

Due February 17, 2025

Generals prep. Make sure you can provide brief definitions of the following terms: Klimontovich distribution, Liouville theorem, Bogoliubov timescale hierarchy, BBGKY hierarchy

1. **Practice with particle correlations.** Recall the definition of the Klimontovich distribution

$$F_\alpha(t, \mathbf{x}) \doteq \sum_{i=1}^{N_\alpha} \delta(\mathbf{x} - \mathbf{X}_{\alpha i}(t)),$$

where $\mathbf{x} = (\mathbf{r}, \mathbf{v})$ is shorthand for the phase-space variables and $\mathbf{X}_{\alpha i} = (\mathbf{R}_{\alpha i}, \mathbf{V}_{\alpha i})$ is shorthand for the phase-space location of particle i of species α ; $N_\alpha \gg 1$ denotes the total number of particles of species α . Also recall the definitions of the one-, two-, and three-particle reduced distribution functions:¹

$$f_\alpha(t, \mathbf{x}) \doteq N_\alpha \int d\mathbf{X}_{\text{all}} \frac{P_N}{d\mathbf{X}_{\alpha 1}},$$

$$f_{\alpha\beta}(t, \mathbf{x}, \mathbf{x}') \doteq N_\alpha N_\beta \int d\mathbf{X}_{\text{all}} \frac{P_N}{d\mathbf{X}_{\alpha 1} d\mathbf{X}_{\beta 1}}, \quad (1)$$

$$f_{\alpha\beta\gamma}(t, \mathbf{x}, \mathbf{x}', \mathbf{x}'') \doteq N_\alpha N_\beta N_\gamma \int d\mathbf{X}_{\text{all}} \frac{P_N}{d\mathbf{X}_{\alpha 1} d\mathbf{X}_{\beta 1} d\mathbf{X}_{\gamma 1}}, \quad (2)$$

respectively, where P_N is the Liouville distribution and the differential

$$d\mathbf{X}_{\text{all}} \doteq d\mathbf{X}_{\alpha 1} d\mathbf{X}_{\alpha 2} \dots d\mathbf{X}_{\alpha N_\alpha} d\mathbf{X}_{\beta 1} d\mathbf{X}_{\beta 2} \dots d\mathbf{X}_{\beta N_\beta} d\mathbf{X}_{\gamma 1} d\mathbf{X}_{\gamma 2} \dots d\mathbf{X}_{\gamma N_\gamma} \dots$$

indicates integration over all of the ‘ Γ space’ (including all species $\alpha, \beta, \gamma, \dots$). Denote an average over the Liouville distribution by angle brackets:

$$\langle G(F_\alpha, F_\beta, F_\gamma, \dots) \rangle \doteq \int d\mathbf{X}_{\text{all}} P_N G(F_\alpha, F_\beta, F_\gamma, \dots).$$

In many cases, one is interested in the differences between the fine-grained (Klimontovich) and coarse-grained (reduced) distribution functions, *viz.*, $\delta F_\alpha(t, \mathbf{x}) \doteq F_\alpha(t, \mathbf{x}) - f_\alpha(t, \mathbf{x})$. For example, in class we used the above definitions to calculate the correlation $\langle \delta F_\alpha \delta \mathbf{E} \rangle$, where $\delta \mathbf{E}$ is the fluctuating electric field given by Coulomb’s law,

$$\delta \mathbf{E}(t, \mathbf{r}) = -\frac{\partial}{\partial \mathbf{r}} \sum_{\beta} q_{\beta} \int d\mathbf{x}' \frac{\delta F_{\beta}(t, \mathbf{x}')}{|\mathbf{r} - \mathbf{r}'|}.$$

Calculating $\langle \delta F_\alpha \delta \mathbf{E} \rangle \propto \langle \delta F_\alpha \delta F_\beta \rangle$ was necessary to obtain the right-hand side of the kinetic equation governing the evolution of the one-particle distribution function. Ultimately, we

¹If α and β refer to the same species, then $\beta_1 \rightarrow \alpha_2$ and $N_\alpha N_\beta \rightarrow N_\alpha(N_\alpha - 1)$ in the definition of the two-particle distribution function (eq. 1). If $\alpha, \beta,$ and γ refer to the same species, then $\beta_1 \rightarrow \alpha_2, \gamma_1 \rightarrow \alpha_3,$ and $N_\alpha N_\beta N_\gamma \rightarrow N_\alpha(N_\alpha - 1)(N_\alpha - 2)$ in the definition of the three-particle distribution function (eq. 2). But, because the number of particles is asymptotically large, there is really no practical difference between $N_\alpha, N_\alpha - 1,$ and $N_\alpha - 2$.

found that

$$\begin{aligned}\langle \delta F_\alpha(t, \mathbf{x}) \delta F_\beta(t, \mathbf{x}') \rangle &= \langle F_\alpha(t, \mathbf{x}) F_\beta(t, \mathbf{x}') \rangle - f_\alpha(t, \mathbf{x}) f_\beta(t, \mathbf{x}') \\ &= g_{\alpha\beta}(t, \mathbf{x}, \mathbf{x}') + \delta_{\alpha\beta} \delta(\mathbf{x} - \mathbf{x}') f_\alpha(t, \mathbf{x}),\end{aligned}\quad (3)$$

where $g_{\alpha\beta} \doteq f_{\alpha\beta} - f_\alpha f_\beta$ is the two-particle correlation function. In other words, the evolution of the one-particle distribution function depends on correlations between two particles instigated by the Coulomb interaction. (The last term in (3) describes self-interaction, and thus is ultimately dropped.)

In this problem, you will calculate a few other correlations that we needed along the way in deriving the evolution equation for $g_{\alpha\beta}$.

- (a) Follow steps similar to those leading up to equation (II.5.9) in the lecture notes to show that

$$\begin{aligned}\langle F_\alpha(\mathbf{x}) F_\beta(\mathbf{x}') F_\gamma(\mathbf{x}'') \rangle &= f_{\alpha\beta\gamma}(\mathbf{x}, \mathbf{x}', \mathbf{x}'') + \delta_{\alpha\beta} \delta(\mathbf{x} - \mathbf{x}') f_{\alpha\gamma}(\mathbf{x}, \mathbf{x}'') \\ &\quad + \delta_{\beta\gamma} \delta(\mathbf{x}' - \mathbf{x}'') f_{\alpha\gamma}(\mathbf{x}, \mathbf{x}'') + \delta_{\alpha\gamma} \delta(\mathbf{x} - \mathbf{x}'') f_{\alpha\beta}(\mathbf{x}, \mathbf{x}') \\ &\quad + \delta_{\alpha\beta} \delta_{\beta\gamma} \delta(\mathbf{x} - \mathbf{x}') \delta(\mathbf{x}' - \mathbf{x}'') f_\alpha(\mathbf{x}),\end{aligned}\quad (4)$$

where I've omitted the time arguments for notational ease. Such a correlation is needed to calculate the term

$$-\frac{q_\alpha}{m_\alpha} \left\langle F_\beta(\mathbf{x}') \delta \mathbf{E}(\mathbf{r}) \cdot \frac{\partial F_\alpha(\mathbf{x})}{\partial \mathbf{v}} \right\rangle - \frac{q_\beta}{m_\beta} \left\langle F_\alpha(\mathbf{x}) \delta \mathbf{E}(\mathbf{r}') \cdot \frac{\partial F_\beta(\mathbf{x}')}{\partial \mathbf{v}'} \right\rangle, \quad (5)$$

in equation (II.5.15) of the lecture notes. These two terms represent the change in the correlation between a particle of species α at position \mathbf{x} and a particle of species β at position \mathbf{x}' due to electric-field fluctuations from the bath accelerating each one of the particles. With $\delta \mathbf{E}$ given by

$$\delta \mathbf{E}(t, \mathbf{r}) = -\frac{\partial}{\partial \mathbf{r}} \sum_\gamma q_\gamma \int d\mathbf{x}'' \frac{\delta F_\gamma(t, \mathbf{x}'')}{|\mathbf{r} - \mathbf{r}''|}, \quad (6)$$

it is clear that we need to compute $\langle F_\alpha F_\beta \delta F_\gamma \rangle = \langle F_\alpha F_\beta F_\gamma \rangle - \langle F_\alpha F_\beta \rangle f_\gamma \dots$

- (b) Use the Mayer cluster expansions for $f_{\alpha\beta}$ and $f_{\alpha\beta\gamma}$ in (4) to show that

$$\begin{aligned}\langle F_\alpha(\mathbf{x}) F_\beta(\mathbf{x}') \delta F_\gamma(\mathbf{x}'') \rangle &= h_{\alpha\beta\gamma}(\mathbf{x}, \mathbf{x}', \mathbf{x}'') + f_\alpha(\mathbf{x}) g_{\beta\gamma}(\mathbf{x}', \mathbf{x}'') + f_\beta(\mathbf{x}') g_{\alpha\gamma}(\mathbf{x}, \mathbf{x}'') \\ &\quad + \delta_{\alpha\beta} \delta(\mathbf{x} - \mathbf{x}') [g_{\alpha\gamma}(\mathbf{x}, \mathbf{x}'') + \delta_{\beta\gamma} \delta(\mathbf{x}' - \mathbf{x}'') f_\alpha(\mathbf{x})] \\ &\quad + \delta_{\beta\gamma} \delta(\mathbf{x}' - \mathbf{x}'') [g_{\alpha\gamma}(\mathbf{x}, \mathbf{x}'') + f_\alpha(\mathbf{x}) f_\gamma(\mathbf{x}'')] \\ &\quad + \delta_{\alpha\gamma} \delta(\mathbf{x} - \mathbf{x}'') [g_{\alpha\beta}(\mathbf{x}, \mathbf{x}') + f_\alpha(\mathbf{x}) f_\beta(\mathbf{x}')].\end{aligned}\quad (7)$$

The first term on the right-hand side of (7) describes three-particle correlations; we discussed in class why it can be dropped. The next two terms on that first line, when substituted into (5), yield important shielding terms (see the last part of §II.6 in the lecture notes). When substituted into (5), the second line of (7) ultimately vanishes, since it describes self-interactions. As for the final two lines of (7)...

(c) ... substitute them into (5) with (6) to obtain what becomes the source term in the evolution equation for $g_{\alpha\beta}$:

$$\frac{\partial}{\partial \mathbf{r}} \frac{q_\alpha q_\beta}{|\mathbf{r} - \mathbf{r}'|} \cdot \left(\frac{1}{m_\alpha} \frac{\partial}{\partial \mathbf{v}} - \frac{1}{m_\beta} \frac{\partial}{\partial \mathbf{v}'} \right) [f_\alpha(\mathbf{x}) f_\beta(\mathbf{x}') + g_{\alpha\beta}(\mathbf{x}, \mathbf{x}')]. \quad (8)$$

State under what conditions the second term ($g_{\alpha\beta}$) can be neglected compared to the first term ($f_\alpha f_\beta$), and explain physically why the first term generates two-particle correlations.

2. Landau damping via Hermite polynomials. Consider the following (1+1)-dimensional model of a homogeneous plasma perturbed about a Maxwellian equilibrium:

$$\frac{\partial g}{\partial t} + \underbrace{v \frac{\partial g}{\partial z}}_{\text{phase mixing}} + \underbrace{v F_0 \frac{\partial \varphi}{\partial z}}_{\text{electric field}} = \underbrace{C[g]}_{\text{collisions}}, \quad (9)$$

$$\varphi = \alpha \int_{-\infty}^{\infty} dv g. \quad (10)$$

Equation (9) is a linearized kinetic equation for the perturbed distribution function $g(t, z, v)$, which includes its advection by the particles (“phase mixing”), the acceleration of its constituent particles by the gradient of an electrostatic potential $\varphi(t, z)$ (“electric field”), and the removal of free energy by a collision operator $C[g]$ (“collisions”); the Maxwellian equilibrium distribution $F_0(v) = e^{-v^2}/\sqrt{\pi}$. For convenience, I have normalized the velocity v (in the z direction) to the thermal speed $v_{\text{th}} = \sqrt{2T/m}$, the spatial coordinate z to an arbitrary lengthscale L , and time t to L/v_{th} .

Equation (10) can be variously interpreted, depending upon the value of α . For example, if g is taken to be the perturbed ion distribution function in a plasma and the electrons are assumed to have a Boltzmann response, then $\alpha = T_e/T_i$ (the ratio of the electron and ion temperatures) and the resulting system of equations describes Landau-damped ion-acoustic waves propagating in the z direction. Equation (10) is then just a statement of quasineutrality. On the other hand, if g is taken to be the perturbed electron distribution function and the ions are assumed to have no response, then $\alpha = 2/k^2 \lambda_D^2$ and the resulting system of equations describes collisionlessly damped Langmuir waves (this case was originally considered by Landau in 1946; here, λ_D is the Debye length). Equation (10) then is the Gauss–Poisson equation. Finally, for

$$\alpha^\pm = - \left[-\frac{T_i}{T_e} + \frac{1}{\beta_i} \pm \sqrt{\left(1 + \frac{T_i}{T_e}\right)^2 + \frac{1}{\beta_i^2}} \right]^{-1},$$

equations (9) and (10) describe the evolution of compressive perturbations in a magnetized plasma at scales much larger than the ion Larmor radius (Schekochihin et al. 2009). In this case, there are two corresponding decoupled fluctuations g^+ and g^- , which are certain linear combinations of the zeroth and w_\perp^2 moments of the perturbed distribution function. All this is to say that equations (9) and (10) capture a variety of interesting kinetic physics.

(a) It is fruitful to recast the kinetic equation (9) in Hermite space by writing

$$g(v) = \sum_{m=0}^{\infty} \frac{H_m(v)F_0(v)}{\sqrt{2^m m!}} g_m, \quad g_m = \int dv \frac{H_m(v)}{\sqrt{2^m m!}} g(v), \quad (11)$$

where

$$H_m(v) \doteq (-1)^m e^{v^2} \frac{d^m}{dv^m} e^{-v^2} \quad (12)$$

is the Hermite polynomial of (integer) order m . Before we do so, first make a plot of $H_m(v)F_0(v)/\sqrt{2^m m!}$ versus $v \in [-4, 4]$ for $m \in [0, 8]$. Note that increasing m corresponds to increasingly fine-scale structure in velocity (the same way that increasing k in Fourier space corresponds to increasingly fine-scale structure in configuration space).

Note that the Hermite polynomials are orthogonal with respect to a Maxwellian weight:

$$\int dv \frac{H_m(v)H_{m'}(v)}{2^m m!} F_0(v) = \delta_{mm'}, \quad (13)$$

where $\delta_{mm'}$ is the Kronecker delta function. It will also be helpful in what follows to note that $g_0 = \int dv g(v)$ is the zeroth moment of the perturbed distribution function (i.e., the perturbed number density), $g_1/\sqrt{2} = \int dv v g(v)$ is the first moment (i.e., the perturbed momentum density), and $g_2/\sqrt{2} + g_0/2 = \int dv v^2 g(v)$ is the second moment. Higher m 's are often referred to as the “kinetic moments”.

(b) Use the recursion relation $dH_m/dv = 2vH_m - H_{m+1} = 2mH_{m-1}$ to show that the linearized Lenard–Bernstein collision operator (about which you’ll learn later),

$$C[g] = \nu \left[\frac{\partial}{\partial v} \left(v g + \frac{1}{2} \frac{\partial g}{\partial v} \right) + 2\nu u F_0 \right] \quad (14)$$

with ν being the collision frequency and $u \equiv \int dv v g(v)$, becomes $-\nu m g_m$ ($m \geq 2$) in Hermite space. That’s rather convenient.

(c) Show that equation (10) implies $\varphi = \alpha g_0$. Use this in equation (9) to derive the following set of equations coupling higher- m moments of g to the lower- m ones:

$$\frac{\partial g_0}{\partial t} = -\frac{\partial}{\partial z} \frac{g_1}{\sqrt{2}}, \quad (15a)$$

$$\frac{\partial g_1}{\partial t} = -\frac{\partial}{\partial z} \left(g_2 + \frac{1+\alpha}{\sqrt{2}} g_0 \right), \quad (15b)$$

$$\frac{\partial g_m}{\partial t} = -\frac{\partial}{\partial z} \left(\sqrt{\frac{m+1}{2}} g_{m+1} + \sqrt{\frac{m}{2}} g_{m-1} \right) - \nu m g_m, \quad m \geq 2. \quad (15c)$$

Given your plot from part (a) and your understanding of Landau damping, explain each of equations (15a,b,c) physically.

Because equations (15a,b,c) are linear, whatever free energy is in one wavenumber k cannot be transferred to another wavenumber. As such, the “cascade” is entirely in velocity space

(i.e., to higher m), with each wavenumber evolving independently of all the others. Therefore, we need only solve (15) for a single wavenumber k ; the solution is then easily rescaled for different wavenumbers. So write

$$g_m(t, z) = \sum_k g_{m,k}(t) \exp(ikz) \quad (16)$$

and proceed...

(d) Show that the free energy in wavenumber k ,

$$W_k \doteq \frac{1}{2} \sum_{m=0}^{\infty} |g_{m,k}|^2 + \frac{\alpha}{2} |g_{0,k}|^2 \quad (17)$$

satisfies

$$\frac{dW_k}{dt} = -\nu \sum_{m=2}^{\infty} m |g_{m,k}|^2. \quad (18)$$

Interpret this equation physically, making contact with the kinetic free-energy conservation law derived in class, in which

$$W_{\text{Vlasov}} = \int d\mathbf{r} \left[\sum_{\alpha} \int d\mathbf{v} \frac{T \langle \delta f_{\alpha}^2 \rangle}{2f_{0\alpha}} + \frac{\langle E^2 \rangle}{8\pi} \right]. \quad (19)$$

In particular, what must happen for irreversible dissipation to occur as $\nu \rightarrow 0$?

(e) Download `landau_damping.f90` from

<https://www.astro.princeton.edu/~kunz/Site/AST554>

This code solves the following implicit set of equations (cf. equations (15)):²

$$\begin{aligned} \frac{g_{0,k}^{(n+1)} - g_{0,k}^{(n)}}{\Delta t} &= -ik \frac{g_{1,k}^{(n+1)}}{\sqrt{2}}, \\ \frac{g_{1,k}^{(n+1)} - g_{1,k}^{(n)}}{\Delta t} &= -ik \left(g_{2,k}^{(n+1)} + \frac{1+\alpha}{\sqrt{2}} g_{0,k}^{(n+1)} \right), \\ \frac{g_{m,k}^{(n+1)} - g_{m,k}^{(n)}}{\Delta t} &= -ik \left(\sqrt{\frac{m+1}{2}} g_{m+1,k}^{(n+1)} + \sqrt{\frac{m}{2}} g_{m-1,k}^{(n+1)} \right) - \nu m^4 g_{m,k}^{(n+1)}, \quad 2 \leq m \leq N_m \end{aligned}$$

where time has been discretized as $t^{(n)} = n\Delta t$ with (fixed) timestep Δt and the integer $n \geq 0$. The default free parameters in the code are as follows: $N_m = 128$, $\alpha = 1$, $k = 2\pi$, $\nu = 10^{-9}$, and $g_{0,k}^{(0)} = 1$ (which is arbitrary, because the equations are linear). Note that the collision term $-\nu m g_{m,k}$ has been replaced by $-\nu m^4 g_{m,k}$, which is a sort of “hyper-collisionality”; the higher power of m here is useful for allowing a larger “collisionless” range of m before the cascade to higher values of m is affected by collisions. You can compile the code using

²If you’re interested in such details, the resulting matrix equation can be cast in the form $\sum_{m=0}^{N_m} \mathcal{A}_{\ell m} g_{m,k}^{(n+1)} = g_{\ell,k}^{(n)}$, where $\mathcal{A}_{\ell m}$ is a tri-diagonal matrix. There is an $\mathcal{O}(N_m)$ algorithm known as the Thomas algorithm for solving tri-diagonal matrix equations (see https://en.wikipedia.org/wiki/Tridiagonal_matrix_algorithm).

```
gfortran -o landau_damping landau_damping.f90
```

Run it. It will output two files: `landau_damping.phik2` and `landau_damping.gkm`. The former contains $|\varphi_k|^2$ versus time; the latter contains the real and imaginary parts of $g_{m,k}$ versus time. These files can be read using the following python script:

```
import numpy as np

fname = 'landau_damping.phik2'
data = np.genfromtxt(fname,autostrip=True)
tphi = data[:,0] ; phik2 = data[:,1]

fname = 'landau_damping.gkm'
data = np.genfromtxt(fname,autostrip=True)
time = data[:,0] ; mlab = data[:,1]
gkmr = data[:,2] ; gkmi = data[:,3]

nm = 128 ; nt = int(time.size/nm)
m = np.arange(0,nm) ; indx = np.arange(0,nt)*nm
t = time[indx]
gkm = np.zeros((nm,nt),dtype=complex)
for m in range(nm):
    gkm[m,:] = gkmr[indx+m] + 1j*gkmi[indx+m]
```

Make semilogy plots of $|\varphi_k|^2$ and W_k vs time. Be sure to label your axes. Comment on what you observe.

- (f) Use $g_{m,k}$ and equations (11) and (16) to reconstruct and plot $g(t, z, v)$ at $t = 0$, $t = 0.5$, and $t = 1$. (Did you remember to take the real part?) Comment on what you observe.
- (g) Re-run the code with $\nu = 10^{-2}$ and make plots of $|\varphi_k|^2$ vs time and $g(t, z, v)$ at $t = 1$. Comment on what you observe.

We're going to revisit this Landau-Hermite problem in HW05, so keep your notes and code.