Useful collection of mathy stuff

Plemlj's formula: $$\lim_{\epsilon\to +0} \frac{1}{x-a\pm i\epsilon} = PV\left(\frac{1}{x-a}\right) \mp i\pi\delta(x-a),$$
where PV denotes the principal value. As a consequence of this expression, we also have
$$\lim_{\epsilon\to +0} \frac{\epsilon}{(x-a)^2+\epsilon^2} = \pi\delta(x-a).$$

Delta functions: $$\delta[f(x)] = \sum_i \frac{\delta(x-x_i)}{|f'(x_i)|}, \quad \text{where } f(x_i)=0$$

$$\delta(x-a) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{\pm ik(x-a)}$$

$$\int_{-\infty}^{\infty} \delta(ax)\,dx = \frac{1}{|a|} \Rightarrow \delta(ax) = \frac{\delta(x)}{|a|}$$

$$\delta(x^2-a^2) = \frac{1}{2|a|}\left[\delta(x+a)+\delta(x-a)\right]$$

$$\delta(x) = \lim_{\epsilon\to +0} \frac{1}{\pi}\frac{\sin(x/\epsilon)}{\epsilon}$$

$$\delta(\vec{r}) = \frac{\delta(r)}{2\pi r^2} \text{ (spherical)}, \quad \frac{\delta(R)\delta(z)}{\pi R} \text{ (cylindrical)}$$

Bessel functions:

$$e^{ia\sin\vartheta} = \sum_{n=-\infty}^{\infty} J_n(a)\, e^{in\vartheta}$$

$$\sin\vartheta\, e^{ia\sin\vartheta} = \sum_{n=-\infty}^{\infty} (-i)\, J_n'(a)\, e^{in\vartheta}$$

$$\cos\vartheta\, e^{ia\sin\vartheta} = \sum_{n=-\infty}^{\infty} \frac{n J_n(a)}{a}\, e^{in\vartheta}$$

$$\sin^2\vartheta\, e^{ia\sin\vartheta} = -\sum_{n=-\infty}^{\infty} J_n''(a)\, e^{in\vartheta}$$

$$\sin\vartheta\cos\vartheta\, e^{ia\sin\vartheta} = -i\sum_{n=-\infty}^{\infty} \frac{n}{a}\left(J_n'(a) - \frac{J_n(a)}{a}\right) e^{in\vartheta}$$

$$e^{ia\cos\vartheta} = \sum_{n=-\infty}^{\infty} i^n J_n(a)\, e^{in\vartheta}$$

$$\cos\vartheta\, e^{ia\cos\vartheta} = \sum_{n=-\infty}^{\infty} i^{n-1} J_n'(a)\, e^{in\vartheta}$$

$$\sin\vartheta\, e^{ia\cos\vartheta} = \sum_{n=-\infty}^{\infty} i^n \left(-\frac{n J_n(a)}{a}\right) e^{in\vartheta}$$

$$\cos^2\vartheta\, e^{ia\cos\vartheta} = \sum_{n=-\infty}^{\infty} i^{n-2} J_n'(a)\, e^{in\vartheta}$$

$$\sin\vartheta\cos\vartheta\, e^{ia\cos\theta} = \sum_{n=-\infty}^{\infty} i^{n-1}\left(\frac{J_n(a)\,n}{a^2} - \frac{nJ_n'(a)}{a}\right) e^{in\vartheta}$$

$$J_{-n}(a) = (-1)^n J_n(a)$$

$$\frac{2n}{a} J_n(a) = J_{n-1}(a) + J_{n+1}(a)$$

$$2\frac{dJ_n(a)}{da} = J_{n-1}(a) - J_{n+1}(a)$$

$$\left(\frac{1}{a}\frac{d}{da}\right)^m \left[a^n J_n(a)\right] = a^{n-m} J_{n-m}(a)$$

$$\left(\frac{1}{a}\frac{d}{da}\right)^m \left[\frac{J_n(a)}{a^n}\right] = (-1)^m \frac{J_{n+m}(a)}{a^{n+m}}$$

$$J_n(a) \sim \frac{1}{\Gamma(n+1)}\left(\frac{a}{2}\right)^n \quad \text{for } 0 < a \ll \sqrt{n+1} \text{ for } n \geq 0$$

$$J_n(a) \sim \frac{(-1)^n}{(-n)!}\left(\frac{2}{a}\right)^n \quad \text{for } 0 < a \ll \sqrt{n+1} \text{ for } n < 0$$

Plasma dispersion function: $Z(\zeta) \equiv \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-t^2}\,dt}{t-\zeta}$

$$Z'(\zeta) = -2\left[1+\zeta Z(\zeta)\right] \Rightarrow \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \frac{t e^{-t^2}\,dt}{t-\zeta} = 1+\zeta Z(\zeta)$$

$$Z''(\zeta) = -2Z(\zeta) + 4\zeta\left[1+\zeta Z(\zeta)\right]$$

$$\Rightarrow \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \frac{t^2 e^{-t^2}\,dt}{t-\zeta} = \zeta\left[1+\zeta Z(\zeta)\right]$$

$$Z'''(\zeta) = 8 + 12\zeta Z(\zeta) - 8\zeta^2\left[1+\zeta Z(\zeta)\right]$$

$$\Rightarrow \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \frac{t^3 e^{-t^2}\,dt}{t-\zeta} = \frac{1}{2} + \zeta^2\left[1+\zeta Z(\zeta)\right]$$

Fourier transforms: $$f(\vec{k}) = \int \frac{d\vec{r}}{(2\pi)^n}\, e^{-i\vec{k}\cdot\vec{r}} f(\vec{r})$$

$$f(\vec{r}) = \int d\vec{k}\, e^{i\vec{k}\cdot\vec{r}} f(\vec{k})$$

in 3D, $$f(k) = \int \frac{d\vec{r}}{(2\pi)^3}\, e^{-i\vec{k}\cdot\vec{r}} f(r)$$

$$= \frac{1}{2\pi^2 k}\int_0^\infty dr\; r \sin kr\, f(r)$$

where $r \equiv |\vec{r}|$ in 3D

in 2D, $$f(k) = \int \frac{d\vec{r}}{(2\pi)^2}\, e^{-i\vec{k}\cdot\vec{r}} f(R)$$

$$= \frac{1}{2\pi}\int_0^\infty dR\; R J_0(kR) f(R)$$

where $R \equiv |\vec{r}|$ in 2D

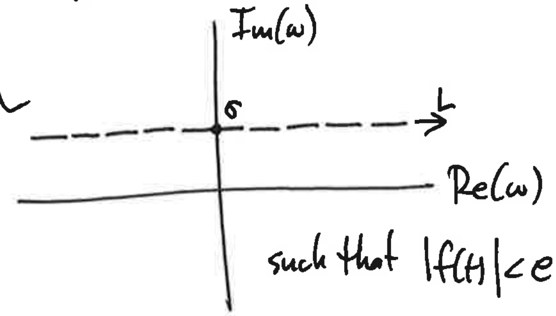
$$\int \frac{d\vec{r}}{(2\pi)^n}\, e^{-i\vec{k}\cdot\vec{r}} \int \frac{d\vec{r}\,'}{(2\pi)^n}\, e^{-i\vec{k}'\cdot\vec{r}\,'} f(\vec{r}-\vec{r}\,')$$

$$= f(\vec{k})\, \delta(\vec{k}+\vec{k}')$$

$f(\vec{r})$	$f(\vec{k}) \equiv \int \frac{d\vec{r}}{(2\pi)^n} e^{-i\vec{k}\cdot\vec{r}} f(\vec{r})$
1	$\delta(\vec{k})$
$(2\pi)^n \delta(\vec{r})$	1
$e^{\pm i\vec{q}\cdot\vec{r}}$	$\delta(\vec{k} \mp \vec{q})$
$\frac{1}{\lvert\vec{r}\rvert}$	$\frac{1}{2\pi^2 k^2}$ in 3D
$\frac{e^{-a\lvert\vec{r}\rvert}}{\lvert\vec{r}\rvert}$	$\frac{1}{2\pi^2(k^2+a^2)}$ in 3D
$\frac{1}{r^2+a^2}$	$\frac{e^{-\lvert ka\rvert}}{4\pi\lvert k\rvert}$ in 3D $\frac{1}{2\pi} K_0(ka)$ in 2D

Laplace transforms: $f(\omega) = \int_0^\infty dt\, e^{i\omega t} f(t)$

$$f(t) = \int_L \frac{d\omega}{2\pi} e^{-i\omega t} f(\omega)$$



such that $|f(t)| < e^{\sigma t}$ as $t \to \infty$

Half-integer factorials: $\Gamma\left(\frac{1}{2}+n\right) = \left(-\frac{1}{2}+n\right)!$

$$= \frac{(2n-1)!}{2^{2n-1}(n-1)!}\sqrt{\pi}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi}$$

$$\Gamma\left(\frac{5}{2}\right) = \frac{3}{4}\sqrt{\pi}$$

$$\Gamma\left(\frac{7}{2}\right) = \frac{15}{8}\sqrt{\pi}$$

$$\Gamma\left(\frac{9}{2}\right) = \frac{105}{16}\sqrt{\pi}$$

Associated Laguerre polynomials: Rodrigues representation

$$L_n^{(k)}(x) = \frac{e^x x^{-k}}{n!} \frac{d^n}{dx^n}\left(e^{-x} x^{n+k}\right)$$

$$= \frac{1}{n!} \Sigma_{i=0}^{n} \frac{n!}{i!} \binom{k+n}{n-i} (-x)^i$$

$$\int_0^\infty e^{-x} x^k L_n^{(k)}(x) L_m^{(k)}(x)\, dx = \frac{(n+k)!}{n!} \delta_{mn}$$

$$\int_0^\infty e^{-x} x^{k+1} \left[L_n^k(x)\right]^2 dx = \frac{(n+k)!}{n!} (2n+k+1)$$

$$L_0^{(k)}(x) = 1$$

$$L_1^{(k)}(x) = -x + k + 1$$

$$L_2^{(k)}(x) = \frac{1}{2}\left[x^2 - 2(k+2)x + (k+1)(k+2)\right]$$

Legendre polynomials: Rodrigues representation

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}\left[(x^2-1)^n\right]$$

$$\frac{1}{\sqrt{1-2xt+t^2}} = \Sigma_{n=0}^{\infty} P_n(x) t^n \quad \text{(generating function)}$$

$$\int_{-1}^{1} P_m(x) P_n(x)\, dx = \frac{2}{2n+1} \delta_{mn}$$

$$\delta(y-x) = \sum_{l=0}^{\infty} \left(l + \frac{1}{2}\right) P_l(y) P_l(x)$$

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$