0$) see an enhanced potential, whereas observers in back ($\hat{\boldsymbol{\mathscr{r}}} \boldsymbol{\cdot} \boldsymbol{V}_0 < 0$) see a reduced potential.

(c) For $\hat{\boldsymbol{\mathscr{r}}} \boldsymbol{\cdot} \boldsymbol{V}_0 = 0$ (i.e., directions perpendicular to the velocity of the test charge), the first-order potential (6) vanishes. Whoops. If I were mean, I would make you calculate the second-order correction to (3). But I'm not, so it's up to you whether you want to torture yourself with six different contour integrals. Here I'll just give the answer:

$$\delta\varphi^{(2)}(\boldsymbol{\mathscr{r}}) = -\frac{q_{\rm T} k_{\rm De}}{(k_{\rm De}\mathscr{r})^3}\, 2(\pi-2)\, \mathsf{T}(k_{\rm De}\mathscr{r}) \boldsymbol{:} \frac{\boldsymbol{V}_0\boldsymbol{V}_0}{v_{\rm the}^2}, \tag{8}$$

where

$$\mathsf{T}(x) \equiv -\big(3\hat{\boldsymbol{\mathscr{r}}}\hat{\boldsymbol{\mathscr{r}}} - \mathsf{I}\big)\left[1 - {\rm e}^{-x}\left(1 + x + \frac{x^2}{2}\right)\right] + {\rm e}^{-x}\frac{x^3}{8}\left[4\,\hat{\boldsymbol{\mathscr{r}}}\hat{\boldsymbol{\mathscr{r}}} + \big(x\,\hat{\boldsymbol{\mathscr{r}}}\hat{\boldsymbol{\mathscr{r}}} - \mathsf{I}\big)\frac{\pi}{\pi-2}\right]$$

and I is the unit dyad. Note that, at large distances ($x \gg 1$), $\mathsf{T} \to -(3\hat{\boldsymbol{\mathscr{r}}}\hat{\boldsymbol{\mathscr{r}}} - \mathsf{I})$, and so $q_{\rm T}\delta\varphi(x \gg 1)$ is negative in the direction perpendicular to $\boldsymbol{V}_0$. Why is this, physically?

(d) Section V.1 of the lecture notes includes five plots of equipotential contours of $(\varphi/q_{\rm T}k_{\rm De})$ in the rest frame of the test particle for $V_0/v_{\rm the} = 0.1, 0.3, 1, 2, 3, 10$. These were obtained by numerically performing the integral in (1) using (5) for the dielectric function.

[2] You won't earn extra points for doing it, but the mathematically inclined might enjoy the asymptotics challenge of proving this analytically.

Explain what you see there in physical terms. For example, why is the potential negative behind the moving test charge (if $q_\mathrm{T} > 0$)? How is this related to polarization drag? What's happening in front of the test charge? What happens to the Debye cloud at $V_0/v_\mathrm{the} = 10$ and why? Is the potential in the direction perpendicular to the test charge's velocity qualitatively consistent with (8)?

(e) The perturbed charge density,

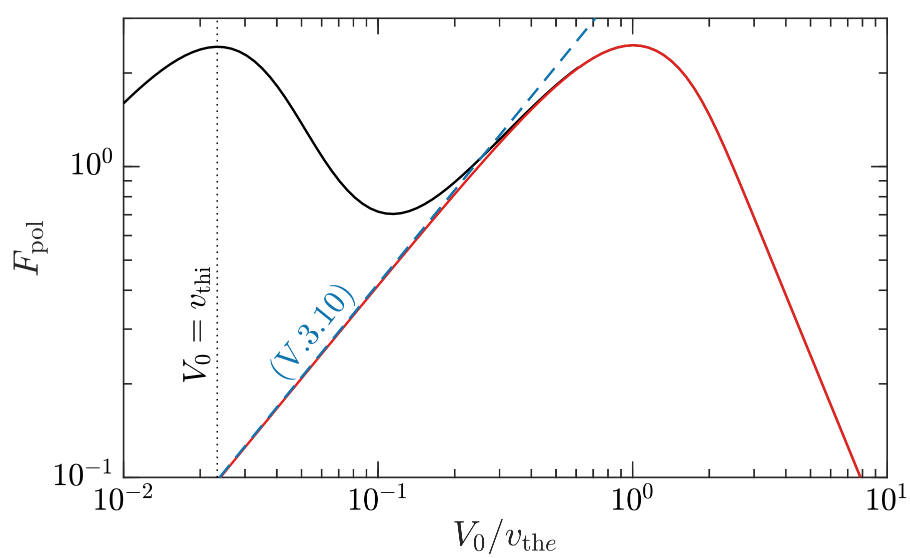
$$\sum_\alpha q_\alpha \delta n_\alpha(t, \boldsymbol{r}) = \frac{q_\mathrm{T}}{(2\pi)^3} \int \mathrm{d}\boldsymbol{k}\, \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}) \left[\frac{1}{\mathcal{D}(\boldsymbol{k}\cdot\boldsymbol{V}_0, \boldsymbol{k})} - 1 \right], \tag{9}$$

is given by equation (V.1.13) in the lecture notes. Numerically evaluating this integral for $V_0/v_\mathrm{the} = 10$ results in the isodensity contours shown on pg. 69 of the lecture notes. You can see a nice Mach cone trailing behind the test charge, with the density fluctuations confined inside $|y/x| \approx C/V_0 \ll 1$ where $C = \sqrt{3}v_\mathrm{the}$. Explain this result physically (including the factor of $\sqrt{3}$).

(f) In the above, we've assumed that the ions are cold and immobile. Equation (V.3.10) of the lecture notes states that the polarization drag force on a test charge $q_\mathrm{T} > 0$ of velocity $\boldsymbol{V}_0$ in a bath of electrons of number density n_e and temperature T_e is given by

$$\boldsymbol{F}_\mathrm{pol} = -q_\mathrm{T}^2 k_\mathrm{De}^2 \ln\lambda_\mathrm{ie} \frac{2}{3\sqrt{\pi}} \frac{\boldsymbol{V}_0}{v_\mathrm{the}}.$$

This expression is traced by the blue dashed line in the figure below. As you can see, it does rather well for $V_0/v_\mathrm{the} \ll 1$ at approximating the red line, which is the full velocity-dependent drag force (equation (V.3.2) in the lecture notes) when the ions are taken to be cold and immobile.[3] You can also see that the drag force drops off for $V_0/v_\mathrm{the} \gtrsim 1$; physically, this is because the charge is out-running its Debye cloud of electrons. Ah, but what's happening with the black line for $V_0/v_\mathrm{the} \lesssim 0.3$? That line is the full drag force (V.3.2) *with* active ions having $m_\mathrm{i}/m_\mathrm{e} = 1836$ and $T_\mathrm{i}/T_\mathrm{e} = 1$. Explain physically why this curve rises and then falls as V_0 approaches v_thi (the vertical dotted line) from above. Perhaps some drawings might help.



[3] I took $k_\mathrm{max} = 10^5 k_\mathrm{De}$ in numerically evaluating (V.3.2).

2. **Thermal Bremsstrahlung**. In class, we computed the radiation emitted as a test particle moves through a continuous plasma—so-called *Cerenkov radiation*. In this problem, you will use similar test-particle methods to compute the radiation due to collisions between discrete particles, which is called *Bremsstrahlung* (German for "braking radiation"). In particular, you will focus on collisions between ions and electrons. Consistent with the rest of the course thus far, only electrostatic fluctuations are considered, so that the wavevector $\boldsymbol{k}$ is parallel to $\boldsymbol{E}_{\boldsymbol{k}}$. (At the end of this problem, I comment on the arguably more important case of transverse polarization.) Your first goal is to compute the radiation emitted as a shielded test electron scatters off a test ion, which is given by

$$W = \lim_{\mathcal{T}\to\infty} \int_{-\mathcal{T}/2}^{\mathcal{T}/2} \mathrm{d}t \int \mathrm{d}\boldsymbol{r}\, \boldsymbol{E}(t,\boldsymbol{r}) \boldsymbol{\cdot} \boldsymbol{J}_{\mathrm{T}}(t,\boldsymbol{r}), \tag{10}$$

where $\boldsymbol{J}_{\mathrm{T}}(t,\boldsymbol{r})$ is the current density of the test particles and $\mathcal{T}$ is the time over which the radiation is emitted. I'll lead you through the process.

(a) Write the electric field and current density in Fourier space,

$$\boldsymbol{E}(t,\boldsymbol{r}) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\boldsymbol{k}\, \boldsymbol{E}(\omega,\boldsymbol{k}) \exp[\mathrm{i}(\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{r} - \omega t)],$$

$$\boldsymbol{J}_{\mathrm{T}}(t,\boldsymbol{r}) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega'}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\boldsymbol{k}'\, \boldsymbol{J}_{\mathrm{T}}(\omega',\boldsymbol{k}') \exp[\mathrm{i}(\boldsymbol{k}'\boldsymbol{\cdot}\boldsymbol{r} - \omega' t)],$$

and use Maxwell's equations with $\boldsymbol{E}_{\boldsymbol{k}} \parallel \boldsymbol{k}$ to show that (10) becomes

$$\begin{aligned} W &= (2\pi)^3\, \mathrm{Re} \int \frac{\mathrm{d}\omega}{2\pi} \int \mathrm{d}\boldsymbol{k}\, \boldsymbol{E}(\omega,\boldsymbol{k}) \boldsymbol{\cdot} \boldsymbol{J}_{\mathrm{T}}^{*}(\omega,\boldsymbol{k}) \\ &= 2(2\pi)^4\, \mathrm{Im} \int \frac{\mathrm{d}\omega}{2\pi} \int \mathrm{d}\boldsymbol{k}\, \frac{|\hat{\boldsymbol{k}}\boldsymbol{\cdot}\boldsymbol{J}_{\mathrm{T}}(\omega,\boldsymbol{k})|^2}{\omega \mathcal{D}(\omega,\boldsymbol{k})} \equiv \int \frac{\mathrm{d}\omega}{2\pi} W_\omega, \end{aligned} \tag{11}$$

where $\hat{\boldsymbol{k}} = \boldsymbol{k}/k$ and $\mathcal{D}$ is given by

$$\mathcal{D}(\omega,\boldsymbol{k}) = 1 + \sum_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{k^2} \frac{1}{n_\alpha} \int \mathrm{d}\boldsymbol{v}\, \frac{\boldsymbol{k}\boldsymbol{\cdot}\partial f_\alpha/\partial \boldsymbol{v}}{\omega - \boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{v} + \mathrm{i}0}. \tag{12}$$

You should prove this step by step. But, once you've become an expert at these things, you could just surmise that the current density of the undressed test particles $\boldsymbol{J}_{\mathrm{T}}$ ought to be "dressed" with a $1/\mathcal{D}(\omega,\boldsymbol{k})$ factor. On pg. 76 of the lecture notes, this formula was used to compute the power emitted by Cerenkov radiation from a single test charge; see equation (V.3.9) there.

(b) Apparently, we need to compute the square of the test-particle current in the ω-$\boldsymbol{k}$ space. To do so, write the current of the test ion-electron pair as

$$\boldsymbol{J}_{\mathrm{T}}(t,\boldsymbol{r}) = Ze\boldsymbol{V}_{\mathrm{i}}(t)\, \delta(\boldsymbol{r} - \boldsymbol{R}_{\mathrm{i}}(t)) - e\boldsymbol{V}_{\mathrm{e}}(t)\, \delta(\boldsymbol{r} - \boldsymbol{R}_{\mathrm{e}}(t)), \tag{13}$$

where Ze ($-e$) is the charge of the ion (electron) and $(\boldsymbol{R}_\alpha, \boldsymbol{V}_\alpha)$ are the position and velocity of species α at time t. Assuming that $kv_{\mathrm{the}} \ll \omega$ and $m_{\mathrm{e}}/m_{\mathrm{i}} \ll 1$, use (13) along with Newton's second and third laws to obtain

$$|\hat{\boldsymbol{k}}\boldsymbol{\cdot}\boldsymbol{J}_{\mathrm{T}}(\omega,\boldsymbol{k})|^2 = \frac{Z^2 e^4}{\omega^2 m_{\mathrm{e}}^2} \frac{|\hat{\boldsymbol{k}}\boldsymbol{\cdot}\boldsymbol{E}_{\mathrm{e}}(\omega,\boldsymbol{R}_{\mathrm{i}})|^2}{(2\pi)^6}, \tag{14}$$

where $\boldsymbol{E}_\mathrm{e}(\omega, \boldsymbol{R}_\mathrm{i})$ is the electric field due to the (dressed!) test electron evaluated at the location of the ion. Thus,

$$W_\omega = \frac{1}{2\pi^2}\,\mathrm{Im}\int \mathrm{d}\boldsymbol{k}\,\frac{Z^2 e^4}{\omega^2 m_\mathrm{e}^2}\frac{|\hat{\boldsymbol{k}}\cdot\boldsymbol{E}_\mathrm{e}(\omega, \boldsymbol{R}_\mathrm{i})|^2}{\omega\mathcal{D}(\omega, \boldsymbol{k})}. \tag{15}$$

(c) Show that, in the long-wavelength limit for which $\omega \gg k v_\mathrm{the}$,

$$\mathcal{D}(\omega, \boldsymbol{k}) \approx 1 - \frac{\omega_\mathrm{pe}^2}{\omega^2}\left(1 + \frac{3}{2}\frac{k^2 v_\mathrm{the}^2}{\omega^2}\right) + \mathrm{i}\,\mathrm{Im}\,\mathcal{D}. \tag{16}$$

Also show that $\mathrm{Im}\,\mathcal{D} \ll 1$ when $\omega \gtrsim \omega_\mathrm{pe}$. Use these properties in (15) and perform the $\boldsymbol{k}$-space integration to obtain

$$W_\omega = \frac{2Z^2 e^4}{9 m_\mathrm{e}^2 v_\mathrm{the}^3}\sqrt{\frac{2}{3}\left(1 - \frac{\omega_\mathrm{pe}^2}{\omega^2}\right)}\,|\boldsymbol{E}_\mathrm{e}(\omega, \boldsymbol{R}_\mathrm{i})|^2. \tag{17}$$

You'll need the following tricks:

$$\lim_{\epsilon\to+0}\frac{\epsilon}{x^2+\epsilon^2} = \pi\delta(x),$$

$$\delta(f(x)) = \sum_i \frac{\delta(x - x_i)}{|f'(x_i)|}, \quad \text{where } f(x_i) = 0.$$

(d) In class, we calculated the electric field excited by a test charge moving through a responsive plasma. Taking the electron to be that test charge, the electric field evaluated at the position of the ion is

$$\boldsymbol{E}_\mathrm{e}(\omega, \boldsymbol{R}_\mathrm{i}) = \frac{\mathrm{i}e}{\pi}\int \mathrm{d}\boldsymbol{k}\,\frac{\boldsymbol{k}\,\delta(\omega - \boldsymbol{k}\cdot\boldsymbol{V}_\mathrm{e})}{k^2\,\mathcal{D}(\omega, \boldsymbol{k})}\,\exp[\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{R}_\mathrm{i} - \boldsymbol{R}_\mathrm{e})]. \tag{18}$$

Plug this into (17). The result is the total radiation emitted when a single test electron, dressed by the surrounding plasma, scatters off a single ion. Now invoke the test-particle superposition principle to show that the total power radiated per unit volume by a distribution $f_\mathrm{e}(\boldsymbol{v})$ of electrons is

$$P = \frac{4Z^2 e^6 n_\mathrm{i}}{9\pi m_\mathrm{e}^2 v_\mathrm{the}^3}\int \mathrm{d}\omega\,\sqrt{\frac{2}{3}\left(1 - \frac{\omega_\mathrm{pe}^2}{\omega^2}\right)}\iint \mathrm{d}\boldsymbol{k}\,\mathrm{d}\boldsymbol{v}\,f_\mathrm{e}(\boldsymbol{v})\,\frac{\delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v})}{k^2|\mathcal{D}(\omega, \boldsymbol{k})|^2}, \tag{19}$$

where n_i is the number density of ions.

Hint: It might help you to set up a coordinate system with the impact parameter $\boldsymbol{b} \equiv \boldsymbol{R}_\mathrm{e} - \boldsymbol{R}_\mathrm{i} = b\hat{\boldsymbol{x}}$, the electron velocity $\boldsymbol{V}_\mathrm{e} = V_\mathrm{e}\hat{\boldsymbol{z}}$, and the wavevector $\boldsymbol{k} = k_\parallel\hat{\boldsymbol{z}} + \boldsymbol{k}_\perp$. If you do this, then you'll have to recall that $\delta(\boldsymbol{k}_\perp - \boldsymbol{k}'_\perp) = 2\,\delta(k_\perp^2 - k'^2_\perp)\,\delta(\phi_k - \phi_{k'})$ in cylindrical coordinates. Alternatively, keep $\boldsymbol{b}$ as a general vector and eventually use $\delta^2(\omega) = (\mathcal{T}/2\pi)\delta(\omega)$ – see equation (III.7.15) in the lecture notes for an explanation.

(e) Using the equations for Cerenkov radiation (e.g., equation (V.3.9) in the lecture notes), argue that (19) is a factor $\sim 1/\Lambda$ smaller than its Cerenkov counterpart. Physically, why is this? (Think in terms of the particle trajectories in the two calculations and how and why they differ.) Does the Balescu–Lenard operator capture Bremsstrahlung emission? If so, where is it? If not, what assumption precluded Bremsstrahlung from entering?

(f) Evaluate (19) for a Maxwellian electron distribution,

$$f_{\mathrm{e}}(\boldsymbol{v}) = \frac{n_{\mathrm{e}}}{\pi^{3/2} v_{\mathrm{the}}^3} \exp\left(-\frac{v^2}{v_{\mathrm{the}}^2} \right). \tag{20}$$

For simplicity, just take $\mathcal{D}(\omega, \boldsymbol{k}) = 1$ and cut off the wavenumber integration using a maximum wavenumber k_{max}. (Note that you need not specify a minimum wavenumber.) You should be able to massage your answer into the following form:

$$P = \frac{16 Z^2 e^6 n_{\mathrm{i}} n_{\mathrm{e}}}{9 m_{\mathrm{e}}^2 v_{\mathrm{the}}^3} \sqrt{\frac{2}{3}}\, k_{\mathrm{max}}\, G(\xi_{\mathrm{min}}), \tag{21}$$

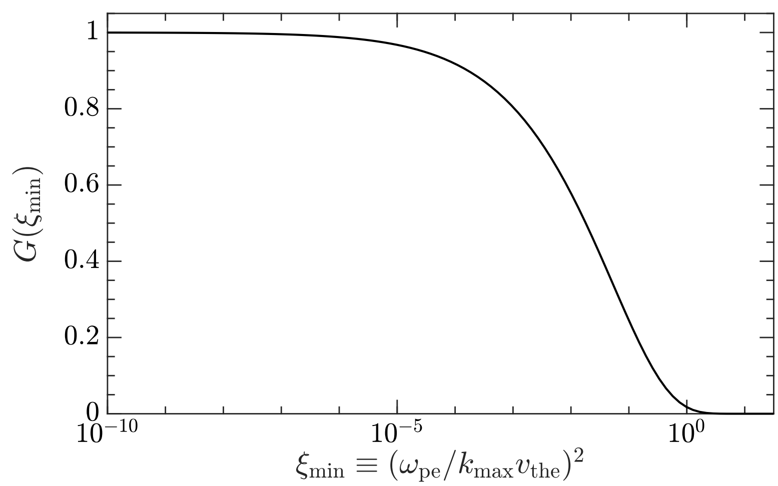
where $\xi_{\mathrm{min}} \equiv (\omega_{\mathrm{pe}}/k_{\mathrm{max}} v_{\mathrm{the}})^2$,

$$G(\xi_{\mathrm{min}}) \equiv \frac{1}{2\sqrt{\pi}} \int_{\xi_{\mathrm{min}}}^{\infty} \mathrm{d}\xi\, E_1(\xi) \sqrt{\frac{1}{\xi}\left(1 - \frac{\xi_{\mathrm{min}}}{\xi}\right)}, \tag{22}$$

and the exponential integral

$$E_1(\xi) = \int_0^1 \mathrm{d}t\, \frac{\mathrm{e}^{-\xi/t}}{t}. \tag{23}$$

The function $G(\xi_{\mathrm{min}}) = 1$ at $\xi_{\mathrm{min}} = 0$ and decreases sharply thereafter:



For plasmas with $T_{\mathrm{e}} > 27.2 Z^2$ eV (the Hartree energy), the electron thermal de Broglie wavelength provides the maximum wavenumber: $k_{\mathrm{max}} = \sqrt{m_{\mathrm{e}} T_{\mathrm{e}}/\hbar^2}$. In this case, $P \propto n_{\mathrm{i}} n_{\mathrm{e}} T_{\mathrm{e}}^{-1}$. Otherwise, $k_{\mathrm{max}} \sim T_{\mathrm{e}}/e^2$ (the "Landau length", or the distance of closest approach of thermal electrons) and $P \propto n_{\mathrm{i}} n_{\mathrm{e}} T_{\mathrm{e}}^{-1/2}$.

An additional note for your own edification (please read):

One can repeat the ion-electron thermal Bremsstrahlung calculation outlined above but with transverse (rather than longitudinal) electromagnetic fluctuations (i.e., light). Now, $\boldsymbol{k}\cdot\boldsymbol{E}_{\boldsymbol{k}} = 0$ but $\boldsymbol{k}\times\boldsymbol{E}_{\boldsymbol{k}} \neq 0$, and so there are magnetic-field fluctuations. This is the kind of Bremsstrahlung radiation you can (and one does) observe, e.g., at radio wavelengths from HII regions in the interstellar medium and in the X-ray band from hot astrophysical plasmas, such as the intracluster medium of galaxy clusters and the accretion flow onto the black hole at our Galactic center. (Of course, a test current $\boldsymbol{J}_{\mathrm{T}}$ of accelerated charges will excite both longitudinal and transverse electromagnetic fluctuations. Problem 2 focused on the former; the following focuses on the latter.) One can straightforwardly generalize the calculation from Problem 2 for this case by using the relevant dielectric function,

$$\mathcal{D}(\omega,\boldsymbol{k}) = 1 - \frac{\omega^2}{k^2c^2} + \frac{\omega_{\mathrm{pe}}^2\omega}{k^2c^2}\int \mathrm{d}\boldsymbol{v}\,\frac{f_{\mathrm{e}}(\boldsymbol{v})}{\omega - \boldsymbol{k}\cdot\boldsymbol{v} + \mathrm{i}0} \equiv 1 - \frac{\omega^2}{k^2c^2} - \frac{4\pi\mathrm{i}\omega}{k^2c^2}\sigma, \qquad (24)$$

where σ is the conductivity, and revising equation (11) for the new polarization. The derivation will be given in my solutions; the result is that

$$P = \frac{4Z^2e^6n_{\mathrm{i}}}{3\pi m_{\mathrm{e}}^2c^3}\int \mathrm{d}\omega\,\sqrt{1-\frac{\omega_{\mathrm{pe}}^2}{\omega^2}}\iint \mathrm{d}\boldsymbol{k}\,\mathrm{d}\boldsymbol{v}\,f_{\mathrm{e}}(\boldsymbol{v})\,\frac{\delta(\omega-\boldsymbol{k}\cdot\boldsymbol{v})}{k^2|\mathcal{D}(\omega,\boldsymbol{k})|^2}, \qquad (25a)$$

$$= \frac{16Z^2e^6n_{\mathrm{i}}n_{\mathrm{e}}}{3m_{\mathrm{e}}^2c^3}\,k_{\max}\,G(\xi_{\min}) \text{ for a Maxwellian with } |\mathcal{D}(\omega,\boldsymbol{k})|^2 \simeq 1. \qquad (25b)$$

For $k_{\max} = \sqrt{m_{\mathrm{e}}T_{\mathrm{e}}/\hbar^2}$, the **total power radiated per unit volume is** $\boldsymbol{\propto Z^2 n_{\mathrm{i}} n_{\mathrm{e}} T_{\mathrm{e}}^{1/2}}$. In CGS units with temperature measured in Kelvin, its numerical value is $\approx 10^{-27}\,Z^2n_{\mathrm{i}}n_{\mathrm{e}}T_{\mathrm{e}}^{1/2}$, a formula routinely used to infer number densities of X-ray emitting astrophysical plasmas from the observed emission. The temperature may be obtained from the observed cutoff of the otherwise flat frequency spectrum $\mathrm{d}P/\mathrm{d}\omega \propto E_1(\xi/2)$, where $\xi \doteq (\hbar\omega/k_{\mathrm{B}}T_e)^2$. Namely, taking the cutoff to be at $\xi \approx 1$ implies an emitted frequency $\nu = \omega/2\pi \approx 10^{18}$ Hz $(k_{\mathrm{B}}T_{\mathrm{e}}/5\text{ keV})$, right in the middle of the X-ray band. Here, I've normalized the temperature to that of the most abundant plasma in the Universe – the intracluster medium, a hot and dilute plasma that fills the space between galaxies in so-called galaxy clusters (the largest virialized objects in the Universe). Thermal Bremsstrahlung is the main cooling process in this plasma, despite a characteristic cooling time $\approx$6 Gyr $(k_{\mathrm{B}}T_{\mathrm{e}}/5\text{ keV})^{1/2}\,(n_{\mathrm{e}}/0.01\text{ cm}^{-3})^{-1}$ … do you know the age of the Universe? In fusion plasmas, the density and temperature are known by other means (generals prep: do you know what they are?), and so the amount of Bremsstrahlung emission (in the visible range) provides a measurement of Z_{eff}. This is an important number because radiation from impurities can affect the confinement time. X-ray Bremsstrahlung emission is also used in laser-plasma experiments (where $n_{\mathrm{e}} \sim 10^{20}$ cm^{-3}, $k_{\mathrm{B}}T_{\mathrm{e}} \sim 0.5$ keV) to measure density fluctuations and in ICF experiments to estimate the neutron yield.