Due Monday, Mar 3, 2025

Generals prep. Make sure you can provide brief definitions of the following terms: Balescu–Lenard collision operator, Landau collision operator, Boltzmann's H theorem

1. **Balescu–Lenard in 1D.** The Balescu–Lenard collision operator may be written in the following highly suggestive form:

$$\left(\frac{\partial f_\alpha}{\partial t}\right)_{\rm c} = \sum_\beta \int \frac{{\rm d}\boldsymbol{k}}{(2\pi)^3} \frac{\boldsymbol{k}}{m_\alpha} \cdot \frac{\partial}{\partial \boldsymbol{v}} \times \int {\rm d}\boldsymbol{v}' \left| \frac{4\pi q_\alpha q_\beta}{k^2 \, \mathcal{D}(\boldsymbol{k}, \boldsymbol{k}\cdot\boldsymbol{v})} \right|^2 \pi\delta(\boldsymbol{k}\cdot\boldsymbol{v} - \boldsymbol{k}\cdot\boldsymbol{v}') \left(\frac{\boldsymbol{k}}{m_\alpha} \cdot \frac{\partial}{\partial \boldsymbol{v}} - \frac{\boldsymbol{k}}{m_\beta} \cdot \frac{\partial}{\partial \boldsymbol{v}'} \right) f_\alpha(\boldsymbol{v}) f_\beta(\boldsymbol{v}'). \tag{1}$$

By the end of this course, you should be able to promptly identify the physical meaning of each and every ingredient of this operator and state all the assumptions that went into their derivation. But that's then. For now, simply note that, in a one-dimensional (1D) plasma, the delta function

$$\delta(\boldsymbol{k}\cdot\boldsymbol{v} - \boldsymbol{k}\cdot\boldsymbol{v}') = \delta[k(v - v')] = \frac{1}{|k|}\delta(v - v')$$

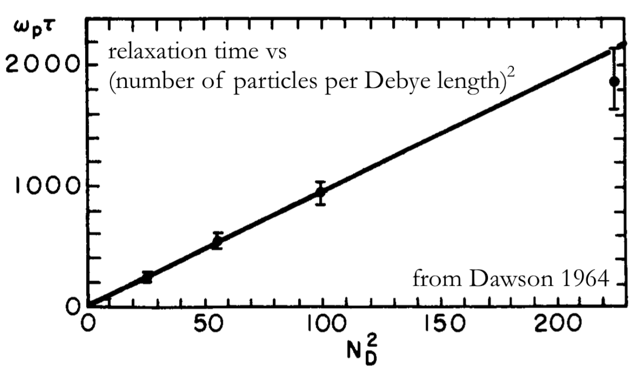
implies that only particles with the same speed can collisionally interact.

(a) Use this fact to prove that (1) vanishes for a single-species plasma in 1D. (Note: the Coulomb potential in 1D is still $\propto k^{-2}$ in Fourier space, just with a different coefficient.)

(b) As shown in the lecture notes (§IV.4), the Balescu–Lenard collision operator independently conserves the total energy and the total momentum of a plasma. Use this to explain in physical terms your answer to part (a).

(c) This led to a puzzle in the early development of kinetic particle-in-cell (PIC) simulations of plasmas (most notably, by John Dawson), which were one-dimensional and yet somehow exhibited relaxation to a Maxwellian. How do you resolve this paradox? Is the Balescu–Lenard operator incomplete in 1D, or were Dawson's simulated plasmas simply relaxing via uncontrolled PIC noise? Explain your answer.

Hint: The following plot taken from Dawson (1964) shows the measured relaxation time of a non-thermal distribution function ($\omega_{\rm p}\tau$) versus the square of the number of particles per Debye length from a series of 1D PIC simulations ($N_{\rm D}^2$).



2. **Equilibrium thermodynamics to $\mathcal{O}(\Lambda^{-1})$.** In thermodynamic equilibrium, the Liouville distribution $P_N(\Gamma)$ reduces to the familiar Gibbs distribution from undergraduate statistical mechanics:

$$D_N(\Gamma) \equiv \frac{1}{\mathcal{Z}} \exp\biggl(-\frac{\mathcal{H}(\Gamma)}{T}\biggr), \tag{2}$$

where T is the (species-independent) temperature, $\mathcal{Z} = \int \mathrm{d}\Gamma \, \exp(-\mathcal{H}(\Gamma)/T)$ is the partition function, and the Hamiltonian

$$\mathcal{H}(\Gamma) = \sum_\alpha \sum_{i=1}^{N_\alpha} \left(\frac{1}{2} m_\alpha V_{\alpha_i}^2 + \sum_\beta \sum_{\substack{j=1 \\ (\beta_j \neq \alpha_i)}}^{N_\beta} \frac{1}{2} \frac{q_\alpha q_\beta}{|\boldsymbol{R}_{\alpha i} - \boldsymbol{R}_{\beta j}|} \right) \tag{3}$$

contains the kinetic and potential energies of the constituent particles. The one- and two-particle equilibrium reduced distribution functions are then

$$\begin{aligned} f_\alpha(\boldsymbol{x}) &\equiv \sum_{i=1}^{N_\alpha} \int \mathrm{d}\Gamma \, D_N(\Gamma) \, \delta(\boldsymbol{x} - \boldsymbol{X}_{\alpha i}), \\ f_{\alpha\beta}(\boldsymbol{x}, \boldsymbol{x}') &\equiv \sum_{i=1}^{N_\alpha} \sum_{j=1}^{N_\beta} \int \mathrm{d}\Gamma \, D_N(\Gamma) \, \delta(\boldsymbol{x} - \boldsymbol{X}_{\alpha i}) \delta(\boldsymbol{x}' - \boldsymbol{X}_{\beta j}) \\ &= f_\alpha(\boldsymbol{x}) f_\beta(\boldsymbol{x}') + g_{\alpha\beta}(\boldsymbol{x}, \boldsymbol{x}') \\ &\equiv f_\alpha(\boldsymbol{v}) f_\beta(\boldsymbol{v}') \bigl[1 + \widehat{g}_{\alpha\beta}(\boldsymbol{r}, \boldsymbol{r}') \bigr], \end{aligned}$$

where $\boldsymbol{x} = (\boldsymbol{r}, \boldsymbol{v})$ is shorthand for the phase-space variables and, in the last line, we have introduced $\widehat{g}_{\alpha\beta} \equiv g_{\alpha\beta}/f_\alpha f_\beta$ as the part of the equilibrium two-particle correlation function that depends only on position. (Recall $f_\alpha(\boldsymbol{x}) = f_\alpha(\boldsymbol{v})$ in equilibrium.) In this problem, you will calculate $\mathcal{O}(\Lambda^{-1})$ corrections to familiar thermodynamic quantities.

(a) The thermodynamic energy

$$U \equiv \langle \mathcal{H} \rangle = \int \mathrm{d}\Gamma \, \mathcal{H}(\Gamma) D_N(\Gamma) \tag{4}$$

is defined as the expectation value of the Hamiltonian. Using the indistinguishability of like-species particles and the definitions of the one- and two-particle equilibrium reduced distribution functions, show that

$$U = \frac{3}{2} NT + \mathcal{V} \sum_\alpha \sum_\beta 2\pi q_\alpha n_\alpha q_\beta n_\beta \int_0^\infty \mathrm{d}\imath \, \imath \, \widehat{g}_{\alpha\beta}(\imath), \tag{5}$$

where $N \equiv \sum_\alpha N_\alpha$ is the total number of particles in the system, $\mathcal{V}$ is the volume of the system, and $\imath \equiv |\boldsymbol{r} - \boldsymbol{r}'|$ is the radial separation between two generic particles of species α and β. The first term in (5) is (obviously) the thermal energy of the plasma; the second term measures the importance of the correlation, or potential, energy.

(b) Remember undergraduate statistical mechanics? Neither did I. The free energy W can be obtained from the partition function via $W = -T \ln \mathcal{Z}$.[1] With this in hand, other thermodynamic quantities may be obtained. Show that the pressure

$$P = -\left(\frac{\partial W}{\partial \mathcal{V}}\right)_{N,T},$$

satisfies

$$P\mathcal{V} = NT + \frac{\mathcal{V}}{3}\sum_\alpha \sum_\beta 2\pi q_\alpha n_\alpha q_\beta n_\beta \int_0^\infty \mathrm{d}\imath\, \imath\, \widehat{g}_{\alpha\beta}(\imath). \tag{6}$$

(Hint: You may be wondering where is the $\mathcal{V}$ in the partition function. Well, $\int \mathrm{d}\Gamma \propto \mathcal{V}^N$ and $|\boldsymbol{R}_{\alpha i} - \boldsymbol{R}_{\beta j}| \propto \mathcal{V}^{1/3}$, so. . .)

(c) Evaluate the thermodynamic energy (5) and pressure (6) for the equilibrium two-particle correlation

$$\widehat{g}_{\alpha\beta}(\imath) = -\frac{q_\alpha q_\beta}{T\imath}\exp(-k_\mathrm{D}\imath), \tag{7}$$

where $k_\mathrm{D}^2 \equiv \sum_\gamma 4\pi q_\gamma^2 n_\gamma / T$, and show explicitly that the corrections to the thermodynamic quantities are $\mathcal{O}(\Lambda^{-1})$.

If you're interested in the above, you might enjoy reading this paper by your student colleagues (T.E. Foster, H. Fetsch & N.J. Fisch, *J. Plasma Phys.* **89**, 905890506), in which they compute the two-particle correlation function from the BBGKY hierarchy and the associated thermodynamics for a moderately coupled plasma, i.e., one in which the plasma parameter satisfies $1 \ll \Lambda/\ln\Lambda \ll (m_i/m_e)^{1/2}$.

3. **Semi-convergent equilibrium pair correlations**. We saw in class that the divergence of the equilibrium two-particle distribution function $f_{\alpha\beta}(\boldsymbol{x}, \boldsymbol{x}')$ as $|\boldsymbol{r} - \boldsymbol{r}'| \to 0$ is related to our neglect of two-particle correlations in the source term for $g_{\alpha\beta}(\boldsymbol{x}, \boldsymbol{x}')$. Dropping that term amounted to an assumption that particles enter into two-body interactions initially uncorrelated. We really should have $f_{\alpha\beta} \to 0$ as $|\boldsymbol{r} - \boldsymbol{r}'| \to 0$, i.e., the probability of finding a particle of species α and a particle of species β at the same location, regardless of what all other particles are probably doing, should vanish. To get this physically reasonable result, we require that $\widehat{g}_{\alpha\beta} \doteq g_{\alpha\beta}/(f_\alpha f_\beta) \to -1$. So, let's retain the contribution of two-particle correlations to the source term and see what happens. With $\varphi_\alpha(\boldsymbol{r}, \boldsymbol{r}') \doteq q_\alpha/|\boldsymbol{r} - \boldsymbol{r}'|$ in 3D, equation (IV.6.10) in the lecture notes should be replaced by

$$(\boldsymbol{v} - \boldsymbol{v}') \cdot \frac{\partial}{\partial \boldsymbol{r}}\left[g_{\alpha\beta}(\boldsymbol{x}, \boldsymbol{x}') + \frac{q_\alpha \varphi_\beta(\boldsymbol{r}, \boldsymbol{r}')}{T} f_\alpha(\boldsymbol{v}) f_\beta(\boldsymbol{v}') + \sum_\gamma \int \mathrm{d}\boldsymbol{x}''\, \frac{q_\alpha \varphi_\gamma(\boldsymbol{r}, \boldsymbol{r}'')}{T} f_\alpha(\boldsymbol{v}) g_{\beta\gamma}(\boldsymbol{x}', \boldsymbol{x}'')\right]$$
$$= -(\boldsymbol{v} - \boldsymbol{v}') \cdot \frac{\partial}{\partial \boldsymbol{r}}\left[\frac{q_\alpha \varphi_\beta(\boldsymbol{r}, \boldsymbol{r}')}{T}\right] g_{\alpha\beta}(\boldsymbol{x}, \boldsymbol{x}'), \tag{8}$$

[1] In some plasma texts, the free energy is denoted A, after the German "Arbeit", meaning "work". (Remember that the free energy is the amount of energy available in the form of useful work.) But I'm going to use A for something different in this course, so it'll have to be W for the English "work".

the additional term having been placed on the right-hand side. You already know from class that the solution to (8) without this extra source term is

$$\widehat{g}_{\alpha\beta}^{(\text{B–L})}(\mathscr{r}) = -\frac{q_\alpha \varphi_\beta(\mathscr{r})}{T} \exp\left(-k_{\text{D}}\mathscr{r}\right), \tag{9}$$

and that its $k_{\text{D}}\mathscr{r} \ll 1$ (Landau) limit is

$$\widehat{g}_{\alpha\beta}^{(\text{Landau})}(\mathscr{r}) = -\frac{q_\alpha \varphi_\beta(\mathscr{r})}{T}. \tag{10}$$

(a) Show that, in the (Boltzmann) limit in which shielding is negligible, the equilibrium two-particle correlation function is

$$\widehat{g}_{\alpha\beta}^{(\text{Boltz})}(\mathscr{r}) = -1 + \exp\left(-\frac{q_\alpha \varphi_\beta(\mathscr{r})}{T}\right). \tag{11}$$

Note that, for like-signed charges, this safely asymptotes to -1 as $\mathscr{r} \to 0$, unlike the Balescu–Lenard correlation function (good), but that the $k_{\text{D}}\mathscr{r} \gg 1$ limit is now wrong (bad). Why is this, physically? Unfortunately, the case with unlike-signed charges blows up as $\mathscr{r} \to 0$. As Greg Hammett put it in his lecture notes: "Nothing classical can prevent electrons from collapsing onto ions with infinite negative potential energy. Only quantum effects can prevent this collapse."

(b) The combination

$$g_{\alpha\beta} = g_{\alpha\beta}^{(\text{B–L})} + g_{\alpha\beta}^{(\text{Boltz})} - g_{\alpha\beta}^{(\text{Landau})} \tag{12}$$

is globally convergent, asymptotes to the correct limits (at least for $\Lambda \to \infty$), and results in a collision operator

$$\frac{q_\alpha}{m_\alpha}\frac{\partial f_\alpha}{\partial \boldsymbol{v}} \cdot \sum_\beta \int \mathrm{d}\boldsymbol{v}'\, f_\beta(\boldsymbol{v}') \int \mathrm{d}\boldsymbol{r}'\, \widehat{g}_{\alpha\beta}(\boldsymbol{r},\boldsymbol{r}')\, \frac{\partial \varphi_\beta(\boldsymbol{r},\boldsymbol{r}')}{\partial \boldsymbol{r}}$$

whose spatial integral exists without artificial regularization (at least for like-signed charges).[2] Sweet! Make a plot of (12) in the same format as Fig. 5 in Section IV.6 of the lecture notes and comment on the differences. Unfortunately, despite these advantages, the internal energy per particle (5) diverges for this choice of $g_{\alpha\beta}$.

(c) We know that the third term in (8) is ultimately responsible for "dressing" the Coulomb potential: $\varphi_\beta(\mathscr{r}) = q_\beta/\mathscr{r} \to (q_\beta/\mathscr{r})\exp(-k_{\text{D}}\mathscr{r})$. Another approach to obtaining a globally convergent collision operator is to solve (8) after neglecting that third term but using this dressed Coulomb potential for φ_β. Do so to derive the two-particle correlation

$$\widehat{g}_{\alpha\beta}(\mathscr{r}) = -1 + \exp\left[-\frac{q_\alpha q_\beta}{T\mathscr{r}}\exp(-k_{\text{D}}\mathscr{r})\right], \tag{13}$$

and show that it asymptotes to the correct limits. Plot it on top of your plot from (b).

[2] Equation (12) is what results by matching solutions of (8) across the various asymptotic limits; see E. A. Frieman & D. L. Book, *Phys. Fluids* **6**, 1700 (1963). If you'd like to read more about globally convergent collision operators, I recommend S. D. Baalrud & J. Daligault, *Phys. Plasmas* **26**, 082106 (2019).

4. **Debye shielding and equilibrium pair correlations in 2D**. Finally, let's return to the original version of (8) featured in class, which has its right-hand side neglected:

$$g_{\alpha\beta}(\boldsymbol{x},\boldsymbol{x}') + \sum_{\gamma} \int \mathrm{d}\boldsymbol{x}'' \, \frac{q_\alpha \varphi_\gamma(\boldsymbol{r},\boldsymbol{r}'')}{T} f_\alpha(\boldsymbol{v}) g_{\beta\gamma}(\boldsymbol{x}',\boldsymbol{x}'') + \frac{q_\alpha \varphi_\beta(\boldsymbol{r},\boldsymbol{r}')}{T} f_\alpha(\boldsymbol{v}) f_\beta(\boldsymbol{v}') = 0, \qquad (14)$$

The solution to this equation may be written for arbitrary dimensionality d $(= 1, 2, 3)$ in Fourier space as follows:

$$\widehat{g}_{\alpha\beta}(\boldsymbol{k}) \equiv \frac{g_{\alpha\beta}(\boldsymbol{k},\boldsymbol{v},\boldsymbol{v}')}{f_\alpha(\boldsymbol{v}) f_\beta(\boldsymbol{v}')} = -\frac{q_\alpha \varphi_\beta(\boldsymbol{k})}{T} \left[1 + (2\pi)^d \sum_{\gamma} \frac{q_\gamma n_\gamma \varphi_\gamma(\boldsymbol{k})}{T} \right]^{-1} . \qquad (15)$$

This may be compared with equation (IV.6.13) in the lecture notes.

Use (15) to calculate the two-particle correlation function for an equilibrium plasma with only two spatial dimensions ($d = 2$), so that the "particles" are infinitely long charged rods aligned with the z axis of a cylindrical-polar coordinate system (R, ϕ, z). You'll first need to calculate the appropriate $\varphi(\boldsymbol{k})$ from Poisson's equation. Plot $\widehat{g}_{\alpha\beta}(R)$ and discuss the limiting cases $k_{\mathrm{D}} R \to 0$ and $\to \infty$ mathematically and physically. Compare with the $d = 3$ case worked out in the lecture notes and given by equation (9) above.[3]

[3] Hint: The two-dimensional Fourier transform of an axisymmetric function, which you derived in HW01 Problem 2(aa.*ii*), is called the *Hankel transform* of order zero. (It's used in spectral gyrokinetic codes.) You'll need its inverse for this problem. Also worth noting: $\delta(\boldsymbol{r}) = \delta(R)/\pi R$ in polar coordinates.