Summary of topics in Irreversible Processes in Plasmas

Matthew W. Kunz†

Department of Astrophysical Sciences, Princeton University, Peyton Hall, Princeton, NJ 08544

Princeton Plasma Physics Laboratory, PO Box 451, Princeton, NJ 08543

(compiled on 12 April 2024)

If you can talk brilliantly about a problem, it can create the consoling illusion that it has been mastered.

Stanley Kubrick

- Klimontovich distribution: $F_\alpha(t,\boldsymbol{r},\boldsymbol{v}) = \sum_{i=1}^{N_\alpha} \delta(\boldsymbol{r}-\boldsymbol{R}_{\alpha i}(t))\delta(\boldsymbol{v}-\boldsymbol{V}_{\alpha i}(t))$

- Liouville equation: $\dfrac{\mathrm{d}P_N}{\mathrm{d}t} \doteq \dfrac{\partial P_N}{\partial t} + \{P_N,\mathcal{H}\} = 0$

- Bogoliubov timescale hierarchy: $\omega_{\mathrm{p}}(\ln\varLambda/\varLambda) \sim \nu_{\mathrm{coll}} \sim (L/\lambda_{\mathrm{mfp}})t^{-1}_{\mathrm{dyn}} \sim (L/\lambda_{\mathrm{mfp}})^2\, t^{-1}_{\mathrm{diff}}$

- BBGKY hierarchy: collisional evolution of f_α depends on $g_{\alpha\beta} \sim f_\alpha f_\beta/\varLambda$, collisional evolution of $g_{\alpha\beta}$ depends on $h_{\alpha\beta\gamma} \sim f_\alpha f_\beta f_\gamma/\varLambda^2$, etc.

- Balescu–Lenard collision operator:
$$\left(\frac{\partial f_\alpha}{\partial t}\right)_{\mathrm{BL}} = \sum_\beta \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\,\frac{\boldsymbol{k}}{m_\alpha}\cdot\frac{\partial}{\partial \boldsymbol{v}} \times \int \mathrm{d}\boldsymbol{v}' \left|\frac{4\pi q_\alpha q_\beta}{k^2\mathcal{D}(\boldsymbol{k}\cdot\boldsymbol{v},\boldsymbol{k})}\right|^2 \pi\delta(\boldsymbol{k}\cdot\boldsymbol{v}-\boldsymbol{k}\cdot\boldsymbol{v}')\left(\frac{\boldsymbol{k}}{m_\alpha}\cdot\frac{\partial}{\partial\boldsymbol{v}} - \frac{\boldsymbol{k}}{m_\beta}\cdot\frac{\partial}{\partial\boldsymbol{v}'}\right) f_\alpha(\boldsymbol{v})f_\beta(\boldsymbol{v}'),$$
which is obtained by solving BBGKY after (i) neglecting $h_{\alpha\beta\gamma}$ and higher-order correlation functions, (ii) taking f_α to be constant and homogeneous in the equation for $g_{\alpha\beta}$, and (iii) adopting a Markov assumption that $g_{\alpha\beta}$ is not correlated with its past self (so as to drop $g_{\alpha\beta}$ in the source term of its evolution equation).

- Landau collision operator:
$$\left(\frac{\partial f_\alpha}{\partial t}\right)_{\mathrm{Landau}} = \sum_\beta \frac{\varGamma_{\alpha\beta}}{2}\frac{\partial}{\partial\boldsymbol{v}}\cdot\int \mathrm{d}\boldsymbol{v}'\,\boldsymbol{U}(\boldsymbol{v}-\boldsymbol{v}')\cdot\left(\frac{1}{m_\alpha}\frac{\partial}{\partial\boldsymbol{v}} - \frac{1}{m_\beta}\frac{\partial}{\partial\boldsymbol{v}'}\right) f_\alpha(\boldsymbol{v})f_\beta(\boldsymbol{v}'),$$
where $\varGamma_{\alpha\beta} \doteq 4\pi q_\alpha^2 q_\beta^2 \ln\lambda_{\alpha\beta}/m_\alpha$ and $\boldsymbol{U}(\boldsymbol{u}) \doteq (\boldsymbol{I}-\hat{\boldsymbol{u}}\hat{\boldsymbol{u}})/|\boldsymbol{u}|$. This is equivalent to the Balescu–Lenard operator after setting $\mathcal{D}=1$ and truncating the k-space integral to get $\int \mathrm{d}k/k \to \ln\lambda_{\alpha\beta}$.

- Boltzmann's H theorem: The entropy $S(t) \doteq -\sum_\alpha \int \mathrm{d}\boldsymbol{x}\, f_\alpha(t,\boldsymbol{x})\ln f_\alpha(t,\boldsymbol{x})$ cannot decrease. If $\int \mathrm{d}\boldsymbol{x}\,\ln f_\alpha(\partial f_\alpha/\partial t)_{\mathrm{c}} = 0$, then f_α is a Maxwell–Boltzmann distribution.

† Email address for correspondence: mkunz@princeton.edu

- Test-particle superposition principle: A weakly coupled plasma may be treated as an ensemble of uncorrelated test charges that are "dressed" by equilibrated Debye clouds.

- Polarization drag: The drag on a moving, charged particle caused by its Debye cloud lagging behind it. The energy lost by the particles from this drag takes on the form of radiated plasma waves that are Landau-resonantly absorbed by other plasma particles, provided a minimum amount of thermal "noise" in the plasma.

- Bremsstrahlung: "Braking radiation" caused by a particle (usually an electron) being decelerated and deflected by the (dressed) Coulomb potential of another charged particle (usually an ion). Its transverse (i.e., electromagnetic) polarization has power $\propto Z^2 n_\mathrm{i} n_\mathrm{e} T_\mathrm{e}^{1/2}$ and its spectrum is cut off at frequency $\nu = \omega/2\pi \approx T_\mathrm{e}/h$. Such formulae are used to determine, e.g., the density and temperature of X-ray-emitting astrophysical plasmas, the Z_eff of fusion plasmas, the density fluctuations in laser-plasma experiments, and the neutron yield in ICF experiments.

- Fokker–Planck equation:
$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial \boldsymbol{v}} \boldsymbol{\cdot} \bigl(\boldsymbol{A} f\bigr) + \frac{1}{2}\frac{\partial}{\partial \boldsymbol{v}}\frac{\partial}{\partial \boldsymbol{v}} \boldsymbol{:} \bigl(\boldsymbol{B} f\bigr),$$
where $\boldsymbol{A} \doteq \lim_{\Delta t \to \text{“}0\text{”}} \langle \Delta \boldsymbol{v} \rangle / \Delta t$ and $\boldsymbol{B} \doteq \lim_{\Delta t \to \text{“}0\text{”}} \langle \Delta \boldsymbol{v} \Delta \boldsymbol{v} \rangle / \Delta t$ are, respectively, the drag and diffusion coefficients. This equation is obtained from the master equation after adopting the Markov assumption and only small jumps in $\boldsymbol{v}$ space. An example Fokker–Planck equation for pitch-angle scattering is
$$\frac{\partial f}{\partial t} = \nu \frac{\partial}{\partial \xi}\left(\frac{1-\xi^2}{2}\frac{\partial f}{\partial \xi}\right),$$
for which $A^\xi = -\nu\xi$ and $B^{\xi\xi} = \nu(1-\xi^2)$.

- Langevin equation:
$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\gamma v + a(t), \quad \text{with} \quad \langle a(t) \rangle = 0 \quad \text{and} \quad \langle a(t) a(t') \rangle = \varepsilon \delta(t-t').$$
The provided statistics for the random acceleration $a(t)$ are called "white noise", and rely on the Markov assumption. The statistical solution to this equation provides $\langle v^2(t) \rangle \to \varepsilon/2\gamma \doteq v_\mathrm{th}^2/2$ as $t \to \infty$, i.e., thermalization. The corresponding Fokker–Planck equation governing the probability $P = P(t, v)$,
$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial v}\bigl(\gamma v P\bigr) + \frac{\varepsilon}{2}\frac{\partial^2 P}{\partial v^2},$$
may be obtained by calculating the short-time jump moments from the Langevin equation.

- Krook (or BGK) operator: $C[f] = -\nu(f - f_\mathrm{M})$ given some specified relaxation rate ν and target Maxwellian f_M.

- Lenard–Bernstein (or Dougherty) operator:
$$C[f] = \nu \frac{\partial}{\partial \boldsymbol{v}} \boldsymbol{\cdot} \left[(\boldsymbol{v}-\boldsymbol{u}) f + \frac{v_\mathrm{th}^2}{2}\frac{\partial f}{\partial \boldsymbol{v}}\right],$$
given some specified relaxation rate ν, target flow velocity $\boldsymbol{u}$, and thermal speed v_th.

- Lorentz (or pitch-angle-scattering) operator:

$$C[f] = \nu(v)\mathcal{L}[f] \doteq \nu(v)\left[\frac{\partial}{\partial\xi}\left(\frac{1-\xi^2}{2}\frac{\partial f}{\partial\xi}\right) + \frac{1}{2(1-\xi^2)}\frac{\partial^2 f}{\partial\phi^2}\right]$$

Note that $\mathcal{L}[P_\ell(\xi)] = -\frac{\ell(\ell+1)}{2}P_\ell(\xi)$, where P_ℓ are the Legendre polynomials.

- Rosenbluth potentials:

$$\varphi_\beta(\boldsymbol{v}) \doteq \int \mathrm{d}\boldsymbol{v}'\, f_\beta(\boldsymbol{v}')\frac{1}{|\boldsymbol{v}-\boldsymbol{v}'|} \quad \text{and} \quad \psi_\beta(\boldsymbol{v}) \doteq \int \mathrm{d}\boldsymbol{v}\, f_\beta(\boldsymbol{v}')|\boldsymbol{v}-\boldsymbol{v}'|;$$

alternatively, $\nabla_v^2\varphi_\beta = -4\pi f_\beta$ and $\nabla_v^2\psi_\beta = 2\varphi_\beta$. The drag and diffusion coefficients are related to these potentials via

$$\boldsymbol{A}_\alpha = \sum_\beta \varGamma_{\alpha\beta}\left(\frac{1}{m_\alpha}+\frac{1}{m_\beta}\right)\frac{\partial\varphi_\beta}{\partial\boldsymbol{v}} \quad \text{and} \quad \boldsymbol{\mathsf{B}}_\alpha = \sum_\beta \varGamma_{\alpha\beta}\,\frac{1}{m_\alpha}\frac{\partial^2\psi_\beta}{\partial\boldsymbol{v}\partial\boldsymbol{v}}$$

- Runaway particles: Because the drag force on a superthermal particle is $\propto v^{-2}$, if some persistent force is applied to the plasma, then this force will always be larger than than the frictional drag force for sufficiently fast particles. Such particles can be accelerated to high energies to form a population of runaways. If an applied electric field is larger than the Dreicer field $E_\mathrm{D} = (e/\lambda_\mathrm{D}^2)\ln\lambda$, then even ordinary thermal electrons can become runaways.

- Spitzer–Härm problem: What is the electrical conductivity of a collisional plasma? Apply electric field E, measure current density j, infer electrical conductivity $\sigma = j/E =$ number $\times\, (e^2 n_e \tau_\mathrm{ei}/m_\mathrm{e})$, where $\tau_\mathrm{ei} \doteq 3\sqrt{m_\mathrm{e}}T_\mathrm{e}^{3/2}/(4\sqrt{2\pi}Z^2e^4n_\mathrm{i}\ln\lambda_\mathrm{ei})$ is the electron–ion collision timescale. For the Krook operator, "number" $= 1$; for the Lorentz operator, "number" $= 32/3\pi$; for the Landau operator, "number" $\simeq 1.95$ when $Z = 1$.

- Chapman–Enskog expansion: A systematic expansion of the kinetic equation using $\epsilon \doteq \lambda_\mathrm{mfp}/L \ll 1$ as an ordering parameter. The leading-order solution (f_0) is always a Maxwellian whose first three moments correspond to number density n, bulk flow velocity $\boldsymbol{u}$, and temperature T. The next-order solution (f_1) provides the distortion in the velocity distribution function that is driven by local gradients in T and $\boldsymbol{u}$ and constrained by collisions. Moments of f_1 provide the heat flux $\boldsymbol{q} \doteq \int \mathrm{d}\boldsymbol{w}\,\frac{1}{2}mw^2\boldsymbol{w}f_1$, the viscous stress $\boldsymbol{\varPi} \doteq \int \mathrm{d}\boldsymbol{w}\, m(\boldsymbol{ww} - \frac{1}{3}w^2\boldsymbol{\mathsf{I}})f_1$, and higher-order moments (if desired).

- Braginskii expansion: A version of the Chapman–Enskog expansion with the following additions: two species (electrons and ions, with $\sqrt{m_\mathrm{e}/m_\mathrm{i}} \ll 1$), magnetization of all species ($\rho_\alpha/\lambda_{\mathrm{mfp},\alpha} \ll 1$), high flows ($u_\alpha/v_\mathrm{th,i} \sim 1$), comparable temperatures ($T_\mathrm{i} \sim T_\mathrm{e}$), and $\beta_\alpha \sim 1$. The result is the parallel, diamagnetic, and perpendicular transport of heat and momentum, $\boldsymbol{q} = \boldsymbol{q}_\parallel + \boldsymbol{q}_\times + \boldsymbol{q}_\perp$ and $\boldsymbol{\varPi} = \boldsymbol{\varPi}_\parallel + \boldsymbol{\varPi}_\times + \boldsymbol{\varPi}_\perp$, with " $\parallel$ " : " $\times$ " : " $\perp$ " $= 1 : \nu/\varOmega : (\nu/\varOmega)^2$.

- Braginskii heat flux:

$$\boldsymbol{q} = -n\kappa_\parallel\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\boldsymbol{\cdot}\boldsymbol{\nabla}T - n\kappa_\times\hat{\boldsymbol{b}}\times\boldsymbol{\nabla}T - n\kappa_\perp\boldsymbol{\nabla}_\perp T,$$

where the conductive diffusion coefficients satisfy $\kappa_\parallel \sim v_\mathrm{th}^2\tau$, $\kappa_\times \sim v_\mathrm{th}^2/\varOmega$, and $\kappa_\perp \sim v_\mathrm{th}^2/(\varOmega^2\tau)$. The electrons dominate the heat flux by a factor $\sim\sqrt{m_\mathrm{i}/m_\mathrm{e}}$. [The full Braginskii solution with the Landau collision operator has additional terms in $\boldsymbol{q}_{\parallel\mathrm{e}}$ and

$\boldsymbol{q}_{\perp \mathrm{e}}$ proportional to the plasma current, which are caused by the speed dependence of the collision frequency: slow electrons acquire the mean ion velocity more quickly than fast electrons. The differential slowing-down of electrons with different energies gives rise to an electron heat flux.]

- Braginskii viscous stress:

$$\begin{aligned}\boldsymbol{\Pi} = & -mn\mu_{\parallel}\left(\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} - \frac{\boldsymbol{I}}{3}\right)\left(\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} - \frac{\boldsymbol{I}}{3}\right) : \boldsymbol{W} \\ & + mn\mu_{\times}\left[\hat{\boldsymbol{b}} \times \boldsymbol{W} \cdot (\boldsymbol{I} + 3\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}) - (\boldsymbol{I} + 3\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}) \cdot \boldsymbol{W} \times \hat{\boldsymbol{b}}\right] \\ & - mn\mu_{\perp}\left[\boldsymbol{I}_{\perp} \cdot \boldsymbol{W}_i \cdot \boldsymbol{I}_{\perp} + \frac{1}{2}\boldsymbol{I}_{\perp}\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} : \boldsymbol{W}_i + 4\boldsymbol{I}_{\perp} \cdot \boldsymbol{W}_i \cdot \hat{\boldsymbol{b}}\hat{\boldsymbol{b}} + 4\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} \cdot \boldsymbol{W}_i \cdot \boldsymbol{I}_{\perp}\right],\end{aligned}$$

where $\boldsymbol{W} \doteq \boldsymbol{\nabla}\boldsymbol{u} + (\boldsymbol{\nabla}\boldsymbol{u})^T - \frac{2}{3}(\boldsymbol{\nabla} \cdot \boldsymbol{u})\boldsymbol{I}$ is the rate-of-strain tensor, $\boldsymbol{I}_{\perp} \doteq \boldsymbol{I} - \hat{\boldsymbol{b}}\hat{\boldsymbol{b}}$, and the viscous diffusion coefficients satisfy $\mu_{\parallel} \sim v_{\mathrm{th}}^2\tau$, $\mu_{\times} \sim v_{\mathrm{th}}^2/\Omega$, and $\mu_{\perp} \sim v_{\mathrm{th}}^2/(\Omega^2\tau)$. The ions dominate the momentum flux by a factor $\sim\sqrt{m_{\mathrm{i}}/m_{\mathrm{e}}}$.

Example #1: Compressible flow in Cartesian coordinates (x, y, z) with $\boldsymbol{u} = u(x)\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{b}} = \hat{\boldsymbol{x}}$. Note that $\boldsymbol{\nabla}u \parallel \hat{\boldsymbol{b}}$, so the transport is entirely *along* the magnetic field. The rate-of-strain tensor is $\boldsymbol{W} = 2(\mathrm{d}u/\mathrm{d}x)(\hat{\boldsymbol{x}}\hat{\boldsymbol{x}} - \boldsymbol{I}/3)$ and so

$$\boldsymbol{\Pi} = \boldsymbol{\Pi}_{\parallel} = -mn\mu_{\parallel}(\hat{\boldsymbol{x}}\hat{\boldsymbol{x}} - \boldsymbol{I}/3)\frac{4}{3}\frac{\mathrm{d}u}{\mathrm{d}x}.$$

Example #2: Incompressible shear flow in Cartesian coordinates (x, y, z) with $\boldsymbol{u} = u(x)\hat{\boldsymbol{y}}$ and $\hat{\boldsymbol{b}} = \hat{\boldsymbol{y}}$. Note that $\boldsymbol{\nabla}u \perp \hat{\boldsymbol{b}}$, so the transport is entirely *across* the magnetic field. The rate-of-strain tensor is $\boldsymbol{W} = (\mathrm{d}u/\mathrm{d}x)(\hat{\boldsymbol{x}}\hat{\boldsymbol{y}} + \hat{\boldsymbol{y}}\hat{\boldsymbol{x}})$ and so

$$\boldsymbol{\Pi} = -4mn\mu_{\times}\frac{\mathrm{d}u}{\mathrm{d}x}(\hat{\boldsymbol{y}}\hat{\boldsymbol{z}} + \hat{\boldsymbol{z}}\hat{\boldsymbol{y}}) - 4mn\mu_{\perp}\frac{\mathrm{d}u}{\mathrm{d}x}(\hat{\boldsymbol{x}}\hat{\boldsymbol{y}} + \hat{\boldsymbol{y}}\hat{\boldsymbol{x}}).$$

Suppose that n, T, and B are functions only of x. Then

$$-\boldsymbol{\nabla} \cdot \boldsymbol{\Pi} = \frac{\mathrm{d}}{\mathrm{d}x}\left(4mn\mu_{\perp}\frac{\mathrm{d}u}{\mathrm{d}x}\right) \propto \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{n^2}{B^2T^{1/2}}\frac{\mathrm{d}u}{\mathrm{d}x}\right).$$

This stress is diffusively transporting linear y momentum across the magnetic field into the x direction.

Example #3: Differentially rotating flow in cylindrical coordinates (R, ϕ, z) with $\boldsymbol{u} = R\varpi(R)\hat{\boldsymbol{\phi}}$ and $\hat{\boldsymbol{b}} = \hat{\boldsymbol{\phi}}$. Note that $\boldsymbol{\nabla}\varpi \perp \hat{\boldsymbol{b}}$, so the transport is entirely *across* the magnetic field. The rate-of-strain tensor is $\boldsymbol{W} = (\mathrm{d}\varpi/\mathrm{d}\ln R)(\hat{\boldsymbol{R}}\hat{\boldsymbol{\phi}} + \hat{\boldsymbol{\phi}}\hat{\boldsymbol{R}})$ and so

$$\boldsymbol{\Pi} = -4mn\mu_{\times}\frac{\mathrm{d}\varpi}{\mathrm{d}\ln R}(\hat{\boldsymbol{\phi}}\hat{\boldsymbol{z}} + \hat{\boldsymbol{z}}\hat{\boldsymbol{\phi}}) - 4mn\mu_{\perp}\frac{\mathrm{d}\varpi}{\mathrm{d}\ln R}(\hat{\boldsymbol{R}}\hat{\boldsymbol{\phi}} + \hat{\boldsymbol{\phi}}\hat{\boldsymbol{R}}).$$

Suppose that n, T, and B are functions only of R. Then

$$-\boldsymbol{\nabla} \cdot \boldsymbol{\Pi} = \frac{1}{R^2}\frac{\mathrm{d}}{\mathrm{d}R}\left(4mn\mu_{\perp}R^2\frac{\mathrm{d}\varpi}{\mathrm{d}\ln R}\right) \propto \frac{1}{R^2}\frac{\mathrm{d}}{\mathrm{d}R}\left(\frac{R^3n^2}{B^2T^{1/2}}\frac{\mathrm{d}\varpi}{\mathrm{d}R}\right).$$

This stress is diffusively transporting angular momentum across the magnetic field into the radial direction.