Due February 3, 2025

1. **Things bumping into things**. In this problem, you'll work through what ought to be a familiar problem: binary Coulomb collisions. This should serve as a remembrance of things past and a preview of things to come. Pluck two charged particles from a fully ionized plasma and label them 1 and 2 (what else?). Their charges are q_1 and q_2 and their masses are m_1 and m_2, respectively. The nonrelativistic equations of motion for the two charged particles are, of course,

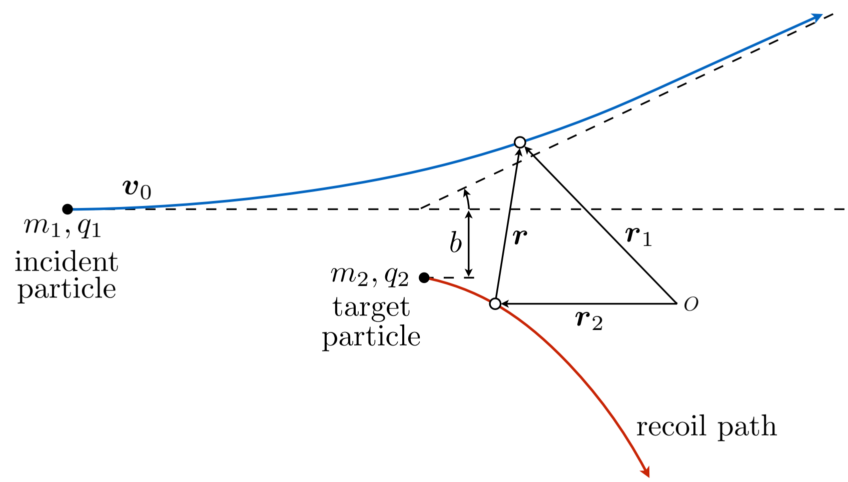
$$m_1 \frac{\mathrm{d}^2 \boldsymbol{r}_1}{\mathrm{d}t^2} = q_1 q_2 \frac{\boldsymbol{r}_1 - \boldsymbol{r}_2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|^3} \quad \text{and} \quad m_2 \frac{\mathrm{d}^2 \boldsymbol{r}_2}{\mathrm{d}t^2} = q_1 q_2 \frac{\boldsymbol{r}_2 - \boldsymbol{r}_1}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|^3}\,, \tag{1}$$

where $\boldsymbol{r}_1$ is the position of charge 1 and $\boldsymbol{r}_2$ is the position of charge 2.

(a) Show that the center of mass $\boldsymbol{R} \equiv (m_1 \boldsymbol{r}_1 + m_2 \boldsymbol{r}_2)/(m_1 + m_2)$ satisfies $\mathrm{d}^2 \boldsymbol{R}/\mathrm{d}t^2 = 0$ and that the relative position $\boldsymbol{r} \equiv \boldsymbol{r}_1 - \boldsymbol{r}_2$ obeys

$$\mu \frac{\mathrm{d}^2 \boldsymbol{r}}{\mathrm{d}t^2} = \frac{q_1 q_2}{|\boldsymbol{r}|^3} \boldsymbol{r}\,, \quad \text{where} \quad \mu \equiv \frac{m_1 m_2}{m_1 + m_2} \tag{2}$$

is the reduced mass.



(b) Now throw one at the other, so that their relative velocity is v_0 and the impact parameter is b. (You may take them to start infinitely far apart; see the figure above.) Show that angular momentum is conserved and, thus, that the solution to (2) may be written parametrically as

$$\frac{b}{r} = -\frac{\cos(\theta + \alpha) + \cos\alpha}{\sin\alpha}\,, \quad \text{where} \quad \tan\alpha \equiv \frac{\mu b v_0^2}{q_1 q_2} \tag{3}$$

and θ is the deflection angle in the center-of-mass frame. [Hint: set $u(\theta) = b/r(\theta)$ and show that $\mathrm{d}^2 u/\mathrm{d}\theta^2 + u = -\cot\alpha$. The initial polar angle $\theta_0 = \pi$, corresponding to particles starting infinitely far apart with m_1 approaching from the left in a standard polar coordinate system.] Use this to show that the asymptotic deflection angle in the center-of-mass frame θ_∞ at $r = \infty$ satisfies

$$\tan\frac{\theta_\infty}{2} = \frac{q_1 q_2}{\mu b v_0^2}. \tag{4}$$

Note that the magnitude of this angle depends only on the impact parameter and initial relative velocity, and not on the sign of the binary force: like and unlike charges are deflected by the same amount. Larger impact parameters and larger initial relative velocities give smaller deflections, which intuitively makes sense. To get a 90° deflection angle requires an impact parameter

$$b_{90^\circ} = \frac{q_1 q_2}{\mu v_0^2}. \tag{5}$$

Scattering an ion-electron pair to 90° requires a smaller impact parameter (by a factor of $\simeq$2) than does scattering a pair of electrons (given the same v_0). Physically, why?

(c) Repeat this experiment many times while varying the impact factor between b and $b + \mathrm{d}b$. You will find that particle 1 will be scattered into asymptotic angles between θ_∞ and $\theta_\infty + \mathrm{d}\theta_\infty$. Use conservation of particles to argue that the ratio of the number of scattered particles per unit solid angle $\mathrm{d}\Omega = 2\pi \sin\theta_\infty \mathrm{d}\theta_\infty$ to the total number of incoming particles per unit area (i.e., the "differential cross section") is given by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{b}{\sin\theta_\infty}\left|\frac{\mathrm{d}b}{\mathrm{d}\theta_\infty}\right| = \left(\frac{q_1 q_2}{2\mu v_0^2}\right)^2 \frac{1}{\sin^4(\theta_\infty/2)}. \tag{6}$$

This should look familiar – it's the *Rutherford scattering cross section*. Use this and (5) to show that the cross section for scattering of a particle by 90° or more in a single encounter is $\pi b_{90^\circ}^2$.

(d) Instead of a single large-angle scattering event, let us now consider the cumulative effect of many small-angle scatterings, for which (4) gives $\theta_\infty \simeq 2q_1 q_2/\mu b v_0^2 \equiv \vartheta(b)$. (Such scatterings could result from a series of large-impact-parameter encounters.) For concreteness, imagine an electron ($m_1 = m_e$, $q_1 = -e$) moving through a bath of ions ($m_2 = m_i \gg m_e$, $q_2 = Ze$, number density n), undergoing such small-angle scatterings.[1] If random, these scatterings will accumulate like a random walk in angle away from the original trajectory of the electron, with an average deflection angle $\langle\theta\rangle = 0$ but with a mean-square deflection angle $\langle\theta^2\rangle$ proportional to the number of scattering events. Use this reasoning to argue that, after the electron has traversed a distance L through the ion bath and scattered many times,

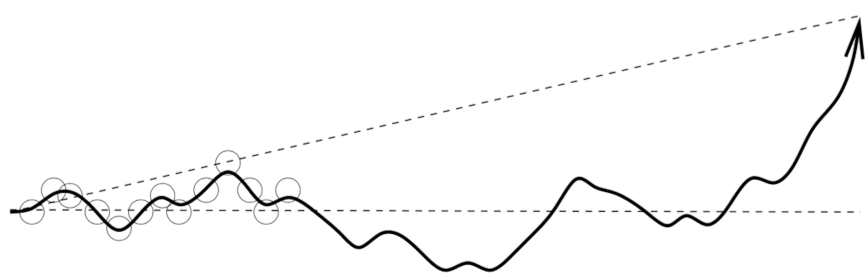
$$\langle\theta^2\rangle = 2\pi n L \int_{b_{\min}}^{b_{\max}} \mathrm{d}b\, b\, \vartheta^2(b) = \frac{8\pi n L Z^2 e^4}{m_e^2 v_0^4} \ln\frac{b_{\max}}{b_{\min}}, \tag{7}$$

[1] The specification to an electron in a bath of ions is not really necessary. One could perform the same calculation without making any assumptions about the characteristics of the charges, as long as q_1 is considered to be a *test particle* – that is, q_1 is assumed not to noticeably disturb the q_2 charges off of whose Coulomb potentials it scatters. Of course, the accuracy of this assumption improves as $m_1 \to 0$; you can't get much better than an electron!

where $b_{\max}$ and $b_{\min}$ are the maximum and minimum impact parameters encountered.

That upper and lower cutoffs for the impact parameter must be enforced should be familiar from GPP1. The divergence as $b_{\max} \to \infty$ is due to the fact that the Coulomb potential used in (1) is long-range. By now, you should know well that the potential between a test particle (electron) and a plasma particle (ion) is not $\varphi = -Ze^2/r$ but rather $\varphi = -(Ze^2/r)\exp(-r/\lambda_{\rm D})$, where $\lambda_{\rm D}^{-2} \equiv \sum_\alpha 4\pi q_\alpha^2 n_\alpha / T_\alpha$ is the Debye length. That is, the ion scatterer is Debye-shielded by the surrounding plasma so that its Coulomb potential is appreciably screened at distances greater than the Debye length. Of course, this screening can be taken into account, and we will do so later in the course. But, for now, let's take $b_{\max} = \lambda_{\rm D}$. The lower cutoff on b is needed to justify the small-angle approximation used in obtaining $\vartheta(b)$. With $\vartheta_{\max} \sim 1$ we have $b_{\min} \sim Ze^2/T_e$, where we have estimated $v_0^2 \sim 2T_e/m_e$. When quantum-mechanical effects are not important, this estimate is almost as good as any (it's in a logarithm!); if the temperature is high enough that $T_e > Z^2 e^4 m_e/(2\pi\hbar^2)$, the $b_{\min}$ should be the de Broglie wavelength. *Anyway*, note that $b_{\max}/b_{\min} \sim \Lambda \equiv n\lambda_{\rm D}^3$ using the classical estimate of $b_{\min}$, so let's just call it Λ and get on with it...

(e) Equation (7) provides an estimate for the mean deflection angle of an electron undergoing many small-angle scatterings over a distance L. Use this formula to derive how far an electron must travel to accumulate a large deflection angle, i.e., $\langle \vartheta^2 \rangle \sim 1$ (see the illustration below, taken from Krommes' notes). Use this to obtain an estimate of the cross section for multiple small-angle scatterings to result in a large-angle deflection. Compare this with the single large-angle scattering cross section, $\pi b_{90^\circ}^2$, found in part (c). Knowing that $\ln\Lambda \sim 10$–30 in most weakly coupled plasmas, what does this say about the statistical importance of small-angle vs large-angle scatterings? What does this say about the ratio of the collision frequency and the plasma frequency?



(f) Use $\ln\Lambda = 20$ and $Z = 1$ in your expression from part (e) to compute the multi-scattering cross section for a 100 keV plasma. Compare this to the cross section for a *d*-*d* fusion reaction and write something intelligent.

2. **Ballistic propagation with Fourier, Laplace, and Green**. In this course, a lot of particles will be zooming around. To get irreversibility (as the course promises), they ought to interact at some point; but let's forget about that for now. Consider the following partial differential equation:

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\right) f(t,\boldsymbol{r},\boldsymbol{v}) = S(t,\boldsymbol{r},\boldsymbol{v}), \qquad t \geq 0 \tag{8}$$

where $S(t, \boldsymbol{r}, \boldsymbol{v})$ is some unspecified phase-space- and time-dependent source term. In this problem, you'll solve it in two different, but equivalent, ways. Again, this should be review.

(a) Define the (multi-dimensional) Fourier and inverse-Fourier transforms as follows:

$$f(\boldsymbol{k}) = \int \frac{\mathrm{d}\boldsymbol{r}}{(2\pi)^3}\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} f(\boldsymbol{r}) \qquad \text{and} \qquad f(\boldsymbol{r}) = \int \mathrm{d}\boldsymbol{k}\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} f(\boldsymbol{k}). \tag{9}$$

Note the 2π convention used in this course. It is different than that used by Klimontovich and Krommes, but is the same as that used by Nicholson, Ichimaru, Montgomery, and Krall and Trivelpiece.[2] I like it this way because, then, the Fourier transform of $f(\boldsymbol{r}) = 1$ is just the delta function $\delta(\boldsymbol{k})$. So, Fourier transform (8) to find

$$\left(\frac{\partial}{\partial t} + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{v}\right) f_{\boldsymbol{k}}(t, \boldsymbol{v}) = S_{\boldsymbol{k}}(t, \boldsymbol{v}), \qquad t \geq 0. \tag{10}$$

I know this is a one-liner and is stupidly easy, but I belabor the point simply to establish conventions and note in passing that, sometimes, I write $f(t, \boldsymbol{k}, \boldsymbol{v})$ instead of $f_{\boldsymbol{k}}(t, \boldsymbol{v})$.

(aa) While you're at it, prove the following:

i. $$\int \frac{\mathrm{d}\boldsymbol{r}}{(2\pi)^3}\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\, f(\mathscr{r}) = \frac{1}{2\pi^2 k}\int_0^\infty \mathrm{d}\mathscr{r}\, \mathscr{r} \sin k\mathscr{r} f(\mathscr{r}), \quad \text{where} \quad \mathscr{r} \equiv |\boldsymbol{r}| \text{ in 3D;} \tag{11a}$$

ii. $$\int \frac{\mathrm{d}\boldsymbol{r}}{(2\pi)^2}\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\, f(R) = \frac{1}{2\pi}\int_0^\infty \mathrm{d}R\, R J_0(kR) f(R), \quad \text{where} \quad R \equiv |\boldsymbol{r}| \text{ in 2D;} \tag{11b}$$

iii. $$\int \frac{\mathrm{d}\boldsymbol{r}}{(2\pi)^3}\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \int \frac{\mathrm{d}\boldsymbol{r}'}{(2\pi)^3}\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}'\cdot\boldsymbol{r}'} f(\boldsymbol{r} - \boldsymbol{r}') = f(\boldsymbol{k})\, \delta(\boldsymbol{k} + \boldsymbol{k}'). \tag{11c}$$

You won't need these for this homework, but they'll come in handy later.

(b) Okay. First, we'll solve (10) using Laplace and inverse-Laplace transforms; my convention for these is the same as Nicholson's and Ichimaru's:

$$f(\omega) = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t} f(t) \qquad \text{and} \qquad f(t) = \int_L \frac{\mathrm{d}\omega}{2\pi}\, \mathrm{e}^{-\mathrm{i}\omega t} f(\omega), \tag{12}$$

where L denotes the Laplace contour (i.e., a straight line in the complex plane parallel to the real ω axis running from $-\infty$ to ∞ and intersecting the imaginary ω axis at $\mathrm{Im}(\omega) = \sigma$, where $\sigma > 0$ is a real number such that $|f(t)| < \exp(\sigma t)$ as $t \to \infty$). The reason I normalize the transforms in this fashion is that the inverse-Laplace transform almost always involves the use of Cauchy's residue theorem; the 2π from the residues neatly cancels the 2π in the denominator of (12). You might be used to seeing the Laplace transform written as $f(s) = \int_0^\infty \mathrm{d}t\, \exp(-st) f(t)$, but pretty much everything we'll look at in this course is dominantly an oscillation, and so I prefer to work directly with the frequency ω.

So, use (12) to show that the solution to (10) is

$$f_{\boldsymbol{k}}(t, \boldsymbol{v}) = \mathrm{i}\int_L \frac{\mathrm{d}\omega}{2\pi} \frac{\mathrm{e}^{-\mathrm{i}\omega t}}{\omega - \boldsymbol{k}\cdot\boldsymbol{v} + \mathrm{i}0} \left[f_{\boldsymbol{k}}(0, \boldsymbol{v}) + S_{\boldsymbol{k},\omega}(\boldsymbol{v})\right], \tag{13}$$

[2]Or, rather, Krall *or* Trivelpiece: Chapter 10 and the latter part of Chapter 11 of their text inexplicably shift the factor of 2π into the inverse transform, so perhaps the authors themselves had conflicting conventions.

where $f_{\boldsymbol{k}}(0, \boldsymbol{v})$ is the initial distribution function and $+\mathrm{i}0$ is shorthand for $\lim_{\epsilon \to 0^+}(\mathrm{i}\epsilon)$. Explain physically why I added $+\mathrm{i}0$ to the denominator. Again, this is simple, but I'm dragging you through it to establish notation.

(c) Let us set the source term $S = 0$ for now. Perform the remaining ω-integral in (13) to obtain the *ballistic response*

$$f_{\boldsymbol{k}}(t, \boldsymbol{v}) = f_{\boldsymbol{k}}(0, \boldsymbol{v})\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{v}t}. \tag{14}$$

Provide the details of your calculation. Namely, (i) draw the original and the shifted Laplace contours in the $\mathrm{Re}(\omega)$-$\mathrm{Im}(\omega)$ plane; (ii) explain why you can analytically continue the integrand into the lower half-ω plane; (iii) explain why (in this case) it's "$-2\pi\mathrm{i}$" times the sum of the residues, and not the usual "$2\pi\mathrm{i}$"; and (iv) show that all the straight pieces of the shifted contour (i.e., those not encircling a pole) vanish.

Finally, perform the inverse Fourier transform to obtain $f(t, \boldsymbol{v}, \boldsymbol{r})$.

(d) There is a mathematically equivalent, but conceptually different, way of solving (8), which is provided by the theory of Green's functions. Recall that a Green's function describes the response due to a unit point source. It is used to propagate initial conditions or the response to a pulse at some time into the future (or the past, if desired, but we won't desire such a thing in this course ... it's named *irreversible* processes, after all). Thus, causality is built into the Green's functions in a more transparent way than for the Laplace transform, which requires you to think about the complex plane as some kind of temporal plane with $\pm\mathrm{i}$ telling you something about the past or future. To remind you of the details: the idea behind the Green's function is that one can divide up the source into a collection of impulses,

$$S(t) = \int_{-\infty}^{\infty} \mathrm{d}t'\, \delta(t - t')S(t'),$$

the response to each being given by the appropriate Green's function. This results from the linearity of the system, i.e., we can just integrate up or "superpose" the responses to stimuli at different times to get the full solution. Thus, if we can solve

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\right) G(t, \boldsymbol{x}; t', \boldsymbol{x}') = \delta(t - t')\delta(\boldsymbol{x} - \boldsymbol{x}'), \tag{15}$$

where $\boldsymbol{x} = (\boldsymbol{r}, \boldsymbol{v})$ is shorthand for the phase-space variables, then our solution to (10) is simply

$$f(t, \boldsymbol{x}) = \int \mathrm{d}\boldsymbol{x}' \left[f(0, \boldsymbol{x}')G(t, \boldsymbol{x}; 0, \boldsymbol{x}') + \int_0^t \mathrm{d}t'\, S(t', \boldsymbol{x}')G(t, \boldsymbol{x}; t', \boldsymbol{x}') \right], \tag{16}$$

where $\mathrm{d}\boldsymbol{x}' = \mathrm{d}\boldsymbol{r}'\mathrm{d}\boldsymbol{v}'$. Physically, the Green's function propagates the initial conditions forward in time while taking into account phase-space stimuli from the source term. By solving (15), prove that the relevant Green's function is

$$G_{\boldsymbol{k},\omega}(\boldsymbol{v}; \boldsymbol{v}') = \frac{\mathrm{i}\,\delta(\boldsymbol{v} - \boldsymbol{v}')}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}} \quad \Longrightarrow \quad G(t, \boldsymbol{x}; t', \boldsymbol{x}') = \delta(\boldsymbol{r} - \boldsymbol{r}' - \boldsymbol{v}(t - t'))\delta(\boldsymbol{v} - \boldsymbol{v}') \tag{17}$$

and so

$$f(t, \boldsymbol{r}, \boldsymbol{v}) = f(0, \boldsymbol{r} - \boldsymbol{v}t, \boldsymbol{v}) + \int_0^t \mathrm{d}t'\, S(t', \boldsymbol{r} - \boldsymbol{v}(t - t'), \boldsymbol{v}). \tag{18}$$