Lecture Notes on Irreversible Processes in Plasmas

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PART I
Introduction

The tendency for entropy to increase in isolated systems is expressed in the second law of thermodynamics – perhaps the most pessimistic and amoral formulation in all human thought.

Gregory Hill and Kerry Thornley
Principia Discordia (1965)

I.1. Bogoliubov's timescale hierarchy

The content of this course is best summarized through the *Bogoliubov hierarchy of timescales*, which is related to the relaxation of an arbitrary perturbation in a plasma. Fill a volume with N particles, each randomly placed and with random velocity. The system will attempt to adjust itself into a statistical equilibrium. There are four stages in this adjustment:[1]

(1) Pair correlations are established and Coulomb potentials are shielded on Debye scales. This occurs on plasma-frequency timescales, $\sim\omega_{\rm p}^{-1}$. This is a *reversible process*.

(2) The velocity distribution relaxes to a local Maxwellian on collisional timescales, $\sim\nu^{-1}$; i.e., local thermodynamic equilibrium is established. Note that $\nu^{-1} = \omega_{\rm p}^{-1}(\mathit{\Lambda}/\ln\mathit{\Lambda}) \gg \omega_{\rm p}^{-1}$ for plasma parameter $\mathit{\Lambda} \doteq n\lambda_{\rm D}^3 \gg 1$. This gives *irreversibility in velocity space*.

(3) Macroscopic force balance emerges on a crossing time $\sim L/v_{\rm th} \sim \nu^{-1}(L/\lambda_{\rm mfp}) \gg \nu^{-1}$. (Note: This timescale was not included in Bogoliubov's original hierarchy, but it appears in the Chapman–Enskog–Braginskii expansion and is thus important in this course.)

(4) Hydrodynamic diffusion occurs on macroscopic spatial and temporal scales, and attempts to relax the system to a global, space- and time-independent Maxwellian. (Boundary conditions that enforce density or temperature gradients prevent this from occurring.) This occurs on a diffusive timescale $\sim L^2/D$; e.g., $\sim\nu^{-1}(L/\lambda_{\rm mfp})^2$ in an unmagnetized plasma, or $\sim\nu^{-1}(L/\rho)^2$ across the magnetic field in a magnetized plasma. This results in *spatial irreversibility*.

Each of these timescales is associated with a part of this course:

(1) At $t = 0$, begin with an arbitrary initial distribution of point particles (Klimontovich, Liouville, BBGKY; §II).

(2) For $\omega_{\rm p}t \gtrsim 1$, Debye-shielding clouds are established (two-particle correlation function; Vlasov physics; §§II,III).

(3) For $\nu t \gtrsim 1$, the particle distribution function approaches a local Maxwellian (Balescu–Lenard and Landau collision operators and their derivatives; discrete particle effects; Fokker–Planck and Langevin equations; test-particle superposition principle; §§IV–VIII).

(4) For $\nu t \gg 1$, the particle distribution function attempts to relax to a global

[1] There is, of course, turbulent transport, but that is a different course.

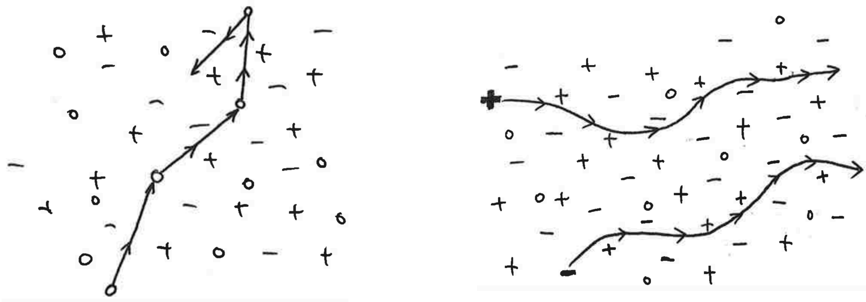


FIGURE 1. Trajectories of (a) a neutral particle and (b) a charged particle.

Maxwellian, but cannot because of macroscopic boundary conditions or source (classical transport theory: Spitzer–Härm, Chapman–Enskog, and Braginskii; §IX).

In a sense, this course is all about the interplay between collective processes and binary processes and how one affects the other.

I.2. Time (Ir)reversibility(?)

The clearest qualitative illustration of this involves the trajectories of particles. Neutral particles in an ionized gas move independently along straight-line trajectories between distinct collisions that occur when particles come within roughly an atomic radius of one another (see figure 1(a)). In contrast, charged particles' trajectories are determined by collective interactions as the weak Coulomb electric field from all nearby charged particles give successive, random, but usually small-angle deflections ("scatterings") of their direction of motion (see figure 1(b)). The cumulative effect of many small-angle Coulomb collisions is examined in HW01.

Now, all of these collisions are, in fact, encoded in Newton's equations of motion for the charged particles:

$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \boldsymbol{v} \qquad \text{and} \qquad \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \frac{q}{m}\left[\boldsymbol{E}(t,\boldsymbol{r}) + \frac{\boldsymbol{v}}{c}\times\boldsymbol{B}(t,\boldsymbol{r})\right]. \tag{I.2.1}$$

But note that setting

$$t \to t' \doteq -t, \quad \boldsymbol{r} \to \boldsymbol{r}' \doteq \boldsymbol{r}, \quad \boldsymbol{v} \to \boldsymbol{v}' \doteq -\boldsymbol{v},$$
$$\boldsymbol{E} \to \boldsymbol{E}' \doteq \boldsymbol{E}(-t,\boldsymbol{r}), \quad \boldsymbol{B} \to \boldsymbol{B}' = -\boldsymbol{B}(-t,\boldsymbol{r}),$$

in (I.2.1), which amounts to a reversal of the time axis, results in

$$\frac{\mathrm{d}\boldsymbol{r}'}{\mathrm{d}t'} = \boldsymbol{v}' \qquad \text{and} \qquad \frac{\mathrm{d}\boldsymbol{v}'}{\mathrm{d}t'} = \frac{q}{m}\left[\boldsymbol{E}'(t',\boldsymbol{r}') + \frac{\boldsymbol{v}'}{c}\times\boldsymbol{B}'(t',\boldsymbol{r}')\right].$$

Thus, the equations of motion for charged particles in electromagnetic fields are time-reversible. This should not be particularly surprising: such equations can be derived from a Hamiltonian, and so the emergence of time reversibility makes sense. But, even though this is a simple consequence, its implications are profound. For then how does irreversibility emerge? How does entropy increase? Why don't we see things running backwards all the time if the governing equations allow for it? These questions have been debated ever since Boltzmann and Maxwell invented kinetic theory.

The main antagonist early on was Josef Loschmidt (1821–1895), who (among others) argued that Boltzmann's H theorem, which implies irreversibility and the second law of thermodynamics, cannot possibly be correct if based on classical mechanics. This is now known as "Loschmidt's paradox". The complaint is not without subtlety, but – to me, at least – Loschmidt seemed a bit of a crackpot. Apparently, he wanted to...

> ...destroy the terroristic nature of the second law [of thermodynamics], which has made it appear to be an annihilating principle for all living beings of the Universe; and at the same time open up the comforting prospect that mankind is not dependent on mineral coal or the Sun for transforming heat into work, but rather may have available forever an inexhaustible supply of transformable heat.

This is quite a leap from simply wondering why particle trajectories cannot simply be reversed. Boltzmann's famous response to this? "Go ahead, reverse them!"

Another levied attack was based on a theory by Poincaré, called the recurrence theorem, which states that a dynamical system with constant energy in a compact phase space must eventually return to its initial state within arbitrary precision for almost all initial conditions. Ernst Zermelo (1871–1953), offering a new and more general proof of this theorem, used it to say that entropy can surely decrease as the system recurs to its initial configuration. Nietzsche loved this idea of eternal recurrence, claiming that "if the Universe has a goal, that goal would have been reached by now". (He believed that the Universe had always existed.) He continued:

> If the Universe may be conceived as a definite quantity of energy, as a definite number of centers of energy – and every other concept remains indefinite and therefore useless – it follows therefrom that the Universe must go through a calculable number of combinations in the great game of chance which constitutes its existence. In infinity, at some moment or other, every possible combination must once have been realized; not only this, but it must have been realized an infinite number of times. And inasmuch as between every one of these combinations and its next recurrence every other possible combination would necessarily have been undergone, and since every one of these combinations would determine the whole series in the same order, a circulate movement of absolutely identical series is thus demonstrated: the Universe is thus shown to be a circular movement which has already repeated itself an infinite number of times, and which plays its game for all eternity.

Such a cycle – and more modest ones – are called "Poincaré cycles", and their period is called the recurrence time. Boltzmann's answer to this was to point out that the recurrence time is incredibly long. He estimated that a system consisting of 10^{18} atoms cm^{-3}, with average velocity 0.5 km s^{-1}, would reproduce all of its coordinates to within 10^{-7} cm precision and all of its velocities to within 100 cm s^{-1} in a time of $10^{10^{19}}$ years (!) Boltzmann put it to Zermelo rather bluntly: "You should live so long."

The difficulty here is that kinetic theory is fundamentally a probabilistic description, and its laws govern trajectories towards the most probable behavior of a system. Irreversibility emerges from the fact that usually the initial state is very unusual, being restricted to a very small part of the total available phase space. In other words,

> There is irreversibility of the basic laws of physics, but there is something special about the initial state of the system that we are considering: this initial state is *very improbable*. By this we mean that it corresponds to a relatively small volume in phase space (or a small entropy). The time evolution then leads to a region with relatively large volume (or large entropy), which corresponds to a very probable state of the system. In principle, after a very long time the system will return to the improbable initial state, but we shall not see this happening... As a physicist, you will want to make an idealization in which the number of particles in your

> system tends to infinity and the time of eternal return also tends to infinity. In this limit you have true irreversibility. (Ruelle 1991)

Or, you could take Maxwell's eventual view (1867):

> I carefully abstain from asking the molecules which enter [the volume under consideration] where they last started from. I only count them and register their mean velocities, avoiding all personal enquiries which would only get me in trouble.

With that, let's get started...

PART II
Klimontovich, Liouville, and the BBGKY hierarchy

> We can measure the globula of matter and the distances between them, but Space plasm itself is incomputable.
>
> Vladimir Nabokov
> *Ada, or Ardor* (1969)

Useful references for concepts presented in this part include Chapter 2.1 of Ichimaru (2004), Chapter 7.1 of Krall & Trivelpiece (1973), Chapter 3 of Nicholson (1983), Section II.4 of Klimontovich (1967), and §§1.3,8 of Krommes (2018).

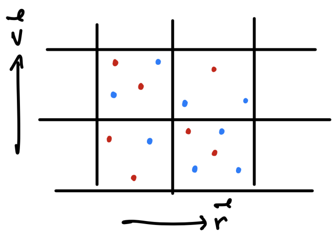
II.1. The Klimontovich equation as a microscopic description of a plasma

A complete description of a plasma would emerge if one were to have knowledge of all the coordinates and momenta of all of the constituent particles, as well as the electromagnetic fields in which they move and which they self-consistently produce. While this description would obviously be untenable with which to work – one is usually only interested in the macroscopic observables such as density, flow velocity, pressure, etc. – let us nevertheless adopt this microscopic standpoint and see where it leads.

Start by defining the function

$$F_\alpha(t,\boldsymbol{r},\boldsymbol{v}) = \sum_{i=1}^{N_\alpha} \delta(\boldsymbol{r}-\boldsymbol{R}_{\alpha i}(t))\delta(\boldsymbol{v}-\boldsymbol{V}_{\alpha i}(t)), \tag{II.1.1}$$

which completely specifies the positions $\boldsymbol{R}_{\alpha i}(t)$ and velocities $\boldsymbol{V}_{\alpha i}(t)$ of N_α particles of species α as functions of time. Graphically,



Note that

$$\lim_{\mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{v}\to 0} \int \mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{v}\, F_\alpha(t,\boldsymbol{r},\boldsymbol{v})$$

is either unity or zero, depending upon whether there is a particle at $(\boldsymbol{r},\boldsymbol{v})$ at time t, so that

$$\int \mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{v}\, F_\alpha(t, \boldsymbol{r}, \boldsymbol{v}) = N_\alpha. \tag{II.1.2}$$

Thus, the microscopic state of the plasma at any time t would be known if one were to know $\boldsymbol{R}_{\alpha i}$ and $\boldsymbol{V}_{\alpha i}$ at $t = 0$ and their temporal evolution. Hamilton's equations of motion provide us with the latter:

$$\frac{\mathrm{d}\boldsymbol{R}_{\alpha i}}{\mathrm{d}t} = \boldsymbol{V}_{\alpha i} \qquad \text{and} \qquad \frac{\mathrm{d}\boldsymbol{V}_{\alpha i}}{\mathrm{d}t} = \frac{q_\alpha}{m_\alpha}\left(\boldsymbol{E}_\mathrm{m} + \frac{\boldsymbol{V}_{\alpha i}}{c} \times \boldsymbol{B}_\mathrm{m}\right), \tag{II.1.3}$$

where q_α and m_α are the charge and mass of species α, and

$$\boldsymbol{E}_\mathrm{m} = \boldsymbol{E}_\mathrm{m}(t, \boldsymbol{R}_{\alpha i}(t)) \qquad \text{and} \qquad \boldsymbol{B}_\mathrm{m} = \boldsymbol{B}_\mathrm{m}(t, \boldsymbol{R}_{\alpha i}(t)) \tag{II.1.4}$$

are the "microphysical" electric and magnetic fields evaluated at the particle position $\boldsymbol{R}_{\alpha i}$ at time t. The adjective "microphysical" here is meant to indicate that $\boldsymbol{E}_\mathrm{m}$ and $\boldsymbol{B}_\mathrm{m}$ are the fields self-consistently generated by the particles themselves. These satisfy Maxwell's equations:

$$\boldsymbol{\nabla} \times \boldsymbol{E}_\mathrm{m} = -\frac{1}{c}\frac{\partial \boldsymbol{B}_\mathrm{m}}{\partial t}, \tag{II.1.5}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B}_\mathrm{m} = \frac{1}{c}\frac{\partial \boldsymbol{E}_\mathrm{m}}{\partial t} + \frac{4\pi}{c}\sum_\alpha q_\alpha \int \mathrm{d}\boldsymbol{v}\, \boldsymbol{v} F_\alpha(t, \boldsymbol{r}, \boldsymbol{v}), \tag{II.1.6}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{E}_\mathrm{m} = 4\pi \sum_\alpha q_\alpha \int \mathrm{d}\boldsymbol{v}\, F_\alpha(t, \boldsymbol{r}, \boldsymbol{v}), \tag{II.1.7}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B}_\mathrm{m} = 0. \tag{II.1.8}$$

Because Maxwell's equations are linear, we can add to these fields any that may be externally imposed: $\boldsymbol{E}_\mathrm{m} \to \boldsymbol{E}_\mathrm{m} + \boldsymbol{E}_\mathrm{ext}$ and $\boldsymbol{B}_\mathrm{m} \to \boldsymbol{B}_\mathrm{m} + \boldsymbol{B}_\mathrm{ext}$. This will be useful for describing magnetized plasmas threaded by an external magnetic field. Before we proceed any further, two things are worth noting:

(1) The electric and magnetic fields in (II.1.3) omit the contribution from particle (αi). In other words, a particle does not interact electromagnetically with itself.

(2) Writing $(\boldsymbol{r}, \boldsymbol{v})$ and $\mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{v}$ all the time is exhausting. Denote $\boldsymbol{x} = (\boldsymbol{r}, \boldsymbol{v})$ and $\mathrm{d}\boldsymbol{x} = \mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{v}$, i.e., $\boldsymbol{x}$ is the phase-space coordinate and $\mathrm{d}\boldsymbol{x}$ is a small volume of phase space. Likewise, $\boldsymbol{X}_{\alpha i} = (\boldsymbol{R}_{\alpha i}, \boldsymbol{V}_{\alpha i})$.

Now, let us consider how $F_\alpha(t,\boldsymbol{x}) \doteq F_\alpha(t,\boldsymbol{r},\boldsymbol{v})$ evolves:[2]

$$
\begin{aligned}
\frac{\partial F_\alpha}{\partial t} &= \frac{\partial}{\partial t}\sum_{i=1}^{N_\alpha} \delta(\boldsymbol{x}-\boldsymbol{X}_{\alpha i}(t)) \\
&= \sum_{i=1}^{N_\alpha} \frac{\mathrm{d}\boldsymbol{X}_{\alpha i}}{\mathrm{d}t}\boldsymbol{\cdot}\frac{\partial}{\partial \boldsymbol{X}_{\alpha i}}\,\delta(\boldsymbol{x}-\boldsymbol{X}_{\alpha i}(t)) \\
&= -\sum_{i=1}^{N_\alpha} \frac{\mathrm{d}\boldsymbol{X}_{\alpha i}}{\mathrm{d}t}\boldsymbol{\cdot}\frac{\partial}{\partial \boldsymbol{x}}\,\delta(\boldsymbol{x}-\boldsymbol{X}_{\alpha i}(t)) \\
&= -\sum_{i=1}^{N_\alpha}\left\{\boldsymbol{V}_{\alpha i}\boldsymbol{\cdot}\boldsymbol{\nabla} + \frac{q_\alpha}{m_\alpha}\left[\boldsymbol{E}_{\mathrm{m}}(t,\boldsymbol{R}_{\alpha i}(t)) + \frac{\boldsymbol{V}_{\alpha i}}{c}\times\boldsymbol{B}_{\mathrm{m}}(t,\boldsymbol{R}_{\alpha i}(t))\right]\boldsymbol{\cdot}\frac{\partial}{\partial\boldsymbol{v}}\right\}\delta(\boldsymbol{x}-\boldsymbol{X}_{\alpha i}(t)) \\
&= -\sum_{i=1}^{N_\alpha}\left\{\boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla} + \frac{q_\alpha}{m_\alpha}\left[\boldsymbol{E}_{\mathrm{m}}(t,\boldsymbol{r}) + \frac{\boldsymbol{v}}{c}\times\boldsymbol{B}_{\mathrm{m}}(t,\boldsymbol{r})\right]\boldsymbol{\cdot}\frac{\partial}{\partial\boldsymbol{v}}\right\}\delta(\boldsymbol{x}-\boldsymbol{X}_{\alpha i}(t)) \\
&= -\left\{\boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla} + \frac{q_\alpha}{m_\alpha}\left[\boldsymbol{E}_{\mathrm{m}}(t,\boldsymbol{r}) + \frac{\boldsymbol{v}}{c}\times\boldsymbol{B}_{\mathrm{m}}(t,\boldsymbol{r})\right]\boldsymbol{\cdot}\frac{\partial}{\partial\boldsymbol{v}}\right\}\sum_{i=1}^{N_\alpha}\delta(\boldsymbol{x}-\boldsymbol{X}_{\alpha i}(t)) \\
&= -\left\{\boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla} + \frac{q_\alpha}{m_\alpha}\left[\boldsymbol{E}_{\mathrm{m}}(t,\boldsymbol{r}) + \frac{\boldsymbol{v}}{c}\times\boldsymbol{B}_{\mathrm{m}}(t,\boldsymbol{r})\right]\boldsymbol{\cdot}\frac{\partial}{\partial\boldsymbol{v}}\right\}F_\alpha(\boldsymbol{t},\boldsymbol{x})
\end{aligned}
$$

$$
\Longrightarrow \quad \boxed{\left[\frac{\partial}{\partial t} + \boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla} + \frac{q_\alpha}{m_\alpha}\left(\boldsymbol{E}_{\mathrm{m}} + \frac{\boldsymbol{v}}{c}\times\boldsymbol{B}_{\mathrm{m}}\right)\boldsymbol{\cdot}\frac{\partial}{\partial\boldsymbol{v}}\right]F_\alpha(t,\boldsymbol{x}) = 0} \tag{II.1.9}
$$

Equation (II.1.9) is called the *Klimontovich equation*. While it is equivalent to phase-space conservation, it is *not* a statistical equation. With proper initial conditions, it is completely deterministic. Together with Maxwell's equations (II.1.5)–(II.1.8), the densities and fields are determined for all time.

The Klimontovich equation (II.1.9) can be thought of as expressing the incompressibility of the substance $F_\alpha(t,\boldsymbol{x})$ as it moves in phase space: $\mathrm{D}F_\alpha/\mathrm{D}t = 0$, where $\mathrm{D}/\mathrm{D}t$ is the phase-space Lagrangian (i.e., comoving) derivative. Nicholson (1983) writes, "is it any wonder that a point particle is incompressible?" Phase-space trajectories that follow the characteristics of (II.1.9) and start from an interval $\mathrm{d}\boldsymbol{x}$ where $F_\alpha = 0$ will carry that null information along with them. Likewise with intervals where $\lim_{\mathrm{d}\boldsymbol{x}\to 0}\int \mathrm{d}\boldsymbol{x}\, F_\alpha(\boldsymbol{x}) = 1$. Thus, the phase space is populated in a very choppy way. For that reason, as well as the simple fact that, despite some mathematics, we haven't actually simplified anything, the Klimontovich equation as a description of the plasma is not worth much practical use. It does, however, form the basis of a *statistical* description of the plasma. But, for that, we need some kind of averaging process...

II.2. The Liouville ("Leé-ooo-ville") distribution

Just as the microscopic state of a plasma is completely specified by the coordinates and momenta of its constituent particles, the statistical properties of the plasma are completely determined by the probabilistic distribution of said particles. Thus, we introduce

[2] Make sure you understand each of these steps. Namely, ask yourself: what if $\mathrm{d}\boldsymbol{X}_{\alpha i}/\mathrm{d}t$ involved a drag force $\propto \boldsymbol{V}_{\alpha i}$? Because the property $X(t)\delta(x - X(t)) = x\delta(x - X(t))$ does *not* imply that $X(t)\delta'(x - X(t)) = x\delta'(x - X(t))$, what is required of $\mathrm{d}\boldsymbol{X}_{\alpha i}/\mathrm{d}t$ for the following to hold?

the distribution function P_N of the coordinates and momenta of all of the $N \doteq \sum_\alpha N_\alpha$ particles in the system. Specifically,

$$P_N \prod_\alpha \mathrm{d}\boldsymbol{X}_{\alpha 1}\mathrm{d}\boldsymbol{X}_{\alpha 2}\dots\mathrm{d}\boldsymbol{X}_{\alpha N_\alpha}$$

gives the probability that, at time t, the phase-space coordinates of the particles of species α have the values $\boldsymbol{X}_{\alpha 1}$, $\boldsymbol{X}_{\alpha 2}$, ..., $\boldsymbol{X}_{\alpha N_\alpha}$ in the range $\mathrm{d}\boldsymbol{X}_{\alpha 1}\mathrm{d}\boldsymbol{X}_{\alpha 2}\dots\mathrm{d}\boldsymbol{X}_{\alpha N_\alpha}$. This $6N$-dimensional phase space is called the "Γ space". The microscopic state of the plasma is expressed in the Γ space by a point $\{\boldsymbol{X}_{\alpha i}\}$. You can read all about this in §2.1.B of Ichimaru (2004) and §7.2 of Krall & Trivelpiece (1973), but let me flag a few important points:

(1) The system points $\{\boldsymbol{X}_{\alpha i}\}$ do not interact with one another and so P_N satisfies a continuity equation of the Liouville kind:

$$\frac{\mathrm{D}P_N}{\mathrm{D}t} \doteq \frac{\partial P_N}{\partial t} + \sum_\alpha \sum_{i=1}^{N_\alpha} \frac{\mathrm{d}\boldsymbol{X}_{\alpha i}}{\mathrm{d}t} \cdot \frac{\partial P_N}{\partial \boldsymbol{X}_{\alpha i}} = 0; \tag{II.2.1}$$

i.e., the probability density is conserved along a characteristic trajectory in phase space.

(2) Because P_N is a probability, we have

$$\int \prod_\alpha \mathrm{d}\boldsymbol{X}_{\alpha 1}\mathrm{d}\boldsymbol{X}_{\alpha 2}\dots\mathrm{d}\boldsymbol{X}_{\alpha N_\alpha}\, P_N \doteq \int \mathrm{d}\boldsymbol{X}_{\mathrm{all}}\, P_N = 1,$$

where I've introduced the shorthand $\mathrm{d}\boldsymbol{X}_{\mathrm{all}}$ to indicate integration over all of the Γ space (including all species).

(3) In thermodynamic equilibrium, P_N equals the Gibbs distribution

$$D_N \doteq \frac{1}{\mathcal{Z}} \exp\left(-\frac{\mathcal{H}}{T}\right), \tag{II.2.2}$$

where $\mathcal{H} = \mathcal{H}(\Gamma)$ is the Hamiltonian (kinetic plus potential energy), T is the (species-independent!) equilibrium temperature (in energy units), and

$$\mathcal{Z} \doteq \int \prod_\alpha \mathrm{d}\boldsymbol{X}_{\alpha 1}\dots\mathrm{d}\boldsymbol{X}_{\alpha N_\alpha} \exp\left(-\frac{\mathcal{H}}{T}\right) \tag{II.2.3}$$

is the partition function. We will primarily be concerned with non-equilibrium systems, and so we will need to know how P_N evolves in time from a given starting distribution $P_N(0)$. We'll return to thermodynamic equilibrium now and then to help develop our intuition (e.g., in §IV.6).

(4) It is profitable to think of P_N in the statistical-mechanics ensemble sense: imagine $\mathcal{N}$ replicas of our plasma, all macroscopically identical but microscopically different, with the system points $\{\boldsymbol{X}_{\alpha i}\}$ scattered over the Γ space. Then P_N can be defined from

$$P_N \prod_\alpha \mathrm{d}\boldsymbol{X}_{\alpha 1}\mathrm{d}\boldsymbol{X}_{\alpha 2}\dots\mathrm{d}\boldsymbol{X}_{\alpha N_\alpha} \doteq \lim_{\mathcal{N}\to\infty} \frac{\mathcal{N}_s}{\mathcal{N}}, \tag{II.2.4}$$

where $\mathcal{N}_s$ is the number of those system points contained in an infinitesimal volume $\prod_\alpha \mathrm{d}\boldsymbol{X}_{\alpha 1}\dots\mathrm{d}\boldsymbol{X}_{\alpha N_\alpha}$ in the Γ space around $\{\boldsymbol{X}_{\alpha i}\}$. (Why can we do this for a plasma? Hint: think about the accuracy of using a statistical description of an

N-body system to describe any one realization of the system. What happens to the model's predictive power when N is not very large?)

II.3. Reduced distribution functions

With a probability distribution in hand, we can perform an ensemble average over all these realizations of the plasma.[3] This will turn our spiky "fine-grained" F_α into the smooth "coarse-grained" distribution. (Greg Hammett's words: "deterministic within any particular realization, stochastic between different realizations".) For example,

$$\int \mathrm{d}\boldsymbol{X}_{\alpha 2} \dots \mathrm{d}\boldsymbol{X}_{\alpha N_\alpha} \prod_\beta \mathrm{d}\boldsymbol{X}_{\beta 1} \mathrm{d}\boldsymbol{X}_{\beta 2} \dots \mathrm{d}\boldsymbol{X}_{\beta N_\beta} \, P_N$$

is the joint probability that particle $\alpha 1$ has coordinates in $(\boldsymbol{X}_{\alpha 1})$ to $(\boldsymbol{X}_{\alpha 1} + \mathrm{d}\boldsymbol{X}_{\alpha 1})$ *irrespective* of the coordinates of particles $\alpha 2$, …, αN_α, $\beta 1$, $\beta 2$, …, βN_β, etc. This *reduced distribution function* is called the *one-particle distribution function*. It can be normalized to one's tastes. I choose the following:[4]

$$f_\alpha(t, \boldsymbol{x}) \doteq N_\alpha \int \mathrm{d}\boldsymbol{X}_{\alpha 2} \dots \mathrm{d}\boldsymbol{X}_{\alpha N_\alpha} \prod_\beta \mathrm{d}\boldsymbol{X}_{\beta 1} \mathrm{d}\boldsymbol{X}_{\beta 2} \dots \mathrm{d}\boldsymbol{X}_{\beta N_\beta} \, P_N , \tag{II.3.1}$$

The operative word here is "irrespective". Of course the probability of, say, an electron being at some phase-space position $\boldsymbol{x}$ is impacted by an ion being nearby at $\boldsymbol{x}' \approx \boldsymbol{x}$, but this information is not in f_α. The influence of a near neighbor on the distribution of a particle is contained in a less reduced description, e.g., the *two-particle distribution function*:

$$f_{\alpha\beta}(t, \boldsymbol{x}, \boldsymbol{x}') \doteq N_\alpha N_\beta \int \mathrm{d}\boldsymbol{X}_{\alpha 2} \dots \mathrm{d}\boldsymbol{X}_{\alpha N_\alpha} \mathrm{d}\boldsymbol{X}_{\beta 2} \dots \mathrm{d}\boldsymbol{X}_{\beta N_\beta} \prod_\gamma \mathrm{d}\boldsymbol{X}_{\gamma 1} \mathrm{d}\boldsymbol{X}_{\gamma 2} \dots \mathrm{d}\boldsymbol{X}_{\gamma N_\gamma} \, P_N . \tag{II.3.2}$$

Then $f_{\alpha\beta}(t, \boldsymbol{x}, \boldsymbol{x}') \mathrm{d}\boldsymbol{x} \mathrm{d}\boldsymbol{x}' / N_\alpha N_\beta$ is the joint probability that particle $\alpha 1$ is at $\boldsymbol{x}$ in interval $\mathrm{d}\boldsymbol{x}$ *and* particle $\beta 1$ is at $\boldsymbol{x}'$ in interval $\mathrm{d}\boldsymbol{x}'$, irrespective of all other particles.

Note three things:

(1) The species labels α and β could refer to the same type of particle ($\alpha = \beta$), in which case $\beta 1 \to \alpha 2$. (The exact numerical labels we affix to a particular particle do not matter.) In this case, $N_\beta \to N_\alpha - 1$.

(2) The two-particle distribution function $f_{\alpha\beta}$ is still a reduced distribution, but, as opposed to the one-particle distribution function f_α, it contains some information about two-body interactions. If the particles do not interact, then $f_{\alpha\beta} = f_\alpha f_\beta$, the product of one-particle distribution functions… simple.

(3) One could of course generalize this process. For example, the three-particle distri-

[3] Krommes (2018) adorns random variables drawn from P_N with a tilde. I find this notation cumbersome and so do not employ it, but you should always be alert when dealing with random variables.

[4] The reason for the N_α is so that $\int \mathrm{d}\boldsymbol{v} \, f_\alpha(t, \boldsymbol{x})$ is the number density n_α, a customary normalization for the one-particle distribution function. Others might introduce a prefactor $\mathcal{V}$ for volume, which makes $\int \mathrm{d}\boldsymbol{v} \, f_\alpha(t, \boldsymbol{x})$ equal to the fraction of the mean number density $\overline{n}_\alpha \doteq N_\alpha / \mathcal{V}$ in that volume.

bution function is

$$f_{\alpha\beta\gamma} \doteq N_\alpha N_\beta N_\gamma \int \frac{\mathrm{d}\boldsymbol{X}_{\rm all}}{\mathrm{d}\boldsymbol{X}_{\alpha 1}\mathrm{d}\boldsymbol{X}_{\beta 1}\mathrm{d}\boldsymbol{X}_{\gamma 1}} P_N; \tag{II.3.3}$$

the four-particle distribution function is

$$f_{\alpha\beta\gamma\delta} \doteq N_\alpha N_\beta N_\gamma N_\delta \int \frac{\mathrm{d}\boldsymbol{X}_{\rm all}}{\mathrm{d}\boldsymbol{X}_{\alpha 1}\mathrm{d}\boldsymbol{X}_{\beta 1}\mathrm{d}\boldsymbol{X}_{\gamma 1}\mathrm{d}\boldsymbol{X}_{\delta 1}} P_N;$$

and so on.

We combine this machinery with the Klimontovich distribution (II.1.1) as follows.

Each term in $F_\alpha = \sum_i \delta(\boldsymbol{x} - \boldsymbol{X}_{\alpha i})$ describes the location of a particle in terms of its initial conditions, and P_N describes the probability of a particle having a certain set of initial conditions, and so the reduced descriptions of P_N can be expressed in terms of the averages of products of F_α over all possible initial conditions. These averages are defined by

$$\langle G(F_\alpha, F_\beta, \ldots, F_\gamma)\rangle \doteq \int \mathrm{d}\boldsymbol{X}_{\rm all}\, P_N\, G(F_\alpha, F_\beta, \ldots, F_\gamma). \tag{II.3.4}$$

Let's put this to work.

II.4. Towards the Vlasov equation

Integrate the Klimontovich distribution (II.1.1) at a particular time over the Liouville distribution (see (II.3.4)):

$$\begin{aligned}
\langle F_\alpha(t, \boldsymbol{x})\rangle &\doteq \sum_{i=1}^{N_\alpha} \int \mathrm{d}\boldsymbol{X}_{\rm all}\, P_N\, \delta(\boldsymbol{x} - \boldsymbol{X}_{\alpha i}(t)) \\
&= N_\alpha \int \mathrm{d}\boldsymbol{X}_{\rm all}\, P_N\, \delta(\boldsymbol{x} - \boldsymbol{X}_{\alpha 1}(t)) \quad \text{(particle labels are arbitrary)} \\
&= N_\alpha \int \mathrm{d}\boldsymbol{X}_{\alpha 2} \ldots \mathrm{d}\boldsymbol{X}_{\alpha N_\alpha} \prod_\beta \mathrm{d}\boldsymbol{X}_{\beta 1}\mathrm{d}\boldsymbol{X}_{\beta 2} \ldots \mathrm{d}\boldsymbol{X}_{\beta N_\beta}\, P_N \\
&\doteq f_\alpha(t, \boldsymbol{x}) \quad \text{(def'n of one-particle distribution function, (II.3.1))}.
\end{aligned} \tag{II.4.1}$$

Similarly, the average electromagnetic fields are obtained by averaging the microscopic fields $\boldsymbol{E}_{\rm m}$ and $\boldsymbol{B}_{\rm m}$, which depend upon the positions of the (point-like) particles, over the probable locations of all of the particles:

$$\boldsymbol{E} \doteq \langle \boldsymbol{E}_{\rm m}\rangle = \int \mathrm{d}\boldsymbol{X}_{\rm all}\, P_N\, \boldsymbol{E}_{\rm m} \qquad \text{and} \qquad \boldsymbol{B} \doteq \langle \boldsymbol{B}_{\rm m}\rangle = \int \mathrm{d}\boldsymbol{X}_{\rm all}\, P_N\, \boldsymbol{B}_{\rm m}. \tag{II.4.2}$$

Using (II.4.1) and (II.4.2) in the Maxwell equations (II.1.5)–(II.1.8) gives

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}, \tag{II.4.3}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} + \frac{4\pi}{c}\sum_\alpha q_\alpha \int \mathrm{d}\boldsymbol{v}\, \boldsymbol{v} f_\alpha(t, \boldsymbol{r}, \boldsymbol{v}), \tag{II.4.4}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = 4\pi \sum_\alpha q_\alpha \int \mathrm{d}\boldsymbol{v}\, f_\alpha(t, \boldsymbol{r}, \boldsymbol{v}), \tag{II.4.5}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0. \tag{II.4.6}$$

Simple. This is because Maxwell's equations are linear.

The difficulty is that the Klimontovich equation (II.1.9) is *not*. It has a quadratic nonlinearity, which it what makes it so hard to solve. Let's see that. The integral of (II.1.9) over the Liouville distribution is

$$\frac{\partial}{\partial t}\langle F_\alpha\rangle + \boldsymbol{v}\cdot\boldsymbol{\nabla}\langle F_\alpha\rangle + \left\langle \frac{q_\alpha}{m_\alpha}\left(\boldsymbol{E}_{\rm m} + \frac{\boldsymbol{v}}{c}\times\boldsymbol{B}_{\rm m}\right)\cdot\frac{\partial F_\alpha}{\partial\boldsymbol{v}}\right\rangle = 0. \tag{II.4.7}$$

The first two terms in (II.4.7) involve only the one-particle distribution function f_α (see (II.4.1)). The third and final term can be manipulated further by decomposing the microscopic electromagnetic fields into their mean and fluctuating parts:

$$\boldsymbol{E}_{\rm m} = \langle\boldsymbol{E}_{\rm m}\rangle + \delta\boldsymbol{E} \doteq \boldsymbol{E} + \delta\boldsymbol{E} \quad\text{and}\quad \boldsymbol{B}_{\rm m} = \langle\boldsymbol{B}_{\rm m}\rangle + \delta\boldsymbol{B} \doteq \boldsymbol{B} + \delta\boldsymbol{B}. \tag{II.4.8}$$

The fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are smooth and coarse-grained; they are the "macroscopic" fields obtained by averaging the microscopic fields over all possible positions of the plasma particles, weighted by the Liouville distribution. The remainders, $\delta\boldsymbol{E}$ and $\delta\boldsymbol{B}$, are spiky and fine-grained; they capture the influence of the discrete nature of the particles on the electromagnetic fields. Using (II.4.8) in the Klimontovich equation (II.4.7) and likewise writing $F_\alpha = f_\alpha + \delta F_\alpha$, we obtain

$$\boxed{\left[\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla} + \frac{q_\alpha}{m_\alpha}\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c}\times\boldsymbol{B}\right)\cdot\frac{\partial}{\partial\boldsymbol{v}}\right] f_\alpha(t,\boldsymbol{x}) = -\left\langle\frac{q_\alpha}{m_\alpha}\left(\delta\boldsymbol{E} + \frac{\boldsymbol{v}}{c}\times\delta\boldsymbol{B}\right)\cdot\frac{\partial\delta F_\alpha}{\partial\boldsymbol{v}}\right\rangle} \tag{II.4.9}$$

If there are externally imposed electric and magnetic fields, they can be added to $\boldsymbol{E}$ and $\boldsymbol{B}$, respectively.

At this point, some derivations discuss the size of the right-hand side of (II.4.9) versus its left-hand side, the latter of which should look quite familiar to you. Let's postpone that for now and just see if we can obtain a general set of equations describing the statistical mechanics of a plasma.

II.5. The BBGKY hierarchy

Let us concern ourselves with non-relativistic plasmas, such that the microscopic magnetic field can be dropped and the Coulomb potential gives an electrostatic field

$$\boldsymbol{E}_{\rm m} = -\frac{\partial}{\partial\boldsymbol{r}}\sum_\alpha q_\alpha \int {\rm d}\boldsymbol{x}'\,\frac{F_\alpha(t,\boldsymbol{x}')}{|\boldsymbol{r}-\boldsymbol{r}'|}. \tag{II.5.1}$$

Thus, $\delta\boldsymbol{B} = 0$, $\boldsymbol{B} = \boldsymbol{B}_{\rm ext}$, and

$$\delta\boldsymbol{E} = -\frac{\partial}{\partial\boldsymbol{r}}\sum_\alpha q_\alpha \int {\rm d}\boldsymbol{x}'\,\frac{\delta F_\alpha(t,\boldsymbol{x}')}{|\boldsymbol{r}-\boldsymbol{r}'|}. \tag{II.5.2}$$

In this case,

$$\boldsymbol{E} = \boldsymbol{E}_{\rm ext} + \langle\boldsymbol{E}_{\rm m}\rangle = \boldsymbol{E}_{\rm ext} - \frac{\partial}{\partial\boldsymbol{r}}\sum_\alpha q_\alpha \int {\rm d}\boldsymbol{x}'\,\frac{f_\alpha(t,\boldsymbol{x}')}{|\boldsymbol{r}-\boldsymbol{r}'|}. \tag{II.5.3}$$

Then, using (II.4.9), the equation governing the one-particle distribution function is

$$\begin{aligned}&\left[\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla} + \frac{q_\alpha}{m_\alpha}\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c}\times\boldsymbol{B}\right)\cdot\frac{\partial}{\partial\boldsymbol{v}}\right] f_\alpha(t,\boldsymbol{x}) \\ &\qquad = \left\langle\frac{\partial}{\partial\boldsymbol{r}}\sum_\beta\frac{q_\alpha q_\beta}{m_\alpha}\int {\rm d}\boldsymbol{x}'\,\frac{\delta F_\beta(t,\boldsymbol{x}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\cdot\frac{\partial}{\partial\boldsymbol{v}}\delta F_\alpha(t,\boldsymbol{x})\right\rangle.\end{aligned} \tag{II.5.4}$$

Recall that, on scales $L \gtrsim \lambda_{\rm D}$, individual particle particles are shielded and what remains are fields due to the collective action of a large number of particles. Also recall that the Coulomb potential is long-range, and so the fields decay on distances long compared to the interparticle spacing ($\lambda_{\rm D} \gg \delta r$). This gives collective behavior: interaction of particles with the mean ("macroscopic") fields generated by all other particles. This means that the entire left-hand side of (II.5.4) consists of terms that vary smoothly in phase space, since it's entirely insensitive to the discrete nature of the plasma. The right-hand side, by contrast, is very sensitive, and is ultimately responsible for collisional effects. Note that it is quadratic in δF. To solve this equation, we must write $\langle \delta F_\beta \delta F_\alpha \rangle$ in terms of f_α.

First, rearrange (II.5.4) to obtain

$$\begin{aligned} \dot{f}_\alpha(t,\boldsymbol{x}) &\doteq \left[\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla} + \frac{q_\alpha}{m_\alpha}\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c}\times\boldsymbol{B}\right)\cdot\frac{\partial}{\partial \boldsymbol{v}}\right] f_\alpha(t,\boldsymbol{x}) \\ &= \sum_\beta \frac{q_\alpha q_\beta}{m_\alpha}\int \mathrm{d}\boldsymbol{x}'\,\frac{\partial}{\partial \boldsymbol{r}}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\cdot\frac{\partial}{\partial \boldsymbol{v}}\big\langle \delta F_\alpha(t,\boldsymbol{x})\delta F_\beta(t,\boldsymbol{x}')\big\rangle. \end{aligned} \tag{II.5.5}$$

Next, write

$$\begin{aligned} \big\langle \delta F_\alpha(t,\boldsymbol{x})\delta F_\beta(t,\boldsymbol{x}')\big\rangle &= \big\langle (F_\alpha - f_\alpha)(F_\beta - f_\beta)\big\rangle \\ &= \big\langle F_\alpha F_\beta\big\rangle - \big\langle f_\alpha F_\beta\big\rangle - \big\langle F_\alpha f_\beta\big\rangle + \big\langle f_\alpha f_\beta\big\rangle \\ &= \big\langle F_\alpha F_\beta\big\rangle - f_\alpha f_\beta - f_\alpha f_\beta + f_\alpha f_\beta \\ &= \big\langle F_\alpha(t,\boldsymbol{x}) F_\beta(t,\boldsymbol{x}')\big\rangle - f_\alpha(t,\boldsymbol{x}) f_\beta(t,\boldsymbol{x}'). \end{aligned} \tag{II.5.6}$$

We must calculate the correlation $\langle F_\alpha F_\beta \rangle$ in (II.5.6) using the Klimontovich distributions $F_\alpha(t,\boldsymbol{x}) = \sum_{i=1}^{N_\alpha} \delta(\boldsymbol{x} - \boldsymbol{X}_{\alpha i}(t))$ and $F_\beta(t,\boldsymbol{x}') = \sum_{j=1}^{N_\beta} \delta(\boldsymbol{x}' - \boldsymbol{X}_{\beta j}(t))$. To do so, first split up the sums into like-particle and unlike-particle pieces:

$$\begin{aligned} \big\langle F_\alpha(t,\boldsymbol{x}) F_\beta(t,\boldsymbol{x}')\big\rangle &= \int \mathrm{d}\boldsymbol{X}_{\rm all} \sum_{i=1}^{N_\alpha}\sum_{j=1}^{N_\beta} P_N\,\delta(\boldsymbol{x} - \boldsymbol{X}_{\alpha i}(t))\delta(\boldsymbol{x}' - \boldsymbol{X}_{\beta j}(t)) \\ &= \delta_{\alpha\beta}\int \mathrm{d}\boldsymbol{X}_{\rm all} \sum_{i=1}^{N_\alpha}\sum_{j=1}^{N_\alpha} P_N\,\delta(\boldsymbol{x} - \boldsymbol{X}_{\alpha i}(t))\delta(\boldsymbol{x}' - \boldsymbol{X}_{\alpha j}(t)) \\ &\quad + (1-\delta_{\alpha\beta})\int \mathrm{d}\boldsymbol{X}_{\rm all} \sum_{i=1}^{N_\alpha}\sum_{j=1}^{N_\beta} P_N\,\delta(\boldsymbol{x} - \boldsymbol{X}_{\alpha i}(t))\delta(\boldsymbol{x}' - \boldsymbol{X}_{\beta j}(t)). \end{aligned} \tag{II.5.7}$$

Next, separate out $i = j$ in the like-particle piece:

$$\begin{aligned} \big\langle F_\alpha(t,\boldsymbol{x}) F_\beta(t,\boldsymbol{x}')\big\rangle &= \delta_{\alpha\beta}\Bigg[\int \mathrm{d}\boldsymbol{X}_{\rm all} \sum_{i=1}^{N_\alpha} P_N\,\delta(\boldsymbol{x} - \boldsymbol{X}_{\alpha i}(t))\delta(\boldsymbol{x}' - \boldsymbol{x}) \\ &\quad + \sum_{i=1}^{N_\alpha}\sum_{j\neq i}^{N_\alpha}\int \mathrm{d}\boldsymbol{X}_{\rm all}\, P_N\,\delta(\boldsymbol{x} - \boldsymbol{X}_{\alpha i}(t))\delta(\boldsymbol{x}' - \boldsymbol{X}_{\alpha j}(t))\Bigg] \\ &\quad + (1-\delta_{\alpha\beta})\int \mathrm{d}\boldsymbol{X}_{\rm all} \sum_{i=1}^{N_\alpha}\sum_{j=1}^{N_\beta} P_N\,\delta(\boldsymbol{x} - \boldsymbol{X}_{\alpha i}(t))\delta(\boldsymbol{x}' - \boldsymbol{X}_{\beta j}(t)). \end{aligned} \tag{II.5.8}$$

Now use the definitions of the one- and two-particle distribution functions (see (II.3.1) and (II.3.2)) to find

$$\begin{aligned}\big\langle F_\alpha(t,\boldsymbol{x})F_\beta(t,\boldsymbol{x}')\big\rangle &= \delta_{\alpha\beta}\left[\delta(\boldsymbol{x}'-\boldsymbol{x})f_\alpha(t,\boldsymbol{x}) + \left(\frac{N_\alpha-1}{N_\alpha}\right)f_{\alpha\alpha}(t,\boldsymbol{x},\boldsymbol{x}')\right] \\ &\quad + (1-\delta_{\alpha\beta})f_{\alpha\beta}(t,\boldsymbol{x},\boldsymbol{x}') \\ &= \delta_{\alpha\beta}\delta(\boldsymbol{x}-\boldsymbol{x}')f_\alpha(t,\boldsymbol{x}) + f_{\alpha\beta}(t,\boldsymbol{x},\boldsymbol{x}') + \mathcal{O}\left(\frac{1}{N_\alpha}\right).\end{aligned} \tag{II.5.9}$$

Substituting (II.5.9) into (II.5.5) and dropping the $\mathcal{O}(1/N_\alpha)$ term, we see that the first term $\propto\delta_{\alpha\beta}\delta(\boldsymbol{x}-\boldsymbol{x}')$ vanishes. This is because self-interactions with the Coulomb force are excluded, i.e.,

$$\int \mathrm{d}\boldsymbol{r}'\,\delta(\boldsymbol{r}-\boldsymbol{r}')\frac{\partial}{\partial\boldsymbol{r}}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = 0.$$

Thus, equation (II.5.5) becomes

$$\dot{f}_\alpha = -\sum_\beta \frac{q_\alpha q_\beta}{m_\alpha}\int \mathrm{d}\boldsymbol{x}'\,\frac{\partial}{\partial\boldsymbol{r}}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\cdot\frac{\partial}{\partial\boldsymbol{v}}\big[f_\alpha(t,\boldsymbol{x})f_\beta(t,\boldsymbol{x}') - f_{\alpha\beta}(t,\boldsymbol{x},\boldsymbol{x}')\big]. \tag{II.5.10}$$

At this point, it's worth reiterating the definitions of f_α and $f_{\alpha\beta}$. f_α is the one-particle distribution function – the probability that a particle of species α has phase-space position $\boldsymbol{x}$ at time t in the interval $\mathrm{d}\boldsymbol{x}$ *regardless* of all other particles. No particle–particle interactions are encoded in f_α. $f_{\alpha\beta}$, on the other hand, is the joint probability that a particle of species α has phase-space position $\boldsymbol{x}$ at time t *and* a particle of species β has phase-space position $\boldsymbol{x}'$ at time t, *regardless* of all other particles. Now, suppose all particles were truly uncorrelated (i.e., no collisions). Then $f_{\alpha\beta}(t,\boldsymbol{x},\boldsymbol{x}') = f_\alpha(t,\boldsymbol{x})f_\beta(t,\boldsymbol{x}')$, and the right-hand side of (II.5.10) would vanish. This would return the Vlasov equation, $\dot{f}_\alpha = 0$. This suggests that we introduce some function, say, $g_{\alpha\beta}(t,\boldsymbol{x},\boldsymbol{x}')$, which captures their difference:

$$f_{\alpha\beta} = f_\alpha f_\beta + g_{\alpha\beta}. \tag{II.5.11}$$

This is the first step in what is known as the *Mayer cluster (or cumulant) expansion*. It splits the statistically independent pieces of $f_{\alpha\beta}$, which have multiplicative probabilities, apart from the statistically dependent piece. It's almost always useful to split off the piece of a joint probability distribution that corresponds to uncorrelated events. Nicholson (1983) on page 54 of his textbook has a cute analogy concerning correlated and uncorrelated coin tosses and die rolls. I prefer Yahtzee: the difference between rolling each die separately versus putting them all in the can and shaking them all and rolling them all out at the same time, so that their mutual collisions influence which side of each die faces up when the system comes to rest.

So, now we have from (II.5.10) and (II.5.11) that

$$\boxed{\dot{f}_\alpha = \sum_\beta \frac{q_\alpha q_\beta}{m_\alpha}\int \mathrm{d}\boldsymbol{x}'\,\frac{\partial}{\partial\boldsymbol{r}}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\cdot\frac{\partial}{\partial\boldsymbol{v}}g_{\alpha\beta}(t,\boldsymbol{x},\boldsymbol{x}')} \tag{II.5.12}$$

This is the first step in what is known as the *BBGKY hierarchy* (Bogoliubov, Born, Green, Kirkwood, Yvon; 1935–1949): the evolution of the one-particle distribution depends on correlations between two particles.

Let us proceed to find an equation for how the two-particle correlation $g_{\alpha\beta}$ evolves:

$$
\begin{aligned}
\frac{\partial}{\partial t} g_{\alpha\beta}(t,\boldsymbol{x},\boldsymbol{x}') &= \frac{\partial}{\partial t}\big[f_{\alpha\beta}(t,\boldsymbol{x},\boldsymbol{x}') - f_\alpha(t,\boldsymbol{x}) f_\beta(t,\boldsymbol{x}')\big] \\
&= \frac{\partial}{\partial t}\big[\big\langle F_\alpha(t,\boldsymbol{x}) F_\beta(t,\boldsymbol{x}')\big\rangle - \delta_{\alpha\beta}\delta(\boldsymbol{x}-\boldsymbol{x}')\, f_\alpha(t,\boldsymbol{x}) - f_\alpha(t,\boldsymbol{x}) f_\beta(t,\boldsymbol{x}')\big] \\
&= \bigg\langle \frac{\partial F_\alpha(t,\boldsymbol{x})}{\partial t} F_\beta(t,\boldsymbol{x}') + F_\alpha(t,\boldsymbol{x}) \frac{\partial F_\beta(t,\boldsymbol{x}')}{\partial t} - \delta_{\alpha\beta}\delta(\boldsymbol{x}-\boldsymbol{x}') \frac{\partial f_\alpha(t,\boldsymbol{x})}{\partial t} \\
&\qquad - \frac{\partial f_\alpha(t,\boldsymbol{x})}{\partial t} f_\beta(t,\boldsymbol{x}') - f_\alpha(t,\boldsymbol{x}) \frac{\partial f_\beta(t,\boldsymbol{x}')}{\partial t} \bigg\rangle. \qquad \text{(II.5.13)}
\end{aligned}
$$

Using the Klimontovich equation (II.1.9) and the kinetic equation for the one-particle distribution function (II.5.12), and defining

$$
\begin{aligned}
\boldsymbol{a} &\doteq \frac{q_\alpha}{m_\alpha}\bigg[\boldsymbol{E}(t,\boldsymbol{r}) + \frac{\boldsymbol{v}}{c} \times \boldsymbol{B}(t,\boldsymbol{r})\bigg], \\
\boldsymbol{a}' &\doteq \frac{q_\beta}{m_\beta}\bigg[\boldsymbol{E}(t,\boldsymbol{r}') + \frac{\boldsymbol{v}'}{c} \times \boldsymbol{B}(t,\boldsymbol{r}')\bigg],
\end{aligned}
$$

equation (II.5.13) becomes

$$
\begin{aligned}
&\frac{\partial g_{\alpha\beta}(t,\boldsymbol{x},\boldsymbol{x}')}{\partial t} \\
&= \bigg\langle \bigg[- \boldsymbol{v}\cdot\boldsymbol{\nabla} F_\alpha(t,\boldsymbol{x}) - \boldsymbol{a}\cdot\frac{\partial F_\alpha(t,\boldsymbol{x})}{\partial \boldsymbol{v}} - \frac{q_\alpha}{m_\alpha}\delta\boldsymbol{E}(t,\boldsymbol{r})\cdot\frac{\partial F_\alpha(t,\boldsymbol{x})}{\partial \boldsymbol{v}}\bigg] F_\beta(t,\boldsymbol{x}') \\
&+ F_\alpha(t,\boldsymbol{x})\bigg[- \boldsymbol{v}'\cdot\boldsymbol{\nabla}' F_\beta(t,\boldsymbol{x}') - \boldsymbol{a}'\cdot\frac{\partial F_\beta(t,\boldsymbol{x}')}{\partial \boldsymbol{v}'} - \frac{q_\beta}{m_\beta}\delta\boldsymbol{E}(t,\boldsymbol{r}')\cdot\frac{\partial F_\beta(t,\boldsymbol{x}')}{\partial \boldsymbol{v}'}\bigg] \\
&- \delta_{\alpha\beta}\delta(\boldsymbol{x}-\boldsymbol{x}')\bigg[- \boldsymbol{v}\cdot\boldsymbol{\nabla} f_\alpha(t,\boldsymbol{x}) - \boldsymbol{a}\cdot\frac{\partial f_\alpha(t,\boldsymbol{x})}{\partial \boldsymbol{v}} \\
&\qquad\qquad + \sum_\gamma \frac{q_\alpha q_\gamma}{m_\alpha}\int \mathrm{d}\boldsymbol{x}''\, \frac{\partial}{\partial \boldsymbol{r}}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}''|}\cdot\frac{\partial g_{\alpha\gamma}(t,\boldsymbol{x},\boldsymbol{x}'')}{\partial \boldsymbol{v}}\bigg] \\
&- \bigg[- \boldsymbol{v}\cdot\boldsymbol{\nabla} f_\alpha(t,\boldsymbol{x}) - \boldsymbol{a}\cdot\frac{\partial f_\alpha(t,\boldsymbol{x})}{\partial \boldsymbol{v}} \\
&\qquad + \sum_\gamma \frac{q_\alpha q_\gamma}{m_\alpha}\int \mathrm{d}\boldsymbol{x}''\, \frac{\partial}{\partial \boldsymbol{r}}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}''|}\cdot\frac{\partial g_{\alpha\gamma}(t,\boldsymbol{x},\boldsymbol{x}'')}{\partial \boldsymbol{v}}\bigg] f_\beta(t,\boldsymbol{x}') \\
&- f_\alpha(t,\boldsymbol{x})\bigg[- \boldsymbol{v}'\cdot\boldsymbol{\nabla}' f_\beta(t,\boldsymbol{x}') - \boldsymbol{a}'\cdot\frac{\partial f_\beta(t,\boldsymbol{x}')}{\partial \boldsymbol{v}'} \\
&\qquad\qquad + \sum_\gamma \frac{q_\beta q_\gamma}{m_\beta}\int \mathrm{d}\boldsymbol{x}''\, \frac{\partial}{\partial \boldsymbol{r}'}\frac{1}{|\boldsymbol{r}'-\boldsymbol{r}''|}\cdot\frac{\partial g_{\beta\gamma}(t,\boldsymbol{x}',\boldsymbol{x}'')}{\partial \boldsymbol{v}'}\bigg]\bigg\rangle. \qquad \text{(II.5.14)}
\end{aligned}
$$

Using (II.5.9) with $f_{\alpha\beta} = f_\alpha f_\beta + g_{\alpha\beta}$ (see (II.5.11)) in (II.5.14) gives

$$
\big\langle F_\alpha(t,\boldsymbol{x}) F_\beta(t,\boldsymbol{x}')\big\rangle = f_\alpha(t,\boldsymbol{x}) f_\beta(t,\boldsymbol{x}') + g_{\alpha\beta}(t,\boldsymbol{x},\boldsymbol{x}') + \delta_{\alpha\beta}\delta(\boldsymbol{x}-\boldsymbol{x}') f_\alpha(t,\boldsymbol{x});
$$

substituting this into (II.5.14) eliminates a lot of terms! Suppressing the time argument for economy of notation,

$$\begin{aligned}\frac{\partial g_{\alpha\beta}(\boldsymbol{x},\boldsymbol{x}')}{\partial t} &= -\frac{q_\alpha}{m_\alpha}\left\langle \delta\boldsymbol{E}(\boldsymbol{r})\cdot\frac{\partial F_\alpha(\boldsymbol{x})}{\partial\boldsymbol{v}}\,F_\beta(\boldsymbol{x}')\right\rangle - \frac{q_\beta}{m_\beta}\left\langle F_\alpha(\boldsymbol{x})\,\delta\boldsymbol{E}(\boldsymbol{r}')\cdot\frac{\partial F_\beta(\boldsymbol{x}')}{\partial\boldsymbol{v}'}\right\rangle \\
&\quad - \left(\boldsymbol{v}\cdot\boldsymbol{\nabla} + \boldsymbol{a}\cdot\frac{\partial}{\partial\boldsymbol{v}} + \boldsymbol{v}'\cdot\boldsymbol{\nabla}' + \boldsymbol{a}'\cdot\frac{\partial}{\partial\boldsymbol{v}'}\right) g_{\alpha\beta}(\boldsymbol{x},\boldsymbol{x}') \\
&\quad - f_\beta(\boldsymbol{x}')\sum_\gamma \frac{q_\alpha q_\gamma}{m_\alpha}\int \mathrm{d}\boldsymbol{x}''\,\frac{\partial}{\partial\boldsymbol{r}}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}''|}\cdot\frac{\partial g_{\alpha\gamma}(\boldsymbol{x},\boldsymbol{x}'')}{\partial\boldsymbol{v}} \\
&\quad - f_\alpha(\boldsymbol{x})\sum_\gamma \frac{q_\beta q_\gamma}{m_\beta}\int \mathrm{d}\boldsymbol{x}''\,\frac{\partial}{\partial\boldsymbol{r}'}\frac{1}{|\boldsymbol{r}'-\boldsymbol{r}''|}\cdot\frac{\partial g_{\beta\gamma}(\boldsymbol{x}',\boldsymbol{x}'')}{\partial\boldsymbol{v}'} \\
&\quad - \delta_{\alpha\beta}\delta(\boldsymbol{x}-\boldsymbol{x}')\sum_\gamma \frac{q_\alpha q_\gamma}{m_\alpha}\int \mathrm{d}\boldsymbol{x}''\,\frac{\partial}{\partial\boldsymbol{r}}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}''|}\cdot\frac{\partial g_{\alpha\gamma}(\boldsymbol{x},\boldsymbol{x}'')}{\partial\boldsymbol{v}},\end{aligned} \tag{II.5.15}$$

where the following identity has been used:

$$\left(\boldsymbol{v}\cdot\boldsymbol{\nabla} + \boldsymbol{a}\cdot\frac{\partial}{\partial\boldsymbol{v}} + \boldsymbol{v}'\cdot\boldsymbol{\nabla}' + \boldsymbol{a}'\cdot\frac{\partial}{\partial\boldsymbol{v}'}\right)\delta_{\alpha\beta}\delta(\boldsymbol{x}-\boldsymbol{x}') = 0.$$

Writing

$$\begin{aligned}\delta\boldsymbol{E}(\boldsymbol{r}) &= -\frac{\partial}{\partial\boldsymbol{r}}\sum_\gamma q_\gamma\int \mathrm{d}\boldsymbol{x}''\,\frac{\delta F_\gamma(\boldsymbol{x}'')}{|\boldsymbol{r}-\boldsymbol{r}''|}, \\
\delta\boldsymbol{E}(\boldsymbol{r}') &= -\frac{\partial}{\partial\boldsymbol{r}'}\sum_\gamma q_\gamma\int \mathrm{d}\boldsymbol{x}''\,\frac{\delta F_\gamma(\boldsymbol{x}'')}{|\boldsymbol{r}'-\boldsymbol{r}''|},\end{aligned}$$

equation (II.5.15) becomes

$$\begin{aligned}&\left[\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla} + \boldsymbol{a}\cdot\frac{\partial}{\partial\boldsymbol{v}} + \boldsymbol{v}'\cdot\boldsymbol{\nabla}' + \boldsymbol{a}'\cdot\frac{\partial}{\partial\boldsymbol{v}'}\right] g_{\alpha\beta}(\boldsymbol{x},\boldsymbol{x}') \\
&\quad = \sum_\gamma \frac{q_\alpha q_\gamma}{m_\alpha}\int \mathrm{d}\boldsymbol{x}''\,\frac{\partial}{\partial\boldsymbol{r}}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}''|}\cdot\frac{\partial}{\partial\boldsymbol{v}}\big\langle F_\alpha(\boldsymbol{x})F_\beta(\boldsymbol{x}')\delta F_\gamma(\boldsymbol{x}'')\big\rangle \\
&\quad + \sum_\gamma \frac{q_\beta q_\gamma}{m_\beta}\int \mathrm{d}\boldsymbol{x}''\,\frac{\partial}{\partial\boldsymbol{r}'}\frac{1}{|\boldsymbol{r}'-\boldsymbol{r}''|}\cdot\frac{\partial}{\partial\boldsymbol{v}'}\big\langle F_\alpha(\boldsymbol{x})F_\beta(\boldsymbol{x}')\delta F_\gamma(\boldsymbol{x}'')\big\rangle \\
&\quad - f_\beta(\boldsymbol{x}')\sum_\gamma \frac{q_\alpha q_\gamma}{m_\alpha}\int \mathrm{d}\boldsymbol{x}''\,\frac{\partial}{\partial\boldsymbol{r}}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}''|}\cdot\frac{\partial g_{\alpha\gamma}(\boldsymbol{x},\boldsymbol{x}'')}{\partial\boldsymbol{v}} \\
&\quad - f_\alpha(\boldsymbol{x})\sum_\gamma \frac{q_\beta q_\gamma}{m_\beta}\int \mathrm{d}\boldsymbol{x}''\,\frac{\partial}{\partial\boldsymbol{r}'}\frac{1}{|\boldsymbol{r}'-\boldsymbol{r}''|}\cdot\frac{\partial g_{\beta\gamma}(\boldsymbol{x}',\boldsymbol{x}'')}{\partial\boldsymbol{v}'} \\
&\quad - \delta_{\alpha\beta}\delta(\boldsymbol{x}-\boldsymbol{x}')\sum_\gamma \frac{q_\alpha q_\gamma}{m_\alpha}\int \mathrm{d}\boldsymbol{x}''\,\frac{\partial}{\partial\boldsymbol{r}}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}''|}\cdot\frac{\partial g_{\alpha\gamma}(\boldsymbol{x},\boldsymbol{x}'')}{\partial\boldsymbol{v}}.\end{aligned} \tag{II.5.16}$$

Things are starting to look better.

Note the appearance in (II.5.16) of the triple correlation $\langle F_\alpha F_\beta\,\delta F_\gamma\rangle$. Following a similar calculation that led to (II.5.9) for $\langle F_\alpha F_\beta\rangle$ gives

$$\begin{aligned}\big\langle F_\alpha(\boldsymbol{x})F_\beta(\boldsymbol{x}')F_\gamma(\boldsymbol{x}'')\big\rangle &= f_{\alpha\beta\gamma}(\boldsymbol{x},\boldsymbol{x}',\boldsymbol{x}'') \\
&\quad + \delta_{\alpha\beta}\delta(\boldsymbol{x}-\boldsymbol{x}')\,f_{\alpha\gamma}(\boldsymbol{x},\boldsymbol{x}'') + \delta_{\alpha\gamma}\delta(\boldsymbol{x}-\boldsymbol{x}'')\,f_{\alpha\beta}(\boldsymbol{x},\boldsymbol{x}') + \delta_{\beta\gamma}\delta(\boldsymbol{x}'-\boldsymbol{x}'')\,f_{\alpha\gamma}(\boldsymbol{x},\boldsymbol{x}'') \\
&\quad + \delta_{\alpha\beta}\delta_{\beta\gamma}\delta(\boldsymbol{x}-\boldsymbol{x}')\delta(\boldsymbol{x}'-\boldsymbol{x}'')f_\alpha(\boldsymbol{x}),\end{aligned} \tag{II.5.17}$$

so that

$$\begin{aligned}\big\langle F_\alpha(\boldsymbol{x})F_\beta(\boldsymbol{x}')\delta F_\gamma(\boldsymbol{x}'')\big\rangle &= f_{\alpha\beta\gamma}(\boldsymbol{x},\boldsymbol{x}',\boldsymbol{x}'') - f_{\alpha\beta}(\boldsymbol{x},\boldsymbol{x}')f_\gamma(\boldsymbol{x}'') \\ &\quad + \delta_{\alpha\beta}\delta(\boldsymbol{x}-\boldsymbol{x}')\big[f_{\alpha\gamma}(\boldsymbol{x},\boldsymbol{x}'') - f_\alpha(\boldsymbol{x})f_\gamma(\boldsymbol{x}'')\big] + \delta_{\alpha\gamma}\delta(\boldsymbol{x}-\boldsymbol{x}'')f_{\alpha\beta}(\boldsymbol{x},\boldsymbol{x}') \\ &\quad + \delta_{\beta\gamma}\delta(\boldsymbol{x}'-\boldsymbol{x}'')f_{\alpha\gamma}(\boldsymbol{x},\boldsymbol{x}'') + \delta_{\alpha\beta}\delta_{\beta\gamma}\delta(\boldsymbol{x}-\boldsymbol{x}')\delta(\boldsymbol{x}'-\boldsymbol{x}'')f_\alpha(\boldsymbol{x}). \end{aligned} \tag{II.5.18}$$

To make further progress, write the three-particle distribution function $f_{\alpha\beta\gamma}$ using the Mayer cluster expansion,

$$\begin{aligned} f_{\alpha\beta\gamma}(\boldsymbol{x},\boldsymbol{x}',\boldsymbol{x}'') &= f_\alpha(\boldsymbol{x})f_\beta(\boldsymbol{x}')f_\gamma(\boldsymbol{x}'') \\ &\quad + f_\alpha(\boldsymbol{x})g_{\beta\gamma}(\boldsymbol{x}',\boldsymbol{x}'') + f_\beta(\boldsymbol{x}')g_{\alpha\gamma}(\boldsymbol{x},\boldsymbol{x}'') + f_\gamma(\boldsymbol{x}'')g_{\alpha\beta}(\boldsymbol{x},\boldsymbol{x}') \\ &\quad + h_{\alpha\beta\gamma}(\boldsymbol{x},\boldsymbol{x}',\boldsymbol{x}''), \end{aligned} \tag{II.5.19}$$

where $h_{\alpha\beta\gamma}$ is the three-particle correlation function. Using (II.5.19) alongside $f_{\alpha\beta} = f_\alpha f_\beta + g_{\alpha\beta}$ (see (II.5.11)), equation (II.5.18) becomes

$$\begin{aligned} &\big\langle F_\alpha(\boldsymbol{x})F_\beta(\boldsymbol{x}')\delta F_\gamma(\boldsymbol{x}'')\big\rangle = f_\alpha(\boldsymbol{x})g_{\beta\gamma}(\boldsymbol{x}',\boldsymbol{x}'') + f_\beta(\boldsymbol{x}')g_{\alpha\gamma}(\boldsymbol{x},\boldsymbol{x}'') \\ &\quad + \delta_{\beta\gamma}\delta(\boldsymbol{x}'-\boldsymbol{x}'')\big[f_\alpha(\boldsymbol{x})f_\gamma(\boldsymbol{x}'') + g_{\alpha\gamma}(\boldsymbol{x},\boldsymbol{x}'')\big] + \delta_{\alpha\gamma}\delta(\boldsymbol{x}-\boldsymbol{x}'')\big[f_\alpha(\boldsymbol{x})f_\beta(\boldsymbol{x}') + g_{\alpha\beta}(\boldsymbol{x},\boldsymbol{x}')\big] \\ &\quad + \delta_{\alpha\beta}\delta(\boldsymbol{x}-\boldsymbol{x}')\big[\delta_{\beta\gamma}\delta(\boldsymbol{x}'-\boldsymbol{x}'')f_\alpha(\boldsymbol{x}) + g_{\alpha\gamma}(\boldsymbol{x},\boldsymbol{x}'')\big] + h_{\alpha\beta\gamma}(\boldsymbol{x},\boldsymbol{x}',\boldsymbol{x}''). \end{aligned} \tag{II.5.20}$$

Plugging (II.5.20) back into (II.5.16) gives, after much simplification,

$$\begin{aligned} &\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla} + \boldsymbol{a}\cdot\frac{\partial}{\partial \boldsymbol{v}} + \boldsymbol{v}'\cdot\boldsymbol{\nabla}' + \boldsymbol{a}'\cdot\frac{\partial}{\partial \boldsymbol{v}'}\right) g_{\alpha\beta}(t,\boldsymbol{x},\boldsymbol{x}') \\ &\quad - \sum_\gamma \frac{q_\alpha q_\gamma}{m_\alpha}\int \mathrm{d}\boldsymbol{x}''\,\frac{\partial}{\partial \boldsymbol{r}}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}''|}\cdot\frac{\partial f_\alpha(t,\boldsymbol{x})}{\partial \boldsymbol{v}}\,g_{\beta\gamma}(t,\boldsymbol{x}',\boldsymbol{x}'') \\ &\quad - \sum_\gamma \frac{q_\beta q_\gamma}{m_\beta}\int \mathrm{d}\boldsymbol{x}''\,\frac{\partial}{\partial \boldsymbol{r}'}\frac{1}{|\boldsymbol{r}'-\boldsymbol{r}''|}\cdot\frac{\partial f_\beta(t,\boldsymbol{x}')}{\partial \boldsymbol{v}'}\,g_{\alpha\gamma}(t,\boldsymbol{x},\boldsymbol{x}'') \\ &\quad = \frac{\partial}{\partial \boldsymbol{r}}\frac{q_\alpha q_\beta}{|\boldsymbol{r}-\boldsymbol{r}'|}\cdot\left(\frac{1}{m_\alpha}\frac{\partial}{\partial \boldsymbol{v}} - \frac{1}{m_\beta}\frac{\partial}{\partial \boldsymbol{v}'}\right)\Big[f_\alpha(t,\boldsymbol{x})f_\beta(t,\boldsymbol{x}') + g_{\alpha\beta}(t,\boldsymbol{x},\boldsymbol{x}')\Big] \\ &\quad + \sum_\gamma q_\gamma \int \mathrm{d}\boldsymbol{x}''\left(\frac{1}{m_\alpha}\frac{\partial}{\partial \boldsymbol{r}}\frac{q_\alpha}{|\boldsymbol{r}-\boldsymbol{r}''|}\cdot\frac{\partial}{\partial \boldsymbol{v}} + \frac{1}{m_\beta}\frac{\partial}{\partial \boldsymbol{r}'}\frac{q_\beta}{|\boldsymbol{r}'-\boldsymbol{r}''|}\cdot\frac{\partial}{\partial \boldsymbol{v}'}\right) h_{\alpha\beta\gamma}(t,\boldsymbol{x},\boldsymbol{x}',\boldsymbol{x}'') \\ &\quad + \text{ self-interaction terms that vanish.} \end{aligned} \tag{II.5.21}$$

Ugh! The evolution of $g_{\alpha\beta}$ depends on $h_{\alpha\beta\gamma}$!

We could keep going, but the set of equations we'll get is just as difficult to solve than the original Klimontovich equation. We must break the hierarchy at some point, in order to obtain a closed system of equations.

II.6. Closing the chain of statistical equations

There is a natural small parameter in a weakly coupled plasma:

$$\Lambda^{-1} \doteq (n\lambda_{\mathrm{D}}^3)^{-1} \lll 1; \tag{II.6.1}$$

i.e., there are many particles in a Debye sphere. Recall that this also means that the average potential energy of the plasma is small compared to the average kinetic energy. To the extent that the potential energy due to interactions can be neglected, the plasma behaves like an ideal gas; thus, Λ^{-1} measures the size of departures of the thermodynamic properties of the plasma from those of an ideal gas.

Before explaining what this means for our BBGKY hierarchy, let us compare this situation with that of a gas of neutral particles. In that situation, the range of the interaction force r_0 is much smaller than the mean spacing δr of the particles $\sim n^{-1/3}$. Then it makes sense to expand particle correlations in the small parameter nr_0^3, and thus neglect the triple correlation. In other words, particle–particle collisions are sufficiently rare due to the small cross section that three-body collisions are much rarer than two-body collisions, with the presence of a third body affecting the collision between two bodies at an asymptotically small level. In a plasma, by contrast, $r_0 \approx \lambda_{\rm D} \gg n^{-1/3}$ implies $nr_0^3 \gg 1$. This is because Debye screening limits the range of the interaction potential, but to a value that is still large compared to the average interparticle separation (i.e., the Coulomb force is long range compared to the scattering force of direct two-body collisions, but has its long range attenuated by Debye screening). However, this does not mean that three-body interactions are more important than two-body interactions, despite $nr_0^3 \gg 1$ for a plasma. This is because, even though a charged particle is interacting with all the particles in its Debye sphere and thus undergoes $\sim\Lambda$ simultaneous Coulomb collisions, such collisions are *weak*, in the sence that the effect of, say, particle A on particle B's orbit is small enough that the collision between particle B and another particle C is practically unaffected. This is because collisions in an ionized plasma result in small-angle (rather than large-angle) deflections. Another way of saying this is that the joint distribution $f_{\alpha\beta}$ of two particles in a small volume ($n^{-1} \ll \mathcal{V} \ll \lambda_{\rm D}^3$) is determined by the many particles outside of the volume rather than by the separation of the two particles from one another; i.e., $f_{\alpha\beta} \approx f_\alpha f_\beta$. We will prove this explicitly in due course, but for now we use these arguments to order

$$\begin{aligned} f_\alpha &\sim \mathcal{O}(1), \\ g_{\alpha\beta} &\sim \mathcal{O}(\Lambda^{-1}), \\ h_{\alpha\beta\gamma} &\sim \mathcal{O}(\Lambda^{-2}), \\ &\dots \end{aligned}$$

Thus, the BBGKY hierarchy can be truncated by dropping, say, three-body interactions ($h_{\alpha\beta\gamma} \to 0$). In this case, our closed set of kinetic equations is:

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla} + \boldsymbol{a}\boldsymbol{\cdot}\frac{\partial}{\partial \boldsymbol{v}}\right) f_\alpha(t,\boldsymbol{x}) = \sum_\beta \frac{q_\alpha q_\beta}{m_\alpha} \int {\rm d}\boldsymbol{x}' \, \frac{\partial}{\partial \boldsymbol{r}} \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} \boldsymbol{\cdot} \frac{\partial g_{\alpha\beta}(t,\boldsymbol{x},\boldsymbol{x}')}{\partial \boldsymbol{v}}, \tag{II.6.2}$$

$$\begin{aligned} &\left(\frac{\partial}{\partial t} + \boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla} + \boldsymbol{a}\boldsymbol{\cdot}\frac{\partial}{\partial \boldsymbol{v}} + \boldsymbol{v}'\boldsymbol{\cdot}\boldsymbol{\nabla}' + \boldsymbol{a}'\boldsymbol{\cdot}\frac{\partial}{\partial \boldsymbol{v}'}\right) g_{\alpha\beta}(t,\boldsymbol{x},\boldsymbol{x}') \\ &\quad - \sum_\gamma \frac{q_\alpha q_\gamma}{m_\alpha} \int {\rm d}\boldsymbol{x}'' \, \frac{\partial}{\partial \boldsymbol{r}} \frac{1}{|\boldsymbol{r}-\boldsymbol{r}''|} \boldsymbol{\cdot} \frac{\partial f_\alpha(t,\boldsymbol{x})}{\partial \boldsymbol{v}} \, g_{\beta\gamma}(t,\boldsymbol{x}',\boldsymbol{x}'') \\ &\quad - \sum_\gamma \frac{q_\beta q_\gamma}{m_\beta} \int {\rm d}\boldsymbol{x}'' \, \frac{\partial}{\partial \boldsymbol{r}'} \frac{1}{|\boldsymbol{r}'-\boldsymbol{r}''|} \boldsymbol{\cdot} \frac{\partial f_\beta(t,\boldsymbol{x}')}{\partial \boldsymbol{v}'} \, g_{\alpha\gamma}(t,\boldsymbol{x},\boldsymbol{x}'') \\ &\quad = \frac{\partial}{\partial \boldsymbol{r}} \frac{q_\alpha q_\beta}{|\boldsymbol{r}-\boldsymbol{r}'|} \boldsymbol{\cdot} \left(\frac{1}{m_\alpha}\frac{\partial}{\partial \boldsymbol{v}} - \frac{1}{m_\beta}\frac{\partial}{\partial \boldsymbol{v}'}\right) \Big[f_\alpha(t,\boldsymbol{x}) f_\beta(t,\boldsymbol{x}') + g_{\alpha\beta}(t,\boldsymbol{x},\boldsymbol{x}')\Big], \end{aligned} \tag{II.6.3}$$

where

$$\boldsymbol{a} \doteq \frac{q_\alpha}{m_\alpha}\left[\boldsymbol{E}(t,\boldsymbol{r}) + \frac{\boldsymbol{v}}{c}\times\boldsymbol{B}(t,\boldsymbol{r})\right],$$
$$\boldsymbol{a}' \doteq \frac{q_\beta}{m_\beta}\left[\boldsymbol{E}(t,\boldsymbol{r}') + \frac{\boldsymbol{v}'}{c}\times\boldsymbol{B}(t,\boldsymbol{r}')\right],$$
$$\boldsymbol{E}(t,\boldsymbol{r}) = \boldsymbol{E}_{\rm ext}(t,\boldsymbol{r}) - \frac{\partial}{\partial\boldsymbol{r}}\sum_\gamma q_\gamma \int {\rm d}\boldsymbol{x}''\,\frac{f_\gamma(t,\boldsymbol{x}'')}{|\boldsymbol{r}-\boldsymbol{r}''|},$$
$$\boldsymbol{B}(t,\boldsymbol{r}) = \boldsymbol{B}_{\rm ext}(t,\boldsymbol{r}).$$

Physical description of each term in (II.6.3):

(1) RHS term $\propto f_\alpha f_\beta$: establishes a two-particle correlation between initially uncorrelated particles α and β caused by a binary Coulomb interaction.

(2) RHS term $\propto g_{\alpha\beta}$: drives changes to an existing two-particle correlation caused by a binary Coulomb interaction between α and β.

(3) LHS first term $\propto \dot{g}_{\alpha\beta}$: conservatively advects the two-particle correlation $g_{\alpha\beta}$ through phase space; the $\boldsymbol{a}\cdot\partial/\partial\boldsymbol{v}$ and $\boldsymbol{a}'\cdot\partial/\partial\boldsymbol{v}'$ terms represent the effect of the mean field on the two-particle correlation.

(4) LHS second term $\propto f_\alpha g_{\beta\gamma}$: modifies correlations between α and β, due to Coulomb interactions between particle α and all other particles γ in the bath that are correlated with β. This is an important *shielding* term, in which a typical particle in the bath both mediates and modifies the correlation between particles α and β.

(5) LHS third term $\propto f_\beta g_{\alpha\gamma}$: modifies correlations between α and β, due to Coulomb interactions between particle β and all other particles γ in the bath that are correlated with α. This is another important shielding term.

II.7. The Vlasov equation

Solutions to (II.6.3) are difficult to come by. We will obtain one solution – the Balescu-Lenard equation – in due course (§IV). For now, let us drop two-particle correlations ($g_{\alpha\beta}\to 0$) to find the *Vlasov equation*:

$$\boxed{\dot{f}_\alpha \doteq \left[\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla} + \frac{q_\alpha}{m_\alpha}\left(\boldsymbol{E}+\frac{\boldsymbol{v}}{c}\times\boldsymbol{B}\right)\cdot\frac{\partial}{\partial\boldsymbol{v}}\right]f_\alpha(t,\boldsymbol{x}) = 0} \tag{II.7.1}$$

Thus, the one-particle distribution function f_α is our old familiar friend, customarily referred to as *the* distribution function of the plasma. The assumption here is that the phase densities at different points in 6D phase space are completely independent (but for collective interactions via electromagnetic fields). Let us use (II.6.3) to see whether this is a good assumption.

First, by ordering $\partial/\partial\boldsymbol{v}\sim v_{\rm th}^{-1}$ and $|\boldsymbol{r}-\boldsymbol{r}'|\sim\lambda_{\rm D}$, we find that the "collision operator" on the right-hand side of (II.6.2) satisfies

$$\left(\frac{\partial f}{\partial t}\right)_{\rm c} \sim \frac{q^2}{m}v_{\rm th}^3\frac{\lambda_{\rm D}}{v_{\rm th}}g \sim \frac{v_{\rm th}^3}{n}\omega_{\rm p}g.$$

Second, adopting $f\sim nv_{\rm th}^{-3}$ and $\partial g/\partial t\sim\omega_{\rm p}g$, where $\omega_{\rm p}$ is the plasma frequency, equation

(II.6.3) implies

$$\omega_{\rm p} g \sim \frac{q^2}{\lambda_{\rm D}^2}\frac{f^2}{mv_{\rm th}} \sim \frac{q^2}{\lambda_{\rm D}^2}\frac{f}{mv_{\rm th}}\frac{n}{v_{\rm th}^3} \sim \frac{\omega_{\rm p}}{\lambda_{\rm D}^3}\frac{1}{v_{\rm th}^3} f.$$

Thus,

$$\left(\frac{\partial f}{\partial t}\right)_{\rm c} \sim \frac{1}{n\lambda_{\rm D}^3}\,\omega_{\rm p} f = \frac{1}{\Lambda}\omega_{\rm p} f \lll \omega_{\rm p} f. \tag{II.7.2}$$

As $\Lambda \to \infty$, we obtain the Vlasov equation.

Before we do anything with the Vlasov equation, a comment is in order. Consider the following excerpt from page 38 of Ichimaru (2004):

> It is interesting to note a formal similarity between the Vlasov equation and the Klimontovich equation... Yet it is quite important also to note the fundamental difference in the physical contents between the two equations: while the Klimontovich equation deals with the microscopic distribution function, containing all the fine structures arising from the individuality of the particles, the Vlasov equation is concerned with a coarse-grained distribution function obtained from a statistical average of the microscopic distribution function. The fluctuations due to discreteness of the particles have not been retained in the Vlasov equation.

PART III
Properties of the Vlasov equation

By now, you should be familiar with the Vlasov equation and how to solve it. But there is a reason for revisiting it. Actually, there are two reasons: (1) we are going to solve the Landau problem in a slightly more sophisticated way than is standard, which will allow for a straightforward solution of the $g_{\alpha\beta}$ equation (II.6.3) to obtain the Balescu-Lenard operator; and (2) it is worthwhile revisiting the notion of phase mixing and the generation of small-scale structure in velocity space. This course is on irreversibility – it's probably best to understand *reversibility* first! This will afford a preview of key concepts in the course: entropy, conservation of free energy, dielectric response to a test particle, etc.

III.1. Time reversibility and entropy

First, let us look at the Vlasov equation (II.7.1), rewritten here:

$$\dot{f}_\alpha(t,\boldsymbol{x}) \doteq \left[\frac{\partial}{\partial t} + \boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla} + \frac{q_\alpha}{m_\alpha}\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c}\times\boldsymbol{B}\right)\boldsymbol{\cdot}\frac{\partial}{\partial\boldsymbol{v}}\right] f_\alpha(t,\boldsymbol{x}) = 0. \tag{III.1.1}$$

Note that setting $t \to t' \doteq -t$, $\boldsymbol{r} \to \boldsymbol{r}' \doteq \boldsymbol{r}$, $\boldsymbol{v} \to \boldsymbol{v}' \doteq -\boldsymbol{v}$, $f_\alpha \to f'_\alpha \doteq f_\alpha(-t,\boldsymbol{r},-\boldsymbol{v})$, $\boldsymbol{E} \to \boldsymbol{E}' \doteq \boldsymbol{E}(-t,\boldsymbol{r})$, and $\boldsymbol{B} \to \boldsymbol{B}' \doteq -\boldsymbol{B}(-t,\boldsymbol{r})$ in (III.1.1) changes nothing:

$$\left[\frac{\partial}{\partial t'} + \boldsymbol{v}'\boldsymbol{\cdot}\boldsymbol{\nabla}' + \frac{q_\alpha}{m_\alpha}\left(\boldsymbol{E}' + \frac{\boldsymbol{v}'}{c}\times\boldsymbol{B}'\right)\boldsymbol{\cdot}\frac{\partial}{\partial\boldsymbol{v}'}\right] f'_\alpha(t',\boldsymbol{x}') = 0.$$

Thus, the Vlasov–Maxwell set of equations is *time-reversible*. All information about the phase-space fluid elements is preserved for all time.

Next, calculate the evolution of the entropy,

$$\mathcal{S} \doteq -\sum_\alpha \int {\rm d}\boldsymbol{x}\, f_\alpha \ln f_\alpha, \tag{III.1.2}$$

in a Vlasov plasma:

$$\frac{\mathrm{d}\mathcal{S}}{\mathrm{d}t} = -\sum_\alpha \int \mathrm{d}\boldsymbol{x}\, \frac{\partial f_\alpha}{\partial t}(1 + \ln f_\alpha) = 0. \tag{III.1.3}$$

Entropy is constant. What a comforting thought.[5] In what follows, we will investigate how this constancy is maintained (and ultimately broken) in a plasma with fluctuations.

III.2. Linearized Vlasov equation

It will be useful for later in the course to consider a spatially homogeneous, charge-neutral plasma in the electrostatic limit (i.e., the $\boldsymbol{v}\times\boldsymbol{B}$ term is dropped). Allowing for fluctuations in the electric field and the distribution function about this equilibrium, denoted by a δ, the Vlasov equation (III.1.1) becomes

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla}\right)\delta f_\alpha + \frac{q_\alpha}{m_\alpha}\delta\boldsymbol{E}\boldsymbol{\cdot}\frac{\partial f_{0\alpha}}{\partial \boldsymbol{v}} = -\frac{q_\alpha}{m_\alpha}\delta\boldsymbol{E}\boldsymbol{\cdot}\frac{\partial \delta f_\alpha}{\partial \boldsymbol{v}}, \tag{III.2.1}$$

where $f_{0\alpha}$ is the equilibrium distribution function of species α. The fluctuating electric field $\delta\boldsymbol{E}$ is determined from Poisson's equation:

$$\delta\boldsymbol{E}(t,\boldsymbol{r}) \doteq -\boldsymbol{\nabla}\varphi(t,\boldsymbol{r}) = -\frac{\partial}{\partial\boldsymbol{r}}\sum_\beta q_\beta \int \mathrm{d}\boldsymbol{x}'\, \frac{\delta f_\beta(t,\boldsymbol{x}')}{|\boldsymbol{r}-\boldsymbol{r}'|}, \tag{III.2.2}$$

where φ is the electrostatic potential and, as before, β is a dummy species index over which to sum. Thus, the right-hand side of (III.2.1) is nonlinear in the fluctuation amplitudes; let us drop it under the assumption that the fluctuations of interest are small. Then, our *linearized Vlasov equation* reads

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla}\right)\delta f_\alpha - \sum_\beta \frac{q_\alpha q_\beta}{m_\alpha}\frac{\partial f_{0\alpha}}{\partial \boldsymbol{v}}\boldsymbol{\cdot}\frac{\partial}{\partial \boldsymbol{r}}\int \mathrm{d}\boldsymbol{x}'\, \frac{\delta f_\beta(t,\boldsymbol{x}')}{|\boldsymbol{r}-\boldsymbol{r}'|} = 0. \tag{III.2.3}$$

This equation is customarily solved using Laplace–Fourier techniques. We will use a Green's function approach, which will also allow us to consider a non-zero right-hand side of (III.2.3) if there are sources/sinks or perhaps initial conditions turned on at some special time (e.g., $t=0$).

III.3. Green's functions

The idea behind the Green's function approach to solving (III.2.3) is to write

$$\left(\frac{\partial}{\partial t} + \mathcal{L}\right)G_{\alpha\beta}(t,\boldsymbol{x};t',\boldsymbol{x}') = \delta_{\alpha\beta}\delta(t-t')\delta(\boldsymbol{x}-\boldsymbol{x}'), \tag{III.3.1}$$

where $\mathcal{L}$ is the Vlasov operator satisfying

$$\mathcal{L}G_{\alpha\beta} = \boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla}G_{\alpha\beta} - \sum_\gamma \frac{q_\alpha q_\gamma}{m_\alpha}\frac{\partial f_{0\alpha}}{\partial \boldsymbol{v}}\boldsymbol{\cdot}\frac{\partial}{\partial \boldsymbol{r}}\int \mathrm{d}\boldsymbol{x}''\, \frac{G_{\gamma\beta}(t,\boldsymbol{x}'';t',\boldsymbol{x}')}{|\boldsymbol{r}-\boldsymbol{r}''|} \tag{III.3.2}$$

for some $f_\alpha = f_\alpha(t,\boldsymbol{x})$, and solve for the *Green's function*, $G_{\alpha\beta}(t,\boldsymbol{x};t',\boldsymbol{x}')$. Once we know $G_{\alpha\beta}$, then

$$\delta f_\alpha(t,\boldsymbol{x}) = \sum_\beta \int \mathrm{d}\boldsymbol{x}'\, G_{\alpha\beta}(t,\boldsymbol{x};0,\boldsymbol{x}')\,\delta f_\beta(0,\boldsymbol{x}') \tag{III.3.3}$$

[5] In fact, *any* integral of a function $Q = Q(f)$ is conserved by the Vlasov equation; such quantities are commonly referred to as *Casimir invariants*.

for some $t > 0$, given the initial condition $\delta f_\alpha(0,\boldsymbol{x})$; or, if there is an additional source/sink $S_{\alpha\beta}(t)$, then

$$\delta f_\alpha(t,\boldsymbol{x}) = \sum_\beta \int \mathrm{d}\boldsymbol{x}'\, G_{\alpha\beta}(t,\boldsymbol{x};0,\boldsymbol{x}')\,\delta f_\beta(0,\boldsymbol{x}') + \int_0^t \mathrm{d}t' \sum_\beta \int \mathrm{d}\boldsymbol{x}'\, G_{\alpha\beta}(t,\boldsymbol{x};t',\boldsymbol{x}')\,S_{\alpha\beta}(t'). \tag{III.3.4}$$

To remind you: the reason you can use this technique is because of the linearity of the operator. One can divide up the source into a collection of impulses:

$$S(t) = \int_{-\infty}^{\infty} \mathrm{d}t'\, \delta(t-t')S(t'),$$

the response to each being given by the appropriate Green's function. Note that the Green's function does not care about the nature of the source. (We're always integrating forwards in time in this course, so backward-propagating solutions will not be discussed.)

We wish to solve (III.2.3) using this technique. Before doing so, let us establish some conventions. . .

III.4. Fourier and Laplace transforms

The (3D spatial) Fourier and inverse-Fourier transforms are, respectively,

$$f(\boldsymbol{k}) = \int \frac{\mathrm{d}\boldsymbol{r}}{(2\pi)^3}\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} f(\boldsymbol{r}), \tag{III.4.1a}$$

$$f(\boldsymbol{r}) = \int \mathrm{d}\boldsymbol{k}\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} f(\boldsymbol{k}). \tag{III.4.1b}$$

The convention here is the same as is used in the textbooks by Nicholson (1983), Krall & Trivelpiece (1973), Ichimaru (2004), and Montgomery (1971) (although K&T are inconsistent – their Chapter 10 and the latter part of Chapter 11 shift the factor of 2π to be in the inverse-Fourier transform). Klimontovich (1967) and Krommes (2018) both use the definitions $f(\boldsymbol{k}) = \int \mathrm{d}\boldsymbol{r}\, \exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}) f(\boldsymbol{r})$ and $f(\boldsymbol{r}) = (2\pi)^{-3}\int \mathrm{d}\boldsymbol{k}\, \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}) f(\boldsymbol{k})$. So, be aware of factors of $(2\pi)^3$ differing between these notes and those texts.

This Fourier convention is such that:

$$f(\boldsymbol{r}) = 1 \quad \Longleftrightarrow \quad f(\boldsymbol{k}) = \delta(\boldsymbol{k}),$$
$$f(\boldsymbol{r}) = \exp(\pm\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}) \quad \Longleftrightarrow \quad f(\boldsymbol{k}) = \delta(\boldsymbol{k}\mp\boldsymbol{q}),$$

which is why I prefer it. Another advantage is that the Dirac delta is then simply related to the Kronecker delta via

$$\delta(\boldsymbol{k}-\boldsymbol{k}_0) \quad \Longleftrightarrow \quad |\varDelta\boldsymbol{k}|^{-1}\delta_{\boldsymbol{k},\boldsymbol{k}_0},$$

where $|\varDelta\boldsymbol{k}|$ ($=\varDelta k_x \varDelta k_y \varDelta k_z$ in 3D) is the spacing between the discrete wavenumbers in the Fourier series

$$f(\boldsymbol{r}) = \sum_{\boldsymbol{k}} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} f_{\boldsymbol{k}}, \quad \text{where } \boldsymbol{k} = \left(\frac{2\pi n_x}{L_x}, \frac{2\pi n_y}{L_y}, \frac{2\pi n_z}{L_z}\right) \tag{III.4.2}$$

with integer (n_x, n_y, n_z) and domain size (L_x, L_y, L_z).

In HW01 you will prove the following:

i. $$\int \frac{\mathrm{d}\boldsymbol{r}}{(2\pi)^3}\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} f(\mathscr{r}) = \frac{1}{2\pi^2 k}\int_0^\infty \mathrm{d}\mathscr{r}\, \mathscr{r} \sin k\mathscr{r}\, f(\mathscr{r}), \quad \text{where} \quad \mathscr{r} \doteq |\boldsymbol{r}| \text{ in 3D;} \tag{III.4.3}$$

ii. $$\int \frac{\mathrm{d}\boldsymbol{r}}{(2\pi)^2}\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} f(R) = \frac{1}{2\pi}\int_0^\infty \mathrm{d}R\, R \mathrm{J}_0(kR)\, f(R), \quad \text{where} \quad R \doteq |\boldsymbol{r}| \text{ in 2D;} \tag{III.4.4}$$

iii. $$\int \frac{\mathrm{d}\boldsymbol{r}}{(2\pi)^3}\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \int \frac{\mathrm{d}\boldsymbol{r}'}{(2\pi)^3}\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}'\cdot\boldsymbol{r}'} f(\boldsymbol{r}-\boldsymbol{r}') = f(\boldsymbol{k})\delta(\boldsymbol{k}+\boldsymbol{k}'). \tag{III.4.5}$$

Thus, the Fourier transform of the 3D Coulomb potential is

$$\int \frac{\mathrm{d}\boldsymbol{r}}{(2\pi)^3}\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \int \frac{\mathrm{d}\boldsymbol{r}'}{(2\pi)^3}\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}'\cdot\boldsymbol{r}'} \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = \frac{1}{2\pi^2 k^2}\delta(\boldsymbol{k}+\boldsymbol{k}'). \tag{III.4.6}$$

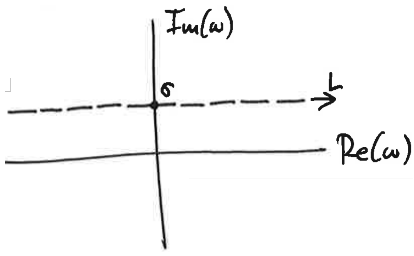
This will come in handy.

The (temporal) Laplace and inverse-Laplace transforms are, respectively,

$$f(\omega) = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t} f(t), \tag{III.4.7a}$$

$$f(t) = \int_L \frac{\mathrm{d}\omega}{2\pi}\, \mathrm{e}^{-\mathrm{i}\omega t} f(\omega), \tag{III.4.7b}$$

where "L" denotes the Laplace (or Bromwich) contour, which is a straight line in the complex plane parallel to the real ω axis and intersecting the imaginary ω axis at $\mathrm{Im}(\omega) = \sigma$:



If there exists a real number $\sigma > 0$ such that $|f(t)| < \exp(\sigma t)$ as $t \to \infty$, then the Laplace transform integral exists for all values of ω such that $\mathrm{Im}(\omega) \geqslant \sigma$. This convention follows Nicholson (1983). The reason I like this convention is that performing the integration along the Laplace contour often brings in a $2\pi\mathrm{i}$ from residues, thus cancelling the 2π in the denominator of (III.4.7b).

III.5. The Vlasov response function

Back to (III.2.3), with our Fourier and Laplace transforms in hand. Note that, since we have taken the background to be spatially homogeneous, $G_{\alpha\beta}$ can only depend on the combination $\boldsymbol{r}-\boldsymbol{r}'$; this follows from translational invariance. Then, writing

$$G_{\alpha\beta}(t,\boldsymbol{x};t',\boldsymbol{x}') = \int \mathrm{d}\boldsymbol{k}\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')} G_{\alpha\beta}(t,\boldsymbol{k},\boldsymbol{v};t',\boldsymbol{v}'), \tag{III.5.1}$$

and likewise for $1/|\boldsymbol{r}-\boldsymbol{r}'|$, equation (III.3.1) becomes

$$\left(\frac{\partial}{\partial t}+\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{v}\right)G_{\alpha\beta}(t,\boldsymbol{k},\boldsymbol{v};t',\boldsymbol{v}')-\sum_{\gamma}\frac{q_{\alpha}q_{\gamma}}{m_{\alpha}}\frac{4\pi\mathrm{i}\boldsymbol{k}}{k^2}\cdot\frac{\partial f_{0\alpha}}{\partial\boldsymbol{v}}\int\mathrm{d}\boldsymbol{v}''\,G_{\gamma\beta}(t,\boldsymbol{k},\boldsymbol{v}'';t',\boldsymbol{v}')$$
$$=\delta_{\alpha\beta}\delta(t-t')\delta(\boldsymbol{v}-\boldsymbol{v}'). \quad \text{(III.5.2)}$$

Next, we Laplace transform in time to transform the $\partial/\partial t$ operator. $G_{\alpha\beta}$ may then be analytically continued into the lower half of the ω plane; there it has poles, corresponding to the roots of the plasma dielectric function $\mathcal{D}(\omega,\boldsymbol{k})$ (defined below).[6] The Laplace contour extends from $-\infty$ to $+\infty$ along a path in the upper half plane in such a way that all poles lie below it. For $t>0$, we can close the contour with an infinite semi-circle in the lower half plane, with Cauchy's theorem used for the inverse transform. For $t<0$, we can close the contour in the upper half plane, giving $G_{\alpha\beta}=0$ for $t<0$.

Proceeding, we find

$$G_{\alpha\beta}(\omega,\boldsymbol{k},\boldsymbol{v};\boldsymbol{v}')=\frac{\delta_{\alpha\beta}\delta(\boldsymbol{v}-\boldsymbol{v}')}{-\mathrm{i}\omega+\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{v}}+\sum_{\gamma}\frac{q_{\alpha}q_{\gamma}}{m_{\alpha}}\frac{4\pi\mathrm{i}\boldsymbol{k}}{k^2}\cdot\frac{\partial f_{0\alpha}/\partial\boldsymbol{v}}{-\mathrm{i}\omega+\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{v}}\int\mathrm{d}\boldsymbol{v}''\,G_{\gamma\beta}(\omega,\boldsymbol{k},\boldsymbol{v}'';\boldsymbol{v}'). \quad \text{(III.5.3)}$$

To eliminate the integral of $G_{\gamma\beta}$, do $\sum_{\alpha}q_{\alpha}\int\mathrm{d}\boldsymbol{v}$ of (III.5.3), which yields

$$\sum_{\alpha}q_{\alpha}\int\mathrm{d}\boldsymbol{v}\,G_{\alpha\beta}(\omega,\boldsymbol{k},\boldsymbol{v};\boldsymbol{v}')=\frac{q_{\beta}}{-\mathrm{i}\omega+\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{v}}+\sum_{\alpha}\frac{q_{\alpha}^2}{m_{\alpha}}\int\mathrm{d}\boldsymbol{v}\,\frac{4\pi\mathrm{i}\boldsymbol{k}}{k^2}\cdot\frac{\partial f_{0\alpha}/\partial\boldsymbol{v}}{-\mathrm{i}\omega+\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{v}}\left[\sum_{\gamma}q_{\gamma}\int\mathrm{d}\boldsymbol{v}''\,G_{\gamma\beta}(\omega,\boldsymbol{k},\boldsymbol{v}'';\boldsymbol{v}')\right]. \quad \text{(III.5.4)}$$

Changing the dummy index α and dummy integration variable $\boldsymbol{v}$ on the left-hand side of (III.5.4) to γ and $\boldsymbol{v}''$, respectively, we obtain

$$\sum_{\gamma}q_{\gamma}\int\mathrm{d}\boldsymbol{v}''\,G_{\gamma\beta}(\omega,\boldsymbol{k},\boldsymbol{v}'';\boldsymbol{v}')=\frac{1}{\mathcal{D}(\omega,\boldsymbol{k})}\frac{q_{\beta}}{-\mathrm{i}\omega+\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{v}}, \quad \text{(III.5.5)}$$

where

$$\boxed{\mathcal{D}(\omega,\boldsymbol{k})\doteq 1+\sum_{\gamma}\frac{q_{\gamma}^2}{m_{\gamma}}\frac{4\pi\boldsymbol{k}}{k^2}\cdot\int\mathrm{d}\boldsymbol{v}''\,\frac{\partial f_{0\gamma}/\partial\boldsymbol{v}''}{\omega-\boldsymbol{k}\cdot\boldsymbol{v}''}} \quad \text{(III.5.6)}$$

is the *dielectric function* describining longitudinal waves in a Vlasov plasma. (It is customary to add a small imaginary positive number, $+\mathrm{i}0$, to the denominator of the integrand when causality must be respected.) Plugging (III.5.5) into (III.5.3), we obtain the *Vlasov response function*

$$\boxed{G_{\alpha\beta}(\omega,\boldsymbol{k},\boldsymbol{v};\boldsymbol{v}')=\frac{\delta_{\alpha\beta}\delta(\boldsymbol{v}-\boldsymbol{v}')}{-\mathrm{i}\omega+\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{v}}+\frac{q_{\alpha}q_{\beta}}{m_{\alpha}}\frac{4\pi\mathrm{i}\boldsymbol{k}}{k^2}\cdot\frac{\partial f_{0\alpha}}{\partial\boldsymbol{v}}\frac{1}{\mathcal{D}(\omega,\boldsymbol{k})}\frac{1}{-\mathrm{i}\omega+\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{v}}\frac{1}{-\mathrm{i}\omega+\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{v}'}} \quad \text{(III.5.7)}$$

The first term on the right-hand side of (III.5.7) represents a free-particle propagator

[6]Here, and throughout these notes, we are assuming that the plasma is stable, so that all the poles of $\mathcal{D}$ lie in the lower half of the complex plane.

(thus, the $\delta_{\alpha\beta}$ factor – a particle doesn't change flavors or interact just by propagating). The second term accounts for the effect of Coulomb interactions; it is the polarization response of the plasma to the test particle represented by the first term. You should know all about $\mathcal{D}(\omega, \boldsymbol{k})$ from the Waves course.

To get $\delta f_\alpha(t, \boldsymbol{x})$, we just integrate $G_{\alpha\beta}(t, \boldsymbol{x}; 0, \boldsymbol{x}')$. Or, better yet, let us obtain $\delta f_\alpha(\omega, \boldsymbol{k}, \boldsymbol{v})$ in terms of $G_{\alpha\beta}(\omega, \boldsymbol{k}, \boldsymbol{v}; \boldsymbol{v}')$ and then inverse-Fourier-Laplace transform that instead:

$$\delta f_\alpha(t, \boldsymbol{x}) = \sum_\beta \int \mathrm{d}\boldsymbol{x}'\, G_{\alpha\beta}(t, \boldsymbol{x}; 0, \boldsymbol{x}')\delta f_\beta(0, \boldsymbol{x}')$$

$$\Longrightarrow \quad \delta f_\alpha(\omega, \boldsymbol{k}, \boldsymbol{v}) = \sum_\beta \int \mathrm{d}\boldsymbol{v}'\, G_{\alpha\beta}(\omega, \boldsymbol{k}, \boldsymbol{v}; \boldsymbol{v}')\delta f_\beta(t = 0, \boldsymbol{k}, \boldsymbol{v}'). \tag{III.5.8}$$

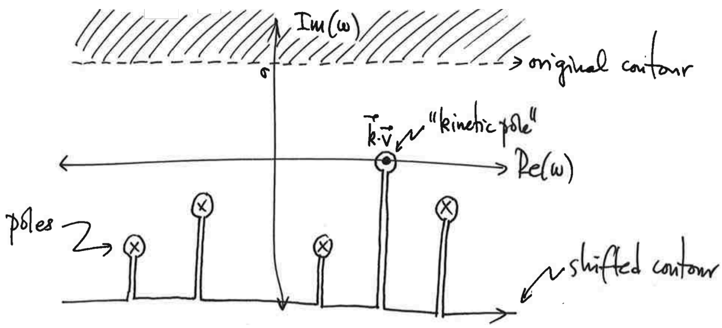
Using (III.5.7) for $G_{\alpha\beta}$, equation (III.5.8) becomes

$$\delta f_\alpha(\omega, \boldsymbol{k}, \boldsymbol{v}) = \frac{\delta f_\alpha(t = 0, \boldsymbol{k}, \boldsymbol{v})}{-\mathrm{i}\omega + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{v}} + \sum_\beta \frac{q_\alpha q_\beta}{m_\alpha}\frac{4\pi\mathrm{i}\boldsymbol{k}}{k^2}\cdot\frac{\partial f_{0\alpha}}{\partial \boldsymbol{v}}\frac{1}{\mathcal{D}(\omega, \boldsymbol{k})}\frac{1}{-\mathrm{i}\omega + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{v}}\int \mathrm{d}\boldsymbol{v}'\,\frac{\delta f_\beta(t = 0, \boldsymbol{k}, \boldsymbol{v}')}{-\mathrm{i}\omega + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{v}'}. \tag{III.5.9}$$

Voila! Then

$$\begin{aligned}\delta f_\alpha(t, \boldsymbol{k}, \boldsymbol{v}) &= \int_L \frac{\mathrm{d}\omega}{2\pi}\,\mathrm{e}^{-\mathrm{i}\omega t}\delta f_\alpha(\omega, \boldsymbol{k}, \boldsymbol{v}) \\ &= \int_L \frac{\mathrm{d}\omega}{2\pi}\,\mathrm{e}^{-\mathrm{i}\omega t}\frac{\delta f_\alpha(t = 0, \boldsymbol{k}, \boldsymbol{v})}{-\mathrm{i}\omega + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{v}} \\ &\quad + \sum_\beta \frac{q_\alpha q_\beta}{m_\alpha}\frac{4\pi\mathrm{i}\boldsymbol{k}}{k^2}\cdot\frac{\partial f_{0\alpha}}{\partial \boldsymbol{v}}\int_L \frac{\mathrm{d}\omega}{2\pi}\frac{\mathrm{e}^{-\mathrm{i}\omega t}}{\mathcal{D}(\omega, \boldsymbol{k})}\frac{1}{-\mathrm{i}\omega + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{v}}\int \mathrm{d}\boldsymbol{v}'\,\frac{\delta f_\beta(t = 0, \boldsymbol{k}, \boldsymbol{v}')}{-\mathrm{i}\omega + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{v}'}.\end{aligned} \tag{III.5.10}$$

To do the integrals in (III.5.10), the Laplace contour must be shifted, with $\delta f_\alpha(\omega, \boldsymbol{k}, \boldsymbol{v})$ being analytically continued everywhere to $\mathrm{Im}(\omega) \to -\infty$ without the contour crossing the poles:



If it's been awhile since you performed such a task, here is a reminder along with some cautionary statements. Push the contour down towards $\mathrm{Im}(\omega) \to -\infty$, draping it over and encircling the poles as you go. The contributions from the horizontal parts of the contour are exponentially small in time (one hopes... keep reading); and the contributions to the integral from the segments leading towards and away from the poles cancel. All that's left are those circles around the poles, and those give the residues by Cauchy's formula: $\pm 2\pi \mathrm{i}$ times the sum of the residues for a counter-clockwise (clockwise) contour. Regarding the "one hopes"... if $\delta f_\beta(t=0,\boldsymbol{k},\boldsymbol{v})$ is proportional to, say, a Maxwellian distribution, then the final integral in (III.5.10) will contain a term $\propto \exp(-\omega^2/k_\parallel^2 v_{\mathrm{th}}^2)$, which can be large at large $\mathrm{Im}(\omega)$. In this case, the integral along the horizontal part of the contour vanishes only at $t\to\infty$, and the above approach isn't necessarily the best way to determine the finite-time behavior of δf_α (it relies on the smallness of $\exp(-\mathrm{i}\omega t)$). Other requirements are that the initial conditions are entire and that there are a finite number of simple poles.

Then the potential

$$\varphi(\omega,\boldsymbol{k}) = \frac{4\pi}{k^2}\frac{1}{\mathcal{D}(\omega,\boldsymbol{k})}\sum_\beta q_\beta \int \mathrm{d}\boldsymbol{v}' \, \frac{\delta f_\beta(t=0,\boldsymbol{k},\boldsymbol{v}')}{-\mathrm{i}\omega + \mathrm{i}\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{v}'} \tag{III.5.11}$$

must have the form

$$\sum_j \frac{c_j}{-\mathrm{i}(\omega-\omega_j)} + A(\omega),$$

where c_j are the residues and $A(\omega)$ is the analytic part of the solution, so that

$$\begin{aligned}\varphi(t,\boldsymbol{k}) &= \int_L \frac{\mathrm{d}\omega}{2\pi}\,\mathrm{e}^{-\mathrm{i}\omega t}\frac{4\pi}{k^2}\frac{1}{\mathcal{D}(\omega,\boldsymbol{k})}\sum_\beta q_\beta\int \mathrm{d}\boldsymbol{v}'\,\frac{\delta f_\beta(t=0,\boldsymbol{k},\boldsymbol{v}')}{-\mathrm{i}\omega+\mathrm{i}\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{v}'}\\ &= \int_L \frac{\mathrm{d}\omega}{2\pi}\,\mathrm{e}^{-\mathrm{i}\omega t}\left[\sum_j \frac{c_j}{-\mathrm{i}(\omega-\omega_j)} + A(\omega)\right]\\ &= \sum_j c_j \mathrm{e}^{-\mathrm{i}\omega_j t}\end{aligned} \tag{III.5.12}$$

by Cauchy's residue theorem. Evidently, the $\boldsymbol{k}$-space electrostatic potential $\varphi(t,\boldsymbol{k})$ is a sum of damped modes. The poles here are the zeros of the dielectric function, $\mathcal{D}(\omega,\boldsymbol{k})=0$ (there can also be a pole due to $\delta f_\beta(t=0,\boldsymbol{k},\boldsymbol{v}')$ if, say, $\delta f_\beta(t=0,\boldsymbol{k},\boldsymbol{v}')\propto\delta(\boldsymbol{v}'-\boldsymbol{V}_0)$). Let us rewrite (III.5.9) using this:

$$\delta f_\alpha(\omega,\boldsymbol{k},\boldsymbol{v}) = \frac{\delta f_\alpha(t=0,\boldsymbol{k},\boldsymbol{v})}{-\mathrm{i}\omega+\mathrm{i}\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{v}} + \frac{q_\alpha}{m_\alpha}\frac{\mathrm{i}\boldsymbol{k}}{-\mathrm{i}\omega+\mathrm{i}\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{v}}\boldsymbol{\cdot}\frac{\partial f_{0\alpha}}{\partial\boldsymbol{v}}\left[\sum_j\frac{c_j}{-\mathrm{i}(\omega-\omega_j)}+A(\omega)\right]. \tag{III.5.13}$$

Here we have the "kinetic pole" ($\omega=\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{v}$) and the poles representing the linear modes ($\omega=\omega_j$). Following through with the inverse-Laplace transform and using the residue theorem, equation (III.5.13) gives

$$\begin{aligned}\delta f_\alpha(t,\boldsymbol{k},\boldsymbol{v}) &= \left[\delta f_\alpha(t=0,\boldsymbol{k},\boldsymbol{v}) - \frac{q_\alpha}{m_\alpha}\mathrm{i}\boldsymbol{k}\boldsymbol{\cdot}\frac{\partial f_{0\alpha}}{\partial\boldsymbol{v}}\sum_j\frac{c_j}{-\mathrm{i}\omega_j+\mathrm{i}\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{v}}\right]\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{v}t}\\ &\quad + \frac{q_\alpha}{m_\alpha}\mathrm{i}\boldsymbol{k}\boldsymbol{\cdot}\frac{\partial f_{0\alpha}}{\partial\boldsymbol{v}}\sum_j\frac{c_j\mathrm{e}^{-\mathrm{i}\omega_j t}}{-\mathrm{i}\omega_j+\mathrm{i}\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{v}}\end{aligned} \tag{III.5.14}$$

$$\begin{aligned}&= \text{ballistic response, which oscillates without decaying}\\ &\quad + \text{eigenmode solution, which is found from } \mathcal{D}(\omega,\boldsymbol{k})=0.\end{aligned}$$

Note that the analytical part of $\varphi(\omega,\boldsymbol{k})$, denoted $A(\omega)$, must satisfy $A(\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{v})=0$ in order for (III.5.14) to return the correct initial value at $t=0$.

Let us summarize. The electrostatic potential $\varphi(t,\boldsymbol{k})$ (see (III.5.12)) is a sum of damped eigenmodes, $\varphi\propto\exp(-\mathrm{i}\omega_j t)$ with $\mathrm{Im}(\omega_j)<0$. However, the solution for δf_α (see

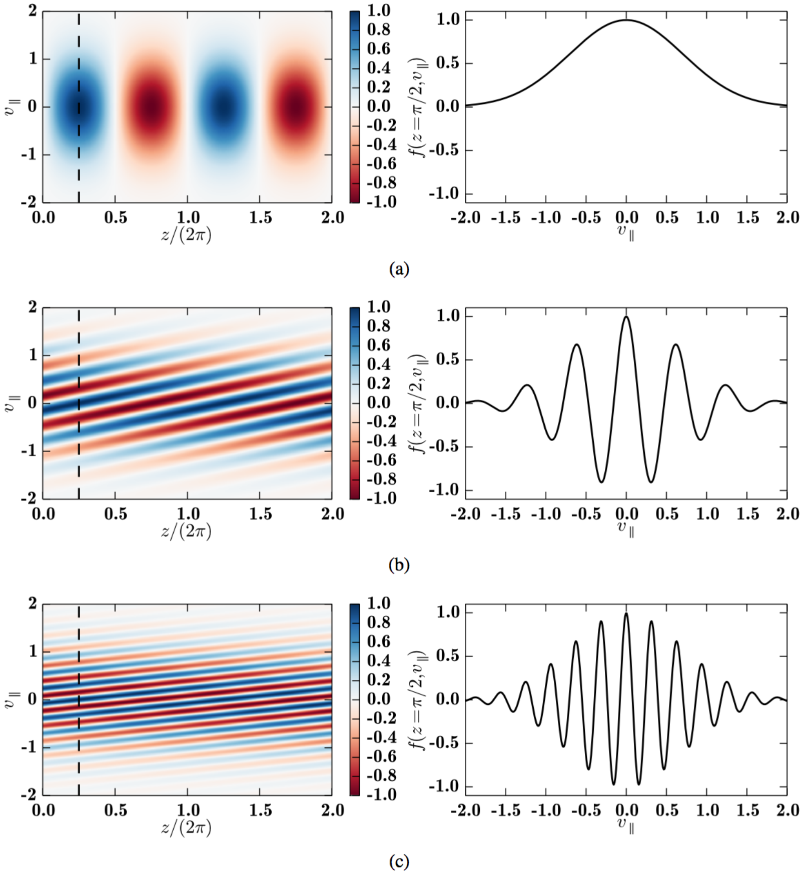


FIGURE 2. Illustration of phase mixing (from Joseph Parker, D.Phil. Thesis at Oxford, 2015). The distribution function $f(t, z, v_\parallel) = \sin(z - v_\parallel t)\exp(-v_\parallel^2)$ solves the free-streaming equation, $\partial f/\partial t + v_\parallel \partial f/\partial z = 0$, thus acquiring fine-scale structure in velocity space.

(III.5.14)) does *not* decay: there is a part of $\delta f_\alpha(t)$ – the "ballistic response" proportional to $\exp(-i\boldsymbol{k}\cdot\boldsymbol{v}t)$ – that oscillates without decaying. (In fact, it has a *growing* part, which keeps the free energy conserved as the potential decays. More on this shortly.) This is *Landau damping*: the transfer of (free) energy from the electric-field fluctuations to the perturbations of the distribution function.

The important thing to note here from the standpoint of irreversibility is that δf_α acquires sharper and sharper structure in velocity space:

$$\frac{1}{\delta f}\frac{\partial \delta f}{\partial v} \sim -ikt \to \infty \quad \text{as} \quad t \to \infty.$$

This is due to phase mixing (see figure 2). Eventually, the velocity-space gradients of δf become so large that collisions can no longer be ignored. We haven't derived the collision integral operator yet, but from (II.6.2) and (II.6.3) we can see that it has a piece involving two applications of $\partial/\partial v$ (the other piece involves just one application of $\partial/\partial v$ and represents a drag force – more later). Thus,

$$\left(\frac{\partial \delta f}{\partial t}\right)_{\rm c} \sim \nu\, v_{\rm th}^2 \frac{\partial^2 \delta f}{\partial v^2} \sim -\nu v_{\rm th}^2 k^2 t^2 \delta f \quad \Longrightarrow \quad \delta f \sim \exp\left(-\frac{1}{3}\nu k^2 v_{\rm th}^2 t^3\right). \qquad \text{(III.5.15)}$$

Collisions decay δf! The characteristic timescale here is $t_{\rm coll} \sim \nu^{-1/3}(kv_{\rm th})^{-2/3}$.

Let us continue with the Vlasov model to see what is conserved during Landau damping.

III.6. Free-energy conservation in the Vlasov treatment

It is straightforward to show by using the Poisson equation (II.4.5), $\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{E} = -\nabla^2\varphi = \sum_\alpha 4\pi q_\alpha \int \mathrm{d}\boldsymbol{v}\, f_\alpha$, and the Vlasov equation (III.1.1) that the rate of change of the electric energy satisfies

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}\boldsymbol{r}\,\frac{E^2}{8\pi} &= \int \mathrm{d}\boldsymbol{r}\,\frac{\boldsymbol{\nabla}\varphi}{4\pi}\boldsymbol{\cdot}\frac{\partial\boldsymbol{\nabla}\varphi}{\partial t} = -\int \mathrm{d}\boldsymbol{r}\,\frac{\varphi}{4\pi}\frac{\partial}{\partial t}\nabla^2\varphi \quad \text{(integrate by parts)} \\
&= -\int \mathrm{d}\boldsymbol{r}\,\frac{\varphi}{4\pi}\frac{\partial}{\partial t}\left(-\sum_\alpha 4\pi q_\alpha \int \mathrm{d}\boldsymbol{v}\, f_\alpha\right) \quad \text{(use Poisson)} \\
&= \int \mathrm{d}\boldsymbol{r}\,\varphi\sum_\alpha q_\alpha \int \mathrm{d}\boldsymbol{v}\,\frac{\partial f_\alpha}{\partial t} \\
&= \int \mathrm{d}\boldsymbol{r}\,\varphi\sum_\alpha q_\alpha \int \mathrm{d}\boldsymbol{v}\left(-\boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla}f_\alpha + \frac{q_\alpha}{m_\alpha}\boldsymbol{\nabla}\varphi\boldsymbol{\cdot}\frac{\partial f_\alpha}{\partial \boldsymbol{v}}\right) \quad \text{(use Vlasov)} \\
&= \underbrace{-\int \mathrm{d}\boldsymbol{r}\int \mathrm{d}\boldsymbol{v}\sum_\alpha q_\alpha\varphi\,\boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla}f_\alpha}_{=\int \mathrm{d}\boldsymbol{r}\,\boldsymbol{\nabla}\varphi\boldsymbol{\cdot}\int \mathrm{d}\boldsymbol{v}\,\sum_\alpha q_\alpha \boldsymbol{v} f_\alpha \text{ by parts}} + \underbrace{\int \mathrm{d}\boldsymbol{r}\,\frac{\boldsymbol{\nabla}\varphi^2}{2}\sum_\alpha\frac{q_\alpha^2}{m_\alpha}\boldsymbol{\cdot}\int \mathrm{d}\boldsymbol{v}\,\frac{\partial f_\alpha}{\partial\boldsymbol{v}}}_{=0 \text{ by parts}} \\
&= -\int \mathrm{d}\boldsymbol{r}\,\boldsymbol{E}\boldsymbol{\cdot}\boldsymbol{j}, \qquad \text{(III.6.1)}
\end{aligned}
$$

where

$$
\boldsymbol{j} \doteq \sum_\alpha q_\alpha \int \mathrm{d}\boldsymbol{v}\,\boldsymbol{v} f_\alpha \qquad \text{(III.6.2)}
$$

is the current density. Thus, the rate of change of the electric energy equals minus the rate at which the electric field does work on the charges ("Joule heating"). Nothing surprising there.

That energy goes into the particles:

$$
\begin{aligned}
\frac{\mathrm{d}U}{\mathrm{d}t} &\doteq \frac{\mathrm{d}}{\mathrm{d}t}\sum_\alpha\int \mathrm{d}\boldsymbol{r}\int \mathrm{d}\boldsymbol{v}\,\frac{m_\alpha v^2}{2} f_\alpha \\
&= \sum_\alpha\int \mathrm{d}\boldsymbol{r}\int \mathrm{d}\boldsymbol{v}\,\frac{m_\alpha v^2}{2}\left(-\boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla}f_\alpha + \frac{q_\alpha}{m_\alpha}\boldsymbol{\nabla}\varphi\boldsymbol{\cdot}\frac{\partial f_\alpha}{\partial\boldsymbol{v}}\right) \quad \text{(use Vlasov)} \\
&= -\sum_\alpha q_\alpha\int \mathrm{d}\boldsymbol{r}\int \mathrm{d}\boldsymbol{v}\, f_\alpha\boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla}\varphi \quad \text{(integrate by parts)} \\
&= \int \mathrm{d}\boldsymbol{r}\,\boldsymbol{E}\boldsymbol{\cdot}\boldsymbol{j} \quad \text{(def'n of current density, (III.6.2))} \qquad \text{(III.6.3)}
\end{aligned}
$$

$$
\Longrightarrow \quad \boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left(U + \int \mathrm{d}\boldsymbol{r}\,\frac{E^2}{8\pi}\right) = 0} \qquad \text{(III.6.4)}
$$

This is just energy conservation. (Collisions do not change this, a fact you'll soon prove. With magnetic fields, the conservation law includes $\int \mathrm{d}\boldsymbol{r}\, B^2/8\pi$.)

If the perturbations are damped, then U must increase. This is sometimes referred to as "heating", but let us examine in detail the evolution of the temperature of the equilibrium. Consider a Maxwellian equilibrium,

$$f_{0\alpha} = \frac{n_\alpha}{\pi^{3/2} v_{\mathrm{th}\alpha}^3} \exp\left(-\frac{v^2}{v_{\mathrm{th}\alpha}^2}\right), \quad v_{\mathrm{th}\alpha}^2 \doteq \frac{2T_\alpha}{m_\alpha}, \tag{III.6.5}$$

which is homogeneous in space but with some possible time dependence, $T_\alpha = T_\alpha(t)$. We say that this time dependence is *slow*, in the sense that the time evolution of the equilibrium temperature occurs on a time scale t_{eq} that is much greater than the characteristic time scale ω^{-1} of the (Landau-damped) fluctuations. Formally,

$$f_{0\alpha}(t, \boldsymbol{v}) = \langle f(t, \boldsymbol{r}, \boldsymbol{v}) \rangle \doteq \frac{1}{\Delta t} \int_{t-\Delta t/2}^{t+\Delta t/2} \mathrm{d}t' \, \frac{1}{\mathcal{V}} \int \mathrm{d}\boldsymbol{r} \, f(t', \boldsymbol{r}, \boldsymbol{v}), \tag{III.6.6}$$

where $\omega^{-1} \ll \Delta t \ll t_{\mathrm{eq}}$ and $\mathcal{V}$ denotes the spatial volume under consideration. Decomposing $f_\alpha = f_{0\alpha} + \delta f_\alpha$, we have

$$U = \mathcal{V} \sum_\alpha \underbrace{\int \mathrm{d}\boldsymbol{v} \, \frac{m_\alpha v^2}{2} f_{0\alpha}}_{= \frac{3}{2} n_\alpha T_\alpha} + \sum_\alpha \int \mathrm{d}\boldsymbol{r} \int \mathrm{d}\boldsymbol{v} \, \frac{m_\alpha v^2}{2} \delta f_\alpha, \tag{III.6.7}$$

Averaging (III.6.7) over time using (III.6.6), the fluctuating part vanishes and we are left with

$$\langle U \rangle = \mathcal{V} \sum_\alpha \frac{3}{2} n_\alpha T_\alpha. \tag{III.6.8}$$

Thus, from (III.6.4), we have

$$\sum_\alpha \frac{3}{2} n_\alpha \frac{\mathrm{d}T_\alpha}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{\mathcal{V}} \int \mathrm{d}\boldsymbol{r} \, \frac{\langle E^2 \rangle}{8\pi}; \tag{III.6.9}$$

that is, the heating rate of the equilibrium equals the rate of decrease of the mean energy of the electromagnetic fluctuations.

Now, we saw from (III.5.12) that $\varphi(t, \boldsymbol{k}) = \sum_j c_j \exp(-\mathrm{i}\omega_j t)$. If we wait long enough, only the slowest-decaying mode will be important, with all others exponentially small; call the frequency and decay rate of this mode $\omega_{\boldsymbol{k}}$ and $\gamma_{\boldsymbol{k}} < 0$, so that $\boldsymbol{E}_{\boldsymbol{k}} = -\mathrm{i}\boldsymbol{k}\varphi_{\boldsymbol{k}} \propto -\mathrm{i}\boldsymbol{k} \exp(-\mathrm{i}\omega_{\boldsymbol{k}} t - |\gamma_{\boldsymbol{k}}| t)$. If $|\gamma_{\boldsymbol{k}}| \ll \omega_{\boldsymbol{k}}$, then (III.6.9) becomes

$$\sum_\alpha \frac{3}{2} n_\alpha \frac{\mathrm{d}T_\alpha}{\mathrm{d}t} = \sum_{\boldsymbol{k}} 2|\gamma_{\boldsymbol{k}}| \frac{|\boldsymbol{E}_{\boldsymbol{k}}|^2}{8\pi} > 0. \tag{III.6.10}$$

In words, the rate of Landau damping of the electric-field fluctuations equals the heating rate of the equilibrium.[7]

There is a subtlety here worth confronting. Recall from (III.1.3) that entropy is a constant in a Vlasov system, but here we have heating of the equilibrium. Indeed, for

[7] Prof. Ilya Dodin, when he was teaching this section of the course in Spring 2022 while I was on sabbatical, rightly pointed out that the above approach (and what follows in the remainder of this section) is a bit dodgy and relies too heavily on the non-trivial assumption that $\omega^{-1} \ll \Delta t \ll t_{\mathrm{eq}}$. Doing the calculation correctly, by rigorously taking into account the evolution of the plasma properties as the Landau damping heats the equilibrium, is not easy but can be done (see Dodin 2022).

each species, the entropy of the background is

$$
\begin{aligned}
S_0 &\doteq -\int \mathrm{d}\boldsymbol{x}\, f_0 \ln f_0 \\
&= -\int \mathrm{d}\boldsymbol{x}\, f_0 \left\{ \ln\left[n \left(\frac{m}{2\pi}\right)^{3/2} \right] - \frac{3}{2}\ln T - \frac{mv^2}{2T} \right\} \\
&= \mathcal{V}\left\{ -n \ln\left[n\left(\frac{m}{2\pi}\right)^{3/2}\right] + \frac{3}{2} n \ln T + \frac{3}{2} n \right\}
\end{aligned}
$$

$$
\Longrightarrow \quad T\frac{\mathrm{d}S_0}{\mathrm{d}t} = \mathcal{V}\,\frac{3}{2} n \frac{\mathrm{d}T}{\mathrm{d}t} > 0. \tag{III.6.11}
$$

Thus, heating is associated with the increase of S_0. But if entropy cannot increase without collisions, where is the missing (negative) entropy?

The mean entropy associated with the *perturbed* distribution function δf is

$$
\begin{aligned}
\langle \delta S \rangle &= -\int \mathrm{d}\boldsymbol{x}\, \bigl\langle (f_0 + \delta f)\ln(f_0+\delta f) - f_0 \ln f_0 \bigr\rangle \\
&= -\int \mathrm{d}\boldsymbol{x} \left\langle (f_0+\delta f)\left(\ln f_0 + \frac{\delta f}{f_0} - \frac{\delta f^2}{2 f_0} + \dots \right) - f_0 \ln f_0 \right\rangle \\
&= -\int \mathrm{d}\boldsymbol{x}\, \frac{\langle \delta f^2 \rangle}{2 f_0} + \dots,
\end{aligned} \tag{III.6.12}
$$

after using $\langle \delta f\rangle = 0$. For the total entropy to be constant, any increase in S_0 must be accompanied by a *decrease* in $\langle \delta S\rangle$. The latter can only be achieved by increasing $\langle \delta f^2\rangle$:

$$
\begin{aligned}
T\frac{\mathrm{d}S}{\mathrm{d}t} &= T\frac{\mathrm{d}S_0}{\mathrm{d}t} + T\frac{\mathrm{d}\langle\delta S\rangle}{\mathrm{d}t} = T\frac{\mathrm{d}S_0}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}t} T\langle \delta S\rangle - \langle\delta S\rangle \frac{\mathrm{d}T}{\mathrm{d}t} \\
&= \left(\mathcal{V}\,\frac{3}{2} n - \langle \delta S\rangle \right)\frac{\mathrm{d}T}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}t} T\langle\delta S\rangle \\
&\approx \mathcal{V}\,\frac{3}{2} n \frac{\mathrm{d}T}{\mathrm{d}t} - \frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}\boldsymbol{x}\, \frac{T\langle \delta f^2\rangle}{2 f_0} \overset{\text{must be}}{=} 0.
\end{aligned} \tag{III.6.13}
$$

If we reaffix the species label α, sum over species, and use (III.6.9), we obtain

$$
\boxed{\frac{\mathrm{d}W}{\mathrm{d}t} \doteq \frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}\boldsymbol{r} \left[\sum_\alpha \int \mathrm{d}\boldsymbol{v}\, \frac{T_\alpha \langle \delta f_\alpha^2\rangle}{2 f_{0\alpha}} + \frac{\langle E^2\rangle}{8\pi} \right] = 0} \tag{III.6.14}
$$

This is the *free energy* of the fluctuations (i.e., energy minus entropy).

Thus, during Landau damping, the free energy is conserved while the electric-field fluctuations decay. This means that the entropy of the fluctuations increases negatively to offset the increase in the entropy of the background equilibrium. In this sense, the phase-mixed fluctuations make the plasma *more* ordered. We will see that, with collisions decaying the δf piece, the overall entropy will increase since then there isn't enough $\langle\delta S\rangle$ to compensate for the rise in S_0. In other words, collisions restore disorder by disrupting the phase-space organization of δf caused by phase mixing.

III.7. Free-energy conservation for a weakly damped plasma oscillation

It is instructive to show explicitly that free energy is conserved for a particular solution to the linear Vlasov equation (III.5.14),

$$\delta f_\alpha(t,\boldsymbol{k},\boldsymbol{v}) = \delta f_\alpha(0,\boldsymbol{k},\boldsymbol{v})\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{v}t} + \frac{q_\alpha}{m_\alpha}\mathrm{i}\boldsymbol{k}\cdot\frac{\partial f_{0\alpha}}{\partial \boldsymbol{v}}\sum_j c_j \mathrm{e}^{-\mathrm{i}\omega_j t}\left[\frac{1-\mathrm{e}^{-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{v}-\omega_j)t}}{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{v}-\omega_j)}\right], \tag{III.7.1}$$

with $f_{0\alpha}$ being Maxwellian so that $\partial f_{0\alpha}/\partial\boldsymbol{v} = -(2\boldsymbol{v}/v_{\mathrm{th}\alpha}^2)f_{0\alpha}$. As in §III.6, we focus on the slowest-decaying mode, for which $|\gamma_{\boldsymbol{k}}| \ll \omega_{\boldsymbol{k}}$. To establish a definite ordering, take $\omega_{\boldsymbol{k}} \sim \omega_{\mathrm{pe}}$ and $\gamma_{\boldsymbol{k}}$ to be no larger than $\sim\epsilon^3\omega_{\mathrm{pe}}$ with $\epsilon \ll 1$ (we will verify this frequency separation *a posteriori*), and consider the coarse-graining time interval Δt to satisfy $\omega_{\mathrm{pe}}\Delta t \sim \epsilon^{-2}$. Thus, $\omega_{\boldsymbol{k}}^{-1} \ll \Delta t \ll \gamma_{\boldsymbol{k}}^{-1} \sim t_{\mathrm{eq}}$, as in §III.6. To ensure weak damping, we adopt $\omega_{\boldsymbol{k}} \sim \epsilon^{-1}kv_{\mathrm{the}}$ and assume that the velocities of the Landau-resonant particles satisfy $v^{(\mathrm{res})} \sim \omega_{\boldsymbol{k}}/k \sim \epsilon^{-1}v_{\mathrm{the}}$, i.e., they are in the tail of the distribution. Then $kv_{\mathrm{the}}\Delta t \sim \epsilon^{-1}$. These orderings focus on free-energy conservation occurring on timescales longer than both the inverse plasma frequency and the inverse phase-mixing rate. As we are seeking a growing $|\delta f_\alpha^2|$, we can drop the initial-value term $\delta f_\alpha(0)$ in (III.7.1), which obviously does not grow. Then

$$\delta f_\alpha(t,\boldsymbol{k},\boldsymbol{v}) \approx -\frac{q_\alpha}{T_\alpha}(\boldsymbol{k}\cdot\boldsymbol{v})f_{0\alpha}\,c_{\boldsymbol{k}}\mathrm{e}^{-\mathrm{i}\omega_{\boldsymbol{k}}t-|\gamma_{\boldsymbol{k}}|t}\left[\frac{1-\mathrm{e}^{-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{v}-\omega_{\boldsymbol{k}})t}}{\boldsymbol{k}\cdot\boldsymbol{v}-\omega_{\boldsymbol{k}}}\right]. \tag{III.7.2}$$

Our goal is to compute

$$\begin{aligned}\sum_\alpha\int\mathrm{d}\boldsymbol{v}\,\frac{T_\alpha|\delta f_\alpha(t,\boldsymbol{k},\boldsymbol{v})|^2}{2f_{0\alpha}} &= \sum_\alpha\int\mathrm{d}\boldsymbol{v}\,\frac{q_\alpha^2}{2T_\alpha}(\boldsymbol{k}\cdot\boldsymbol{v})^2 f_{0\alpha}\underbrace{\left|c_{\boldsymbol{k}}\,\mathrm{e}^{-\mathrm{i}\omega_{\boldsymbol{k}}t-|\gamma_{\boldsymbol{k}}|t}\right|^2}_{=|\varphi_{\boldsymbol{k}}|^2}\left|\frac{1-\mathrm{e}^{-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{v}-\omega_{\boldsymbol{k}})t}}{\boldsymbol{k}\cdot\boldsymbol{v}-\omega_{\boldsymbol{k}}}\right|^2\\ &= \sum_\alpha\int\mathrm{d}\boldsymbol{v}\,\frac{q_\alpha^2}{T_\alpha}(\hat{\boldsymbol{k}}\cdot\boldsymbol{v})^2 f_{0\alpha}\frac{k^2|\varphi_{\boldsymbol{k}}|^2}{2}\left|\frac{1-\mathrm{e}^{-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{v}-\omega_{\boldsymbol{k}})t}}{\boldsymbol{k}\cdot\boldsymbol{v}-\omega_{\boldsymbol{k}}}\right|^2,\end{aligned} \tag{III.7.3}$$

where $\hat{\boldsymbol{k}} \doteq \boldsymbol{k}/k$, and show that its time derivative is equal to $-(8\pi)^{-1}\mathrm{d}|\boldsymbol{E}_{\boldsymbol{k}}|^2/\mathrm{d}t$.

Our ordering makes clear that there are two populations of particles in our distribution function: those which are non-resonant ($\boldsymbol{k}\cdot\boldsymbol{v} \ll \omega_{\boldsymbol{k}}$) and take part in a mean (oscillating) flow of the plasma, and those which are resonant ($\boldsymbol{k}\cdot\boldsymbol{v} \approx \omega_{\boldsymbol{k}}$) and are responsible for the Landau damping. The former are easier to treat, so let's start there. In the limit $\boldsymbol{k}\cdot\boldsymbol{v} \ll \omega_{\boldsymbol{k}}$, the non-resonant contribution to (III.7.2) is

$$\delta f_\alpha^{(\mathrm{n.r.})}(t,\boldsymbol{k},\boldsymbol{v}) \approx \frac{q_\alpha}{T_\alpha}\frac{\boldsymbol{k}\cdot\boldsymbol{v}}{\omega_{\boldsymbol{k}}}f_{0\alpha}\Big(\varphi_{\boldsymbol{k}} - c_{\boldsymbol{k}}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{v}t}\Big), \tag{III.7.4}$$

and so (III.7.3) becomes

$$\begin{aligned}\sum_\alpha\int\mathrm{d}\boldsymbol{v}\,\frac{T_\alpha|\delta f_\alpha^{(\mathrm{n.r.})}(t,\boldsymbol{k},\boldsymbol{v})|^2}{2f_{0\alpha}} \approx{}& \sum_\alpha\int\mathrm{d}\boldsymbol{v}\,\frac{q_\alpha^2}{T_\alpha}\frac{(\hat{\boldsymbol{k}}\cdot\boldsymbol{v})^2}{\omega_{\boldsymbol{k}}^2}f_{0\alpha}\left(\frac{k^2|\varphi_{\boldsymbol{k}}|^2}{2}+\frac{k^2|c_{\boldsymbol{k}}|^2}{2}\right)\\ &-\sum_\alpha\int\mathrm{d}\boldsymbol{v}\,\frac{q_\alpha^2}{T_\alpha}\frac{(\boldsymbol{k}\cdot\boldsymbol{v})^2}{\omega_{\boldsymbol{k}}^2}f_{0\alpha}\cos[(\boldsymbol{k}\cdot\boldsymbol{v}-\omega_{\boldsymbol{k}})t].\end{aligned} \tag{III.7.5}$$

The second line on the right-hand side of (III.7.5) integrates away in the long-time limit of interest (i.e., time enough for many oscillations and for phase mixing but not so much time that the perturbation has completely decayed away), since the integrand becomes rapidly varying in velocity space. The second term on the first line is time-independent to leading order in ϵ, and thus will vanish upon taking the time derivative for (III.6.14). What remains is

$$\begin{aligned}\sum_\alpha \int \mathrm{d}\boldsymbol{v}\, \frac{T_\alpha |\delta f_\alpha^{(\mathrm{n.r.})}(t,\boldsymbol{k},\boldsymbol{v})|^2}{2 f_{0\alpha}} &\approx \sum_\alpha \frac{q_\alpha^2}{T_\alpha} \frac{k^2 |\varphi_{\boldsymbol{k}}|^2}{2} \int \mathrm{d}\boldsymbol{v}\, \frac{(\hat{\boldsymbol{k}}\cdot\boldsymbol{v})^2}{\omega_{\boldsymbol{k}}^2} f_{0\alpha} \\ &= \sum_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{\omega_{\boldsymbol{k}}^2} \frac{k^2|\varphi_{\boldsymbol{k}}|^2}{8\pi} = \sum_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{\omega_{\boldsymbol{k}}^2} \frac{|\boldsymbol{E}_{\boldsymbol{k}}|^2}{8\pi},\end{aligned} \tag{III.7.6}$$

where

$$\omega_{\mathrm{p}\alpha}^2 \doteq \frac{4\pi q_\alpha^2 n_\alpha}{m_\alpha} \tag{III.7.7}$$

is the (square of the) plasma frequency of species α. What is this energy? Returning to (III.7.4), its first term from which (III.7.6) stems has an interesting first moment:

$$\delta\boldsymbol{u}_\alpha(t,\boldsymbol{k}) \doteq \frac{q_\alpha}{T_\alpha} \frac{\boldsymbol{k}\varphi_{\boldsymbol{k}}}{\omega_{\boldsymbol{k}}} \cdot \underbrace{\frac{1}{n_\alpha}\int \mathrm{d}\boldsymbol{v}\, \boldsymbol{v}\boldsymbol{v} f_{0\alpha}}_{=\,(T_\alpha/m_\alpha)\mathsf{I}} = \frac{q_\alpha}{m_\alpha}\frac{\boldsymbol{k}\varphi_{\boldsymbol{k}}}{\omega_{\boldsymbol{k}}} = \frac{q_\alpha}{m_\alpha}\frac{\boldsymbol{E}_{\boldsymbol{k}}}{(-\mathrm{i}\omega_{\boldsymbol{k}})}. \tag{III.7.8}$$

This equation states that the rate of change of the fluid velocity (i.e., $-\mathrm{i}\omega_{\boldsymbol{k}}\delta\boldsymbol{u}_\alpha$) is equal to the (electric) force divided by the mass – Newton's second law of motion. Equation (III.7.6) then captures the kinetic energy of the bulk plasma as it is accelerated by the oscillating electric field:

$$\sum_\alpha \int \mathrm{d}\boldsymbol{v}\, \frac{T_\alpha |\delta f_\alpha^{(\mathrm{n.r.})}(t,\boldsymbol{k},\boldsymbol{v})|^2}{2 f_{0\alpha}} \approx \sum_\alpha \frac{1}{2} m_\alpha n_\alpha |\delta\boldsymbol{u}_\alpha(t,\boldsymbol{k})|^2. \tag{III.7.9}$$

Now for the resonant contribution to (III.7.3). The difficulty is that last squared term in the long-time limit: for the Landau-resonant particles satisfying $x = \boldsymbol{k}\cdot\boldsymbol{v} - \omega_{\boldsymbol{k}} \approx 0$, the factor

$$\begin{aligned}\frac{1-\mathrm{e}^{-\mathrm{i}xt}}{x} &= \underbrace{\frac{1-\cos xt}{x}}_{\substack{\text{finite as}\\ t\to\infty,\text{ even}\\ \text{at } x=0}} + \mathrm{i}\underbrace{\frac{\sin xt}{x}}_{\substack{=\text{it as}\\ t\to\infty\\ \text{at } x=0}} \approx \mathrm{i}\frac{\sin xt}{x} = \frac{\mathrm{e}^{\mathrm{i}xt}-\mathrm{e}^{-\mathrm{i}xt}}{2x} = \frac{\mathrm{i}}{2}\int_{-t}^{t} \mathrm{d}t'\, \mathrm{e}^{\mathrm{i}xt'} \\ &\to \mathrm{i}\pi\delta(x) \text{ as } t\to\infty\end{aligned} \tag{III.7.10}$$

by definition of the delta function. So we must square a delta function... dangerous! Before doing so, note that (III.7.10) implies

$$\delta f_\alpha^{(\mathrm{res})}(t,\boldsymbol{k},\boldsymbol{v}) \propto \mathrm{e}^{-\mathrm{i}\omega_{\boldsymbol{k}}t}\left[\frac{1-\mathrm{e}^{-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{v}-\omega_{\boldsymbol{k}})t}}{\boldsymbol{k}\cdot\boldsymbol{v}-\omega_{\boldsymbol{k}}}\right] \to \mathrm{e}^{-\mathrm{i}\omega_{\boldsymbol{k}}t}\,\mathrm{i}\pi\delta(\boldsymbol{k}\cdot\boldsymbol{v}-\omega_{\boldsymbol{k}}) \text{ as } t\to\infty \tag{III.7.11}$$

for the resonant particles. This delta-function solution is an example of a *Case–van Kampen mode* (Van Kampen 1955; Case 1959). Now, back to squaring that delta function.

Here is one way to do that:

$$\frac{\partial}{\partial t}\left|\frac{1-\mathrm{e}^{-\mathrm{i}xt}}{x}\right|^2 = \frac{\partial}{\partial t}\left[\frac{2-2\cos(xt)}{x^2}\right] = 2\frac{\sin xt}{x} \xrightarrow[t\to\infty]{} 2\pi\delta(x)$$
$$\Longrightarrow \left|\frac{1-\mathrm{e}^{-\mathrm{i}xt}}{x}\right|^2 \xrightarrow[t\to\infty]{} 2\pi t\delta(x). \tag{III.7.12}$$

Apologies for this interruption and what will be a long aside, but one must be *incredibly* careful here about what is meant by $|\delta(x)|^2$. Because such a thing will always appear inside of an integral, and that integral has limits that we are taking to infinity in order to make sense of that delta function, the context matters. Here's an example, based on equations (E.31)–(E.35) of Krommes' opus. Consider the function $f(t) = \cos t$, and compute its square over a time interval $\mathcal{T}$, which we let tend to infinity:

$$\lim_{\mathcal{T}\to\infty}\int_{-\mathcal{T}/2}^{\mathcal{T}/2}\mathrm{d}t\,\cos^2 t = \lim_{\mathcal{T}\to\infty}\frac{\mathcal{T}}{2\pi}\int_0^{2\pi}\mathrm{d}t\,\cos^2 t = \lim_{\mathcal{T}\to\infty}\frac{\mathcal{T}}{2}. \tag{III.7.13}$$

So, we may write

$$\int_{-\infty}^{\infty}\mathrm{d}t\,|f(t)|^2 = \frac{\mathcal{T}}{2}. \tag{III.7.14}$$

Alternatively, in Fourier space we have

$$f(\omega) = \int_{-\infty}^{\infty}\mathrm{d}t\,\mathrm{e}^{\mathrm{i}\omega t}\cos t = \frac{1}{2}\,2\pi\bigl[\delta(\omega+1)+\delta(\omega-1)\bigr].$$

Now square it:

$$|f(\omega)|^2 = \pi^2\bigl[\delta^2(\omega+1)+\delta^2(\omega-1)\bigr].$$

What are we to make of these squared delta functions? Using the formula $\delta(x-a)f(x) = \delta(x-a)f(a)$, we have $\delta^2(\omega-1) = \delta(\omega-1)\delta(\omega=0)$ and $\delta^2(\omega+1) = \delta(\omega+1)\delta(\omega=0)$, and so

$$\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\,|f(\omega)|^2 = \pi\delta(\omega=0).$$

By Parseval's theorem, this must be equal to (III.7.14), implying that

$$\boxed{\delta(\omega=0) = \frac{\mathcal{T}}{2\pi} \quad\Longrightarrow\quad |\delta(\omega)|^2 = \frac{\mathcal{T}}{2\pi}\delta(\omega)} \tag{III.7.15}$$

Now, return to (III.7.13) and let the integral run instead from $-\mathcal{T}$ to $\mathcal{T}$; surely the result is twice our previous answer. In this case, $|\delta(\omega)|^2 = (\mathcal{T}/\pi)\delta(\omega)$. (See why the context matters?) Apparently, sending $\mathcal{T}\to\infty$ is different than sending $\mathcal{T}/2\to\infty$, and we chose the former when deriving (III.7.12). Indeed, setting $x=0$ in (III.7.10) gives $\delta(0) = t/\pi$. When dealing with delta functions, be extremely attentive to the context. Back to our regularly scheduled program…

Using (III.7.15) to compute the resonant contribution in (III.7.3), we find

$$\begin{aligned}
\sum_\alpha\int\mathrm{d}\boldsymbol{v}\,\frac{T_\alpha|\delta f_\alpha^{(\mathrm{res})}(t,\boldsymbol{k},\boldsymbol{v})|^2}{2f_{0\alpha}} &\approx \sum_\alpha\int\mathrm{d}\boldsymbol{v}\,\frac{q_\alpha^2}{T_\alpha}(\hat{\boldsymbol{k}}\cdot\boldsymbol{v})^2 f_{0\alpha}\frac{k^2|\varphi_{\boldsymbol{k}}|^2}{2}\,2\pi t\delta(\boldsymbol{k}\cdot\boldsymbol{v}-\omega_{\boldsymbol{k}})\\
&= \sum_\alpha\int\mathrm{d}\boldsymbol{v}\,\frac{q_\alpha^2}{T_\alpha}(\hat{\boldsymbol{k}}\cdot\boldsymbol{v})^2 f_{0\alpha}\frac{k^2|\varphi_{\boldsymbol{k}}|^2}{2}\frac{2\pi t}{k}\delta(\hat{\boldsymbol{k}}\cdot\boldsymbol{v}-\omega_{\boldsymbol{k}}/k)\\
&= \sum_\alpha\frac{2\omega_{\boldsymbol{k}}^2\omega_{\mathrm{p}\alpha}^2}{k^3v_{\mathrm{th}\alpha}^2}\frac{\pi}{n_\alpha}F_{0\alpha}\left(\frac{\omega_{\boldsymbol{k}}}{k}\right)\frac{|\boldsymbol{E}_{\boldsymbol{k}}|^2}{8\pi}\,2t,
\end{aligned} \tag{III.7.16}$$

where in the last line we have introduced the one-dimensional distribution function,

$$F_{0\alpha}(u) \doteq \int \mathrm{d}\boldsymbol{v}\, f_{0\alpha}(\boldsymbol{v})\delta(\hat{\boldsymbol{k}}\boldsymbol{\cdot}\boldsymbol{v}-u) \qquad \text{(III.7.17)}$$
$$= \frac{n_\alpha}{\sqrt{\pi}v_{\mathrm{th}\alpha}}\mathrm{e}^{-u^2/v^2_{\mathrm{th}\alpha}} \quad \text{for a Maxwellian,}$$

obtained by integrating out the velocity space perpendicular to $\boldsymbol{k}$. The time derivative of (III.7.16) is

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_\alpha \int \mathrm{d}\boldsymbol{v}\, \frac{T_\alpha |\delta f^{(\mathrm{res})}_\alpha(t,\boldsymbol{k},\boldsymbol{v})|^2}{2f_{0\alpha}} \approx \sum_\alpha \frac{4\omega^2_{\boldsymbol{k}}\omega^2_{\mathrm{p}\alpha}}{k^3 v^2_{\mathrm{th}\alpha}}\frac{\pi}{n_\alpha}F_{0\alpha}\Big(\frac{\omega_{\boldsymbol{k}}}{k}\Big)\frac{|\boldsymbol{E}_{\boldsymbol{k}}|^2}{8\pi}$$
$$= -\sum_\alpha \frac{2\omega_{\boldsymbol{k}}\omega^2_{\mathrm{p}\alpha}}{k^2}\frac{\pi}{n_\alpha}F'_{0\alpha}\Big(\frac{\omega_{\boldsymbol{k}}}{k}\Big)\frac{|\boldsymbol{E}_{\boldsymbol{k}}|^2}{8\pi}. \qquad \text{(III.7.18)}$$

This is the entropic contribution to the free energy due to the Landau-resonant piece of the perturbed distribution function; note that it is positive, corresponding to an *decreasing* entropy. The combination of pre-factors to the electric-field energy in (III.7.18) may be simplified as follows. When $|\gamma| \ll \omega$ (i.e., slow damping), we can expand our eigenfrequency equation $\mathcal{D}(\omega,\boldsymbol{k})=0$ to obtain

$$\mathcal{D}(\omega,\boldsymbol{k}) + \mathrm{i}\gamma\frac{\partial}{\partial\omega}\mathcal{D}(\omega,\boldsymbol{k}) \approx 0. \qquad \text{(III.7.19)}$$

The imaginary part of this gives

$$\gamma = -\frac{\operatorname{Im}\mathcal{D}(\omega,\boldsymbol{k})}{\dfrac{\partial}{\partial\omega}\operatorname{Re}\mathcal{D}(\omega,\boldsymbol{k})}. \qquad \text{(III.7.20)}$$

Using

$$\mathcal{D}(\omega,\boldsymbol{k}) = 1 + \sum_\alpha \frac{q^2_\alpha}{m_\alpha}\frac{4\pi\boldsymbol{k}}{k^2}\boldsymbol{\cdot}\int \mathrm{d}\boldsymbol{v}'\,\frac{\partial f_{0\alpha}/\partial\boldsymbol{v}'}{\omega-\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{v}'} \quad \text{(by def'n, (III.5.6))}$$
$$= 1 - \sum_\alpha \frac{\omega^2_{\mathrm{p}\alpha}}{k^2}\frac{1}{n_\alpha}\int \mathrm{d}u \frac{F'_{0\alpha}(u)}{u-\omega/k} \quad \text{(by def'n, (III.7.17))}$$
$$= \left[1 - \sum_\alpha \frac{\omega^2_{\mathrm{p}\alpha}}{k^2}\frac{1}{n_\alpha}\,\mathrm{PV}\int \mathrm{d}u\,\frac{F'_{0\alpha}(u)}{u-\omega/k}\right] + \mathrm{i}\left[-\sum_\alpha \frac{\omega^2_{\mathrm{p}\alpha}}{k^2}\frac{\pi}{n_\alpha}F'_{0\alpha}\Big(\frac{\omega}{k}\Big)\right], \qquad \text{(III.7.21)}$$

where the last equality follows from Plemelj's formula ("PV" denotes the principal value), (III.7.20) becomes

$$\gamma = -\frac{\displaystyle\sum_\alpha \frac{\omega^2_{\mathrm{p}\alpha}}{k^2}\frac{\pi}{n_\alpha}F'_{0\alpha}\Big(\frac{\omega}{k}\Big)}{\displaystyle\frac{\partial}{\partial\omega}\sum_\alpha\frac{\omega^2_{\mathrm{p}\alpha}}{k^2}\frac{1}{n_\alpha}\,\mathrm{PV}\int\mathrm{d}u\,\frac{F'_{0\alpha}(u)}{u-\omega/k}}. \qquad \text{(III.7.22)}$$

An analytic expression for the principal value integral can be obtained by expanding[8]

$$\begin{aligned}\int \mathrm{d}u\, \frac{F_{0\alpha}'(u)}{u-\omega/k} &\approx \int \mathrm{d}u\, \frac{F_{0\alpha}'(u)}{(-\omega/k)}\left[1+\frac{ku}{\omega}+\left(\frac{ku}{\omega}\right)^2+\dots\right]\\ &= -\frac{k}{\omega}\int \mathrm{d}u\, F_{0\alpha}'(u)\left[\frac{ku}{\omega}+\left(\frac{ku}{\omega}\right)^2+\dots\right]\\ &\approx \frac{k^2}{\omega^2}n_\alpha. \end{aligned} \tag{III.7.23}$$

Then (III.7.22) becomes

$$\gamma = -\frac{\displaystyle\sum_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{k^2}\frac{\pi}{n_\alpha}F_{0\alpha}'\left(\frac{\omega}{k}\right)}{\displaystyle\frac{\partial}{\partial\omega}\sum_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{k^2}\frac{k^2}{\omega^2}} = -\frac{\displaystyle\sum_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{k^2}\frac{\pi}{n_\alpha}F_{0\alpha}'\left(\frac{\omega}{k}\right)}{\displaystyle -\frac{2}{\omega}\underbrace{\sum_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{\omega^2}}_{\substack{\approx 1 \text{ in}\\ \text{this limit}}}}$$

$$\implies \quad 2\gamma_{\boldsymbol{k}} = \sum_\alpha \frac{\omega_{\boldsymbol{k}}\omega_{\mathrm{p}\alpha}^2}{k^2}\frac{\pi}{n_\alpha}F_{0\alpha}'\left(\frac{\omega_{\boldsymbol{k}}}{k}\right). \tag{III.7.24}$$

Note that, for $F_{0\alpha}$ Maxwellian, this equation implies $\gamma_{\boldsymbol{k}}/\omega_{\boldsymbol{k}} \sim \epsilon^{-3}\exp(-1/\epsilon^2)$, so that the damping rate is exponentially small. Thus, equation (III.7.18) may be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_\alpha \int \mathrm{d}\boldsymbol{v}\, \frac{T_\alpha|\delta f_\alpha^{(\mathrm{res})}(t,\boldsymbol{k},\boldsymbol{v})|^2}{2f_{0\alpha}} \approx -4\gamma_{\boldsymbol{k}}\frac{|\boldsymbol{E}_{\boldsymbol{k}}|^2}{8\pi} = -2\frac{\mathrm{d}}{\mathrm{d}t}\frac{|\boldsymbol{E}_{\boldsymbol{k}}|^2}{8\pi}. \tag{III.7.25}$$

Again, a decreasing entropy is associated with the Landau-resonant particles.

Finally, we combine (III.7.25) with its non-resonant counterpart (III.7.6) to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_\alpha \int \mathrm{d}\boldsymbol{v}\, \frac{T_\alpha|\delta f_\alpha(t,\boldsymbol{k},\boldsymbol{v})|^2}{2f_{0\alpha}} \approx -\frac{\mathrm{d}}{\mathrm{d}t}\frac{|\boldsymbol{E}_{\boldsymbol{k}}|^2}{8\pi}, \tag{III.7.26}$$

which is (by Parseval's theorem) equivalent to the free-energy conservation law, (III.6.14). *Q.E.D.*

Summary: Landau damping is the process of transferring free energy from fluctuations in the electric field to fluctuations in the distribution function. The latter acquire fine-scale structure in velocity space, which gets progressively finer with time: if $\delta f \sim \exp(-\mathrm{i}kvt)$, then $(\mathrm{i}/\delta f)(\partial\delta f/\partial v) \sim kt \to \infty$ as $t \to \infty$. This is called phase mixing, and is a consequence of the shearing of phase space as particles at the same position have differing velocities. Phase mixing explains why δf need not decay for its moments (e.g., the potential φ) to decay:

$$\varphi = \frac{4\pi}{k^2}\sum_\alpha q_\alpha \underbrace{\int \mathrm{d}\boldsymbol{v}\, \delta f_\alpha}_{\substack{\text{fine-scale}\\ \text{structure}\\ \text{cancels}}} \propto \mathrm{e}^{-|\gamma|t} \to 0.$$

[8]This is a good expansion, since it guarantees that the damping is small (because $ku/\omega \ll 1$ implies that not many particles in the distribution are Landau resonant). If $\omega \sim kv_{\mathrm{th}}$, then the damping would be strong since the waves would interact with a majority of the population.

In other words, a perturbation initially "visible" as φ phase-mixes away. This lost (free) energy goes into δf^2, where it gives a negative entropy that exactly offsets the heating of the background. The total entropy only increases once the fast velocity dependence of δf is removed – and *that* is accomplished through collisions.

III.8. Landau damping via Newton's 2nd law

I used to assign the following calculation (taken from Lifshitz & Pitaevskii 1981 and brought to my attention by Alex Schekochihin) as part of a problem set, but it's sufficiently pedagogical that it's worth including it here in my notes.

Imagine an electron moving along the z axis with speed v_0. Slowly turn on a wave-like electric field: $\boldsymbol{E}(t,z) = E_0 \cos(\omega t - kz)\,\mathrm{e}^{\epsilon t}\hat{\boldsymbol{z}}$, where ω is the frequency as k is the wavenumber of the wave. The adverb "slowly" is captured by the $\mathrm{e}^{\epsilon t}$ factor with $\epsilon \ll 1$. We'll take $\epsilon \to +0$ at the end of the calculation; its only purpose is to establish an arrow of time. The goal is to solve perturbatively for the motion of the electron by assuming that E_0 is so small that it changes the electron's trajectory only a little bit over several wave periods. The solution illustrates the physical mechanism of Landau damping.

The equations of motion are

$$\frac{\mathrm{d}z}{\mathrm{d}t} = v_z, \tag{III.8.1}$$

$$\frac{\mathrm{d}v_z}{\mathrm{d}t} = -\frac{e}{m_e} E_0 \cos(\omega t - kz)\mathrm{e}^{\epsilon t}. \tag{III.8.2}$$

The solution to lowest order in E_0 is trivial: $z(t) = v_0 t$ and $v_z(t) = v_0 = \text{const}$. Write $z(t) = v_0 t + \delta z(t)$ and $v_z(t) = v_0 + \delta v_z(t)$ and calculate the first-order changes δz and δv_z. Equation (III.8.1) becomes

$$\frac{\mathrm{d}\delta v_z}{\mathrm{d}t} = -\frac{e}{m_e}E(t,z(t)) \approx -\frac{e}{m_e}E(t,v_0 t) = -\frac{eE_0}{m_e}\,\mathrm{Re}\,\mathrm{e}^{[\mathrm{i}(\omega - kv_0)+\epsilon]t}. \tag{III.8.3}$$

Integrating this gives

$$\begin{aligned}
\delta v_z(t) &= -\frac{eE_0}{m_e}\int_0^t \mathrm{d}t'\,\mathrm{Re}\,\mathrm{e}^{[\mathrm{i}(\omega-kv_0)+\epsilon]t'} \\
&= -\frac{eE_0}{m_e}\,\mathrm{Re}\,\frac{\mathrm{e}^{[\mathrm{i}(\omega-kv_0)+\epsilon]t} - 1}{\mathrm{i}(\omega - kv_0) + \epsilon} \\
&= -\frac{eE_0}{m_e}\frac{\epsilon\,\mathrm{e}^{\epsilon t}\cos[(\omega-kv_0)t] - \epsilon + (\omega - kv_0)\mathrm{e}^{\epsilon t}\sin[(\omega-kv_0)t]}{(\omega-kv_0)^2+\epsilon^2}.
\end{aligned} \tag{III.8.4}$$

Integrating again,

$$\begin{aligned}
\delta z(t) &= \int_0^t \mathrm{d}t'\,\delta v_z(t') = -\frac{eE_0}{m_e}\int_0^t \mathrm{d}t'\,\mathrm{Re}\,\frac{\mathrm{e}^{[\mathrm{i}(\omega-kv_0)+\epsilon]t} - 1}{\mathrm{i}(\omega-kv_0)+\epsilon} \\
&= -\frac{eE_0}{m_e}\left\{\mathrm{Re}\,\frac{\mathrm{e}^{[\mathrm{i}(\omega-kv_0)+\epsilon]t} - 1}{[\mathrm{i}(\omega-kv_0)+\epsilon]^2} - \frac{\epsilon t}{(\omega-kv_0)^2+\epsilon^2}\right\} \\
&= -\frac{eE_0}{m_e}\left\{\frac{[\epsilon^2 - (\omega-kv_0)^2][\mathrm{e}^{\epsilon t}\cos[(\omega-kv_0)t] - 1] + 2\epsilon\,(\omega-kv_0)\,\mathrm{e}^{\epsilon t}\sin[(\omega-kv_0)t]}{[(\omega-kv_0)^2+\epsilon^2]^2}\right. \\
&\qquad \left. - \frac{\epsilon t}{(\omega-kv_0)^2+\epsilon^2}\right\}.
\end{aligned} \tag{III.8.5}$$

The first-order correction to the electric field evaluated at the particle position is

$$\delta E(t,z(t)) = E(t,z(t)) - E(t,v_0 t) = \delta z(t)\frac{\partial E(t,v_0 t)}{\partial z} = \delta z(t)\, k\sin[(\omega - kv_0)t]E_0 \mathrm{e}^{\epsilon t}, \tag{III.8.6}$$

with $\delta z(t)$ given by (III.8.5). The work done by the field on the electron per unit time is the power gained by the electron (and thus lost by the wave). Denoting an average over timescales satisfying $\omega^{-1} \ll t \ll \epsilon^{-1}$ by $\langle\cdot\rangle$, this power is

$$\begin{aligned}
P(v_0) &= -e\langle E(t,z(t))v_z(t)\rangle = -e\langle [E(t,v_0 t) + \delta E(t,z)][v_0 + \delta v_z(t)]\rangle \\
&= -e\big\langle \underbrace{E(t,v_0 t)v_0}_{\substack{\text{vanishes}\\\text{under}\\\text{averaging}}} + \underbrace{E(t,v_0 t)\delta v_z(t)}_{\substack{\text{only }\cos^2\\\text{term survives}\\\text{averaging}}} + \underbrace{\delta E(t,z(t))v_0}_{\substack{\text{only }\sin^2\\\text{term survives}\\\text{averaging}}}\big\rangle + \mathcal{O}(\delta^2) \\
&\approx \frac{e^2 E_0^2}{m_e}\,\mathrm{e}^{2\epsilon t}\left\langle \frac{\epsilon}{(\omega - kv_0)^2 + \epsilon^2}\cos^2[(\omega - kv_0)t] + \frac{2kv_0\epsilon(\omega - kv_0)}{[(\omega - kv_0)^2+\epsilon^2]^2}\sin^2[(\omega - kv_0)t]\right\rangle \\
&= \frac{e^2 E_0^2}{2m_e}\mathrm{e}^{2\epsilon t}\left[\frac{\epsilon}{(\omega - kv_0)^2 + \epsilon^2} + \frac{2kv_0\epsilon(\omega - kv_0)}{[(\omega - kv_0)^2+\epsilon^2]^2}\right]
\end{aligned}$$

$$\Longrightarrow \quad \boxed{P(v_0) = \frac{e^2 E_0^2}{2m_e}\,\mathrm{e}^{2\epsilon t}\frac{\mathrm{d}\chi}{\mathrm{d}v_0} \quad \text{with} \quad \chi \doteq \frac{\epsilon v_0}{(\omega - kv_0)^2 + \epsilon^2}} \tag{III.8.7}$$

If $v_0 \lesssim \omega/k$ (particle lagging the wave), then $\mathrm{d}\chi/\mathrm{d}v_0 > 0$ and so $P(v_0) > 0$, indicating that energy is being transferred from the field to the electron. The wave damps. If $v_0 \gtrsim \omega/k$ (particle leading the wave), then $\mathrm{d}\chi/\mathrm{d}v_0 < 0$ and so $P(v_0) < 0$, indicating that energy is being transferred from the electron to the field. The wave grows.

Suppose there is now a distribution of these electrons, $F(v_0)$. The total power per unit volume going into (or out of) this distribution is

$$P = \int \mathrm{d}v_z\, F(v_z)P(v_z) = \frac{e^2E_0^2}{2m_e}\mathrm{e}^{2\epsilon t}\int \mathrm{d}v_z\, F(v_z)\frac{\mathrm{d}\chi}{\mathrm{d}v_z} \overset{\text{bp}}{=} -\frac{e^2E_0^2}{2m_e}\mathrm{e}^{2\epsilon t}\int \mathrm{d}v_z\, F'(v_z)\chi(v_z). \tag{III.8.8}$$

Take $\epsilon \to +0$ and use Plemelj's formula,

$$\lim_{\epsilon\to+0}\frac{1}{x - \zeta \mp \mathrm{i}\epsilon} = \mathrm{PV}\frac{1}{x-\zeta} \pm \mathrm{i}\pi\delta(x-\zeta),$$

where PV denotes the principal value and $\delta(x)$ is the Dirac delta function, to show that

$$\chi(v_z) = \frac{\epsilon v_z}{(\omega - kv_z)^2 + \epsilon^2} = -\frac{\mathrm{i}}{2}\left(\frac{v_z}{kv_z - \omega - \mathrm{i}\epsilon} - \frac{v_z}{kv_z - \omega + \mathrm{i}\epsilon}\right) \to \pi\frac{\omega}{k^2}\delta(v_z - \omega/k). \tag{III.8.9}$$

Using this limit in (III.8.8) leads to

$$\boxed{P = -\frac{e^2E_0^2}{2m_e k^2}\pi\omega F'(\omega/k)} \tag{III.8.10}$$

If $\omega F'(\omega/k) < 0$ (> 0), there are more resonant particles lagging (leading) the wave than there are leading (lagging) the wave, resulting in a net transfer of energy to the electrons (wave). It's left as an exercise to the reader to show that, for $F'(\omega/k) < 0$, this power comes at the expense of the electric energy (i.e., damping), and that, for $F(\omega/k) > 0$, the energy loss from the electrons goes into growing the electric energy (i.e., instability).

III.9. Plasma echo

While we're on the topic of the Landau damping, here's a fun calculation that demonstrates that nothing is lost in Vlasov dynamics: the *plasma echo*. The classic calculation of the echo is due to Gould *et al.* (1967) and involves all the Landavian gymnastics in the complex plane caused by the dielectric function... but worse because the echo calculation is actually a nonlinear phenomenon, so there are multiple Bromwich contours to close. It turns out, though, that nothing of great pedagogical value is gained by carrying around $\mathcal{D}(\omega, \boldsymbol{k})$, and so (at first) we'll effectively set it equal to unity by ignoring the self-consistent collective response of the plasma. This decision means that we can avoid an unnecessary detour through Laplace space, which is nice, but at the cost of losing the plasma's eigenmodes. No matter, though, because nothing qualitative about the plasma echo depends upon the dielectric response – it's actually just phase mixing (and un-mixing).

The calculation goes as follows. Write $f_\alpha(t, \boldsymbol{v}, \boldsymbol{r}) = f_{0\alpha}(\boldsymbol{v}) + \delta f_\alpha(t, \boldsymbol{v}, \boldsymbol{r})$ and assume an electrostatic plasma with potential $\varphi = \varphi(t, \boldsymbol{r})$. Substituting these fields into the Vlasov equation yields

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla}\right) \delta f_\alpha - \frac{q_\alpha}{m_\alpha} \boldsymbol{\nabla} \varphi \cdot \frac{\mathrm{d} f_{0\alpha}}{\mathrm{d} \boldsymbol{v}} = \frac{q_\alpha}{m_\alpha} \boldsymbol{\nabla} \varphi \cdot \frac{\partial \delta f_\alpha}{\partial \boldsymbol{v}}. \tag{III.9.1}$$

Write $\delta f_\alpha = \sum_{\boldsymbol{k}} \delta f_{\alpha,\boldsymbol{k}}(t, \boldsymbol{v}) \exp(\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{r})$ and $\varphi = \sum_{\boldsymbol{k}} \varphi_{\boldsymbol{k}}(t) \exp(\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{r})$ and Fourier transform the above equation to find

$$\left(\frac{\partial}{\partial t} + \mathrm{i} \boldsymbol{k} \cdot \boldsymbol{v}\right) \delta f_{\alpha,\boldsymbol{k}} - \frac{q_\alpha}{m_\alpha} \varphi_{\boldsymbol{k}} \mathrm{i} \boldsymbol{k} \cdot \frac{\mathrm{d} f_{0\alpha}}{\mathrm{d} \boldsymbol{v}} = \frac{q_\alpha}{m_\alpha} \sum_{\boldsymbol{k}'} \varphi_{\boldsymbol{k}'} \mathrm{i} \boldsymbol{k}' \cdot \frac{\partial \delta f_{\alpha,\boldsymbol{k}-\boldsymbol{k}'}}{\partial \boldsymbol{v}}. \tag{III.9.2}$$

Now, rather than take the potential φ to satisfy Poisson's equation, we'll choose φ to represents two impulsive "hammers" – one occurring at time t_1 and the other a later time $t_2 > t_1$ – that excite sinusoidal (in space) fluctuations in the plasma that are subsequently phase mixed by the free streaming of particles. Denoting this collection of hammers by $\chi(t, \boldsymbol{r})$, we then have that

$$\varphi(t, \boldsymbol{r}) = \chi(t, \boldsymbol{r}) \doteq A_1 \delta(t - t_1) \cos(\boldsymbol{k}_1 \cdot \boldsymbol{r}) + A_2 \delta(t - t_2) \cos(\boldsymbol{k}_2 \cdot \boldsymbol{r}); \tag{III.9.3}$$

or in Fourier space,

$$\varphi_{\boldsymbol{k}}(t) = \chi_{\boldsymbol{k}}(t) \doteq \frac{1}{2} A_1 \delta(t - t_1)(\delta_{\boldsymbol{k},\boldsymbol{k}_1} + \delta_{\boldsymbol{k},-\boldsymbol{k}_1}) + \frac{1}{2} A_2 \delta(t - t_2)(\delta_{\boldsymbol{k},\boldsymbol{k}_2} + \delta_{\boldsymbol{k},-\boldsymbol{k}_2}). \tag{III.9.4}$$

In this case, the linear response is given by

$$\begin{aligned} \delta f^{(\mathrm{lin})}_{\alpha,\boldsymbol{k}}(t, \boldsymbol{v}) &= \cancelto{0}{\delta f_{\alpha,\boldsymbol{k}}(0, \boldsymbol{v})\, \mathrm{e}^{-\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{v} t}} + \frac{q_\alpha}{m_\alpha} \mathrm{i} \boldsymbol{k} \cdot \frac{\mathrm{d} f_{0\alpha}}{\mathrm{d} \boldsymbol{v}} \int_0^t \mathrm{d}t'\, \mathrm{e}^{-\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{v}(t - t')} \chi_{\boldsymbol{k}}(t') \\ &= \frac{q_\alpha}{m_\alpha} \mathrm{i} \boldsymbol{k} \cdot \frac{\mathrm{d} f_{0\alpha}}{\mathrm{d} \boldsymbol{v}} \frac{1}{2} \Big[A_1 \Theta(t - t_1)(\delta_{\boldsymbol{k},\boldsymbol{k}_1} + \delta_{\boldsymbol{k},-\boldsymbol{k}_1})\, \mathrm{e}^{-\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{v}(t - t_1)} \\ &\qquad + A_2 \Theta(t - t_2)(\delta_{\boldsymbol{k},\boldsymbol{k}_2} + \delta_{\boldsymbol{k},-\boldsymbol{k}_2})\, \mathrm{e}^{-\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{v}(t - t_2)} \Big], \end{aligned} \tag{III.9.5}$$

where we have assumed the plasma to be unperturbed at $t = 0$; the Heaviside function $\Theta(\tau) = 1$ for $\tau > 1$ and $= 0$ otherwise. Physically, at times t_1 and t_2, particles are instantaneously accelerated from $f_{0\alpha}$ into δf_α by two impulses; thereafter, each of these kinetic responses is phase mixed. Transforming (III.9.5) back into real space provides

$$\begin{aligned}\delta f_\alpha^{(\mathrm{lin})}(t,\boldsymbol{r},\boldsymbol{v}) = &-\frac{q_\alpha}{m_\alpha}\boldsymbol{k}_1\cdot\frac{\mathrm{d}f_{0\alpha}}{\mathrm{d}\boldsymbol{v}}\,A_1\Theta(t-t_1)\sin\{\boldsymbol{k}_1\cdot[\boldsymbol{r}-\boldsymbol{v}(t-t_1)]\}\\ &-\frac{q_\alpha}{m_\alpha}\boldsymbol{k}_2\cdot\frac{\mathrm{d}f_{0\alpha}}{\mathrm{d}\boldsymbol{v}}\,A_2\Theta(t-t_2)\sin\{\boldsymbol{k}_2\cdot[\boldsymbol{r}-\boldsymbol{v}(t-t_2)]\}.\end{aligned} \tag{III.9.6}$$

Let us suppose for the moment that $f_{0\alpha} = f_{0\alpha}(v)$ is Maxwellian. In this case it is a straightforward exercise to calculate the associated density fluctuation by integrating (III.9.6) over velocity space:

$$\begin{aligned}\frac{\delta n_\alpha^{(\mathrm{lin})}(t,\boldsymbol{r})}{n_{0\alpha}} = &-\frac{q_\alpha}{m_\alpha}k_1^2\,A_1\Theta(t-t_1)(t-t_1)\,\mathrm{e}^{-[k_1 v_{\mathrm{th}}(t-t_1)/2]^2}\cos(\boldsymbol{k}_1\cdot\boldsymbol{r})\\ &-\frac{q_\alpha}{m_\alpha}k_2^2\,A_2\Theta(t-t_2)(t-t_2)\,\mathrm{e}^{-[k_2 v_{\mathrm{th}}(t-t_2)/2]^2}\cos(\boldsymbol{k}_2\cdot\boldsymbol{r}).\end{aligned} \tag{III.9.7}$$

As anticipated, soon after each harmonic is excited the associated density fluctuation grows linearly and then decays rapidly once $k_1 v_{\mathrm{th}}(t-t_1) \gtrsim \sqrt{2}$ (likewise for k_2). Phase mixing creates small-scale, rapidly varying structure in velocity space, which when integrated over leads to cancellations that cause the "damping" of real-space quantities.

The goal of the echo calculation is to compute the *nonlinear* response (i.e., the effect of the potential on the perturbed distribution, represented by the right-hand side of (III.9.2)) and show that there is a time at which the small-scale structure in velocity space caused by the linear phase mixing gets temporally removed by this nonlinear term, leading to the recovery of real-space structure. In general, this nonlinear response cannot be calculated analytically. But if we assume that δf_α is sufficiently small, we can solve (III.9.2) iteratively by evaluating (III.9.5) at $\boldsymbol{k}-\boldsymbol{k}'$, substituting the result into the right-hand side of (III.9.2), and integrating forward in time:

$$\begin{aligned}\delta f_{\alpha,\boldsymbol{k}}^{(\mathrm{n.l.})}(t,\boldsymbol{v}) &= \frac{q_\alpha}{m_\alpha}\sum_{\boldsymbol{k}'}\mathrm{i}\boldsymbol{k}'\cdot\int_0^t\mathrm{d}t'\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{v}(t-t')}\chi_{\boldsymbol{k}'}(t')\frac{\partial\delta f_{\alpha,\boldsymbol{k}-\boldsymbol{k}'}^{(\mathrm{lin})}(t',\boldsymbol{v})}{\partial\boldsymbol{v}}\\ &= -\frac{1}{4}\frac{q_\alpha^2}{m_\alpha^2}\sum_{\boldsymbol{k}'}\boldsymbol{k}'(\boldsymbol{k}-\boldsymbol{k}')\int_0^t\mathrm{d}t'\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{v}(t-t')}\\ &\quad\Big[A_1\delta(t'-t_1)(\delta_{\boldsymbol{k}',\boldsymbol{k}_1}+\delta_{\boldsymbol{k}',-\boldsymbol{k}_1})+A_2\delta(t'-t_2)(\delta_{\boldsymbol{k}',\boldsymbol{k}_2}+\delta_{\boldsymbol{k}',-\boldsymbol{k}_2})\Big]\\ &\quad:\frac{\partial}{\partial\boldsymbol{v}}\bigg\{\frac{\mathrm{d}f_{0\alpha}}{\mathrm{d}\boldsymbol{v}}\Big[A_1\Theta(t'-t_1)(\delta_{\boldsymbol{k}-\boldsymbol{k}',\boldsymbol{k}_1}+\delta_{\boldsymbol{k}-\boldsymbol{k}',-\boldsymbol{k}_1})\,\mathrm{e}^{-\mathrm{i}(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{v}(t'-t_1)}\\ &\qquad+A_2\Theta(t'-t_2)(\delta_{\boldsymbol{k}-\boldsymbol{k}',\boldsymbol{k}_2}+\delta_{\boldsymbol{k}-\boldsymbol{k}',-\boldsymbol{k}_2})\,\mathrm{e}^{-\mathrm{i}(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{v}(t'-t_2)}\Big]\bigg\}.\end{aligned} \tag{III.9.8}$$

This integral has four main pieces. Two of them are nonlinearities associated with hammer 1 (hammer 2) interacting with the plasma's response to hammer 1 (hammer 2); they can be spotted easily by matching up the A coefficients. Physically, these terms must vanish by causality: an impulsive hammer can't interact with something it created. Mathematically, $\int_0^t\mathrm{d}t'\,\delta(t'-t_1)\Theta(t'-t_1) = \Theta(0) = 0$. The remaining two terms in (III.9.8) are cross terms $\propto A_1A_2$ describing the interaction between one hammer and the kinetic response to the other hammer. One of them vanishes: because $t_2 > t_1$, the combination $\delta(t'-t_1)\Theta(t'-t_2)$ kills the integral. Physically, the first hammer cannot interact with the perturbation generated by the second hammer (again, causality). The only nonlinear term of interest is therefore

$$\delta f^{(\mathrm{echo})}_{\alpha,\boldsymbol{k}}(t,\boldsymbol{v}) = -\frac{1}{4}\frac{q_\alpha^2}{m_\alpha^2}\sum_{\boldsymbol{k}'}\boldsymbol{k}'(\boldsymbol{k}-\boldsymbol{k}')\int_0^t \mathrm{d}t'\, A_1 A_2 \delta(t'-t_2)\Theta(t'-t_1)\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{v}(t-t')}$$
$$\boldsymbol{:}\frac{\partial}{\partial\boldsymbol{v}}\left[\frac{\mathrm{d}f_{0\alpha}}{\mathrm{d}\boldsymbol{v}}(\delta_{\boldsymbol{k}',\boldsymbol{k}_2}+\delta_{\boldsymbol{k}',-\boldsymbol{k}_2})(\delta_{\boldsymbol{k}-\boldsymbol{k}',\boldsymbol{k}_1}+\delta_{\boldsymbol{k}-\boldsymbol{k}',-\boldsymbol{k}_1})\,\mathrm{e}^{-\mathrm{i}(\boldsymbol{k}-\boldsymbol{k}')\boldsymbol{\cdot}\boldsymbol{v}(t'-t_1)}\right]$$
$$= -\frac{1}{4}\frac{q_\alpha^2}{m_\alpha^2}\boldsymbol{k}_2\boldsymbol{k}_1 A_1 A_2\Theta(t-t_2)\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{v}(t-t_2)}$$
$$\boldsymbol{:}\frac{\partial}{\partial\boldsymbol{v}}\left\{\frac{\mathrm{d}f_{0\alpha}}{\mathrm{d}\boldsymbol{v}}\left[\delta_{\boldsymbol{k},\boldsymbol{k}_1+\boldsymbol{k}_2}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_1\boldsymbol{\cdot}\boldsymbol{v}(t_2-t_1)} - \delta_{\boldsymbol{k},\boldsymbol{k}_2-\boldsymbol{k}_1}\mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\boldsymbol{\cdot}\boldsymbol{v}(t_2-t_1)}\right.\right.$$
$$\left.\left. - \delta_{\boldsymbol{k},\boldsymbol{k}_1-\boldsymbol{k}_2}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_1\boldsymbol{\cdot}\boldsymbol{v}(t_2-t_1)} + \delta_{\boldsymbol{k},-\boldsymbol{k}_1-\boldsymbol{k}_2}\mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\boldsymbol{\cdot}\boldsymbol{v}(t_2-t_1)}\right]\right\}. \tag{III.9.9}$$

There are two types of terms here: those associated with the sum of the excited wavenumbers, and those associated with the difference of the excited wavenumbers. Let's take the former and inverse-Fourier transform them to find

$$-\frac{1}{2}\frac{q_\alpha^2}{m_\alpha^2}\boldsymbol{k}_2\boldsymbol{k}_1 A_1 A_2\Theta(t-t_2)\boldsymbol{:}\left[\frac{\mathrm{d}^2 f_{0\alpha}}{\mathrm{d}\boldsymbol{v}\mathrm{d}\boldsymbol{v}}\cos\psi(t) + \frac{\mathrm{d}f_{0\alpha}}{\mathrm{d}\boldsymbol{v}}\boldsymbol{k}_1(t_2-t_1)\sin\psi(t)\right], \tag{III.9.10}$$

where $\psi(t) \doteq (\boldsymbol{k}_1+\boldsymbol{k}_2)\boldsymbol{\cdot}[\boldsymbol{r}-\boldsymbol{v}(t-t_{12})]$ and $t_{12} \doteq (k_1t_1+k_2t_2)/(k_1+k_2)$. Note that the sinusoidal variation in velocity space associated with these terms vanishes at

$$t = t_{12} = t_2 - \frac{k_1}{k_1+k_2}(t_2-t_1) < t_2, \tag{III.9.11}$$

but because of the Heaviside function, $\Theta(t-t_2)$, this will never occur. This is *not* the case with the terms in (III.9.9) that feature the difference of the wavenumbers, $\boldsymbol{k}_2-\boldsymbol{k}_1$; inverse-Fourier transforming those provides

$$+\frac{1}{2}\frac{q_\alpha^2}{m_\alpha^2}\boldsymbol{k}_2\boldsymbol{k}_1 A_1 A_2\Theta(t-t_2)\boldsymbol{:}\left[\frac{\mathrm{d}^2 f_{0\alpha}}{\mathrm{d}\boldsymbol{v}\mathrm{d}\boldsymbol{v}}\cos\theta(t) - \frac{\mathrm{d}f_{0\alpha}}{\mathrm{d}\boldsymbol{v}}\boldsymbol{k}_1(t_2-t_1)\sin\theta(t)\right], \tag{III.9.12}$$

where $\theta(t) \doteq (\boldsymbol{k}_2-\boldsymbol{k}_1)\boldsymbol{\cdot}[\boldsymbol{r}-\boldsymbol{v}(t-t_{\mathrm{echo}})]$ and $t_{\mathrm{echo}} \doteq (k_2t_2-k_1t_1)/(k_2-k_1)$. Note that the sinusoidal variation in velocity space associated with these terms vanishes at

$$t = t_{\mathrm{echo}} = t_2 + \frac{k_1}{k_2-k_1}(t_2-t_1) > t_2; \tag{III.9.13}$$

thus, there will always be a time at which this variation vanishes. This is the echo. Assembling (III.9.10) and (III.9.12), we have

$$\delta f^{(\mathrm{echo})}_{\alpha}(t,\boldsymbol{r},\boldsymbol{v}) = \frac{1}{2}\frac{q_\alpha^2}{m_\alpha^2}\boldsymbol{k}_2\boldsymbol{k}_1 A_1 A_2\Theta(t-t_2)$$
$$\boldsymbol{:}\left\{\frac{\mathrm{d}^2 f_{0\alpha}}{\mathrm{d}\boldsymbol{v}\mathrm{d}\boldsymbol{v}}[\cos\theta(t)-\cos\psi(t)] - \frac{\mathrm{d}f_{0\alpha}}{\mathrm{d}\boldsymbol{v}}\boldsymbol{k}_1(t_2-t_1)[\sin\theta(t)+\sin\psi(t)]\right\}. \tag{III.9.14}$$

To offer something more concrete, let us suppose a Maxwellian $f_{0\alpha} = f_{0\alpha}(v)$. In this case it is a straightforward exercise to calculate the density fluctuation associated with the

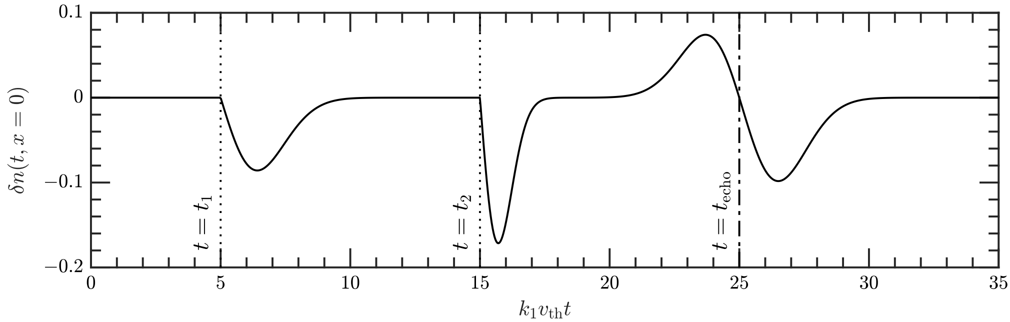


FIGURE 3. Total density perturbation from two impulsive, sinusoidal, electrostatic hammers with dielectric $\mathcal{D} = 1$, exhibiting a plasma echo at $t = t_{\rm echo}$. See equations (III.9.7) and (III.9.15).

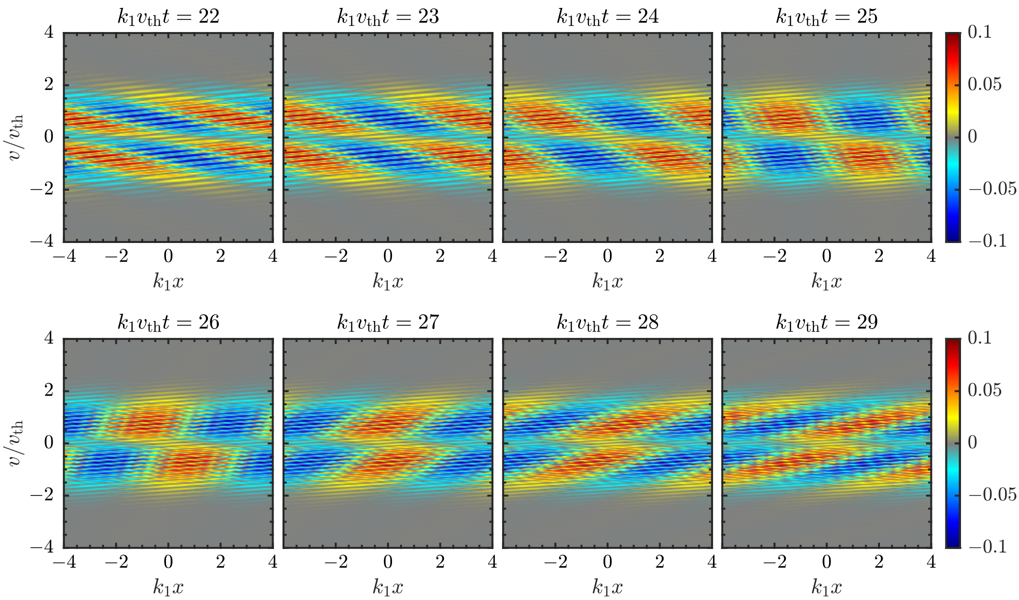


FIGURE 4. Echo distribution function $\delta f^{(\rm echo)}(t, x, v)$ caused by two impulsive, sinusoidal, electrostatic hammers with dielectric $\mathcal{D} = 1$, with $t_{\rm echo} = 25$. See equation (III.9.14).

echo by integrating (III.9.14) over velocity space. The result is that

$$\begin{aligned}\frac{\delta n_\alpha^{(\rm echo)}}{n_{0\alpha}} = &-\frac{1}{2}\frac{q_\alpha^2}{m_\alpha^2} k_1 k_2 A_1 A_2 (t - t_2) \varTheta(t - t_2) \\ &\times \Big\{ (k_2 - k_1)^2 (t - t_{\rm echo})\, {\rm e}^{-[(k_2 - k_1) v_{\rm th}(t - t_{\rm echo})/2]^2} \cos[(\boldsymbol{k}_2 - \boldsymbol{k}_1)\boldsymbol{\cdot}\boldsymbol{r}] \\ &- (k_2 + k_1)^2 (t - t_{12})\, {\rm e}^{-[(k_2 + k_1) v_{\rm th}(t - t_{12})/2]^2} \cos[(\boldsymbol{k}_2 + \boldsymbol{k}_1)\boldsymbol{\cdot}\boldsymbol{r}] \Big\}. \end{aligned} \tag{III.9.15}$$

The first term in the curly brackets contributes the echo, while the second term is the analog of the linear response (III.9.7) at the nonlinearly excited harmonic $\boldsymbol{k}_1 + \boldsymbol{k}_2$. Putting the full solution together, equations (III.9.7) and (III.9.15), and setting the parameters $(k_1, k_2, A_1, A_2, t_1, t_2) = (1, 2, 0.1, 0.1, 5, 15)$ yields the evolution shown in figures 3 and 4.

For completeness... if instead we were to have done this calculation including the self-consistent potential response, such that (III.9.3) becomes

$$\varphi(t,\boldsymbol{r}) = \chi(t,\boldsymbol{r}) + \frac{4\pi}{k^2}\sum_\alpha q_\alpha \int \mathrm{d}\boldsymbol{v}\,\delta f_\alpha, \tag{III.9.16}$$

then the calculation would necessarily venture through the Laplace space. The transformed potential associated with the linear response to the hammers would be dressed by the dielectric function (III.5.6),

$$\varphi_{\boldsymbol{k}}(\omega) = \frac{\chi_{\boldsymbol{k}}(\omega)}{\mathcal{D}(\omega,\boldsymbol{k})} \quad \text{with} \quad \chi_{\boldsymbol{k}}(\omega) = \frac{1}{2}A_1\mathrm{e}^{\mathrm{i}\omega t_1}(\delta_{\boldsymbol{k},\boldsymbol{k}_1}+\delta_{\boldsymbol{k},-\boldsymbol{k}_1}) + \frac{1}{2}A_2\mathrm{e}^{\mathrm{i}\omega t_2}(\delta_{\boldsymbol{k},\boldsymbol{k}_2}+\delta_{\boldsymbol{k},-\boldsymbol{k}_2}),$$

and the linear part of the perturbed distribution function would be as in §III.5:

$$\delta f^{(\mathrm{lin})}_{\alpha,\boldsymbol{k}}(\omega,\boldsymbol{v}) = \frac{q_\alpha}{m_\alpha}\frac{\mathrm{i}\boldsymbol{k}}{-\mathrm{i}\omega+\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{v}}\cdot\varphi_{\boldsymbol{k}}(\omega)\,\frac{\mathrm{d}f_{0\alpha}}{\mathrm{d}\boldsymbol{v}}. \tag{III.9.17}$$

Likewise, equation (III.9.8) for the nonlinear response would be

$$\begin{aligned}\delta f^{(\mathrm{n.l.})}_{\alpha,\boldsymbol{k}}(\omega,\boldsymbol{v}) &= \frac{q_\alpha}{m_\alpha}\sum_{\boldsymbol{k}'}\frac{\mathrm{i}\boldsymbol{k}'}{-\mathrm{i}\omega+\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{v}}\cdot\frac{\partial}{\partial\boldsymbol{v}}\int_0^\infty \mathrm{d}t\,\mathrm{e}^{\mathrm{i}\omega t}\int_{-\infty+\mathrm{i}\sigma}^{\infty+\mathrm{i}\sigma'}\frac{\mathrm{d}\omega'}{2\pi}\int_{-\infty+\mathrm{i}\sigma''}^{\infty+\mathrm{i}\sigma''}\frac{\mathrm{d}\omega''}{2\pi}\\ &\quad\times\varphi_{\boldsymbol{k}'}(\omega')\,\delta f^{(\mathrm{lin})}_{\alpha,\boldsymbol{k}-\boldsymbol{k}'}(\omega'',\boldsymbol{v})\,\mathrm{e}^{-\mathrm{i}(\omega'+\omega'')t}\\ &= -\frac{q_\alpha}{m_\alpha}\sum_{\boldsymbol{k}'}\frac{\mathrm{i}\boldsymbol{k}'}{-\mathrm{i}\omega+\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{v}}\cdot\frac{\partial}{\partial\boldsymbol{v}}\iint\frac{\mathrm{d}\omega'\mathrm{d}\omega''}{(2\pi)^2}\frac{\varphi_{\boldsymbol{k}'}(\omega')\,\delta f^{(\mathrm{lin})}_{\alpha,\boldsymbol{k}-\boldsymbol{k}'}(\omega'',\boldsymbol{v})}{\mathrm{i}(\omega-\omega'-\omega'')},\end{aligned} \tag{III.9.18}$$

with $\mathrm{Im}(\omega) > \sigma'+\sigma''$ to ensure convergence of the Laplace transform. Substituting (III.9.17) and $\varphi_{\boldsymbol{k}}(\omega) = \chi_{\boldsymbol{k}}(\omega)/\mathcal{D}(\omega,\boldsymbol{k})$ into (III.9.18), cleaning up various factors of $\pm\mathrm{i}$, and rearranging ultimately leads to

$$\begin{aligned}\delta f^{(\mathrm{n.l.})}_{\alpha,\boldsymbol{k}}(\omega,\boldsymbol{v}) &= \frac{q_\alpha^2}{m_\alpha^2}\sum_{\boldsymbol{k}'}\frac{\mathrm{i}\boldsymbol{k}'(\boldsymbol{k}-\boldsymbol{k}')}{\omega-\boldsymbol{k}\cdot\boldsymbol{v}}:\frac{\partial}{\partial\boldsymbol{v}}\iint\frac{\mathrm{d}\omega'\mathrm{d}\omega''}{(2\pi)^2}\frac{1}{\omega-\omega'-\omega''}\frac{1}{\omega''-(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{v}}\\ &\quad\times\frac{\chi_{\boldsymbol{k}'}(\omega')\chi_{\boldsymbol{k}-\boldsymbol{k}'}(\omega'')}{\mathcal{D}(\omega',\boldsymbol{k}')\mathcal{D}(\omega'',\boldsymbol{k}-\boldsymbol{k}')}\frac{\mathrm{d}f_{0\alpha}}{\mathrm{d}\boldsymbol{v}},\end{aligned} \tag{III.9.19}$$

We know from the calculation with $\mathcal{D} = 1$ that we're only interested in the terms proportional to the product of the two hammer amplitudes, A_1A_2. We also know that, of those terms, there are two kinds: those associated with the sum of the excited wavenumbers, and those associated with the difference of the excited wavenumbers. The former are:

$$\begin{aligned}&\frac{1}{4}\frac{q_\alpha^2}{m_\alpha^2}\sum_{\boldsymbol{k}'}\frac{\mathrm{i}\boldsymbol{k}'(\boldsymbol{k}-\boldsymbol{k}')}{\omega-\boldsymbol{k}\cdot\boldsymbol{v}}:\frac{\partial}{\partial\boldsymbol{v}}\iint\frac{\mathrm{d}\omega'\mathrm{d}\omega''}{(2\pi)^2}\frac{1}{\omega-\omega'-\omega''}\frac{1}{\omega''-(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{v}}\\ &\quad\times\frac{A_1A_2\,\mathrm{e}^{\mathrm{i}(\omega't_2+\omega''t_1)}}{\mathcal{D}(\omega',\boldsymbol{k}')\mathcal{D}(\omega'',\boldsymbol{k}-\boldsymbol{k}')}(\delta_{\boldsymbol{k}',\boldsymbol{k}_2}\delta_{\boldsymbol{k},\boldsymbol{k}_1+\boldsymbol{k}_2}+\delta_{\boldsymbol{k}',-\boldsymbol{k}_2}\delta_{\boldsymbol{k},-\boldsymbol{k}_1-\boldsymbol{k}_2})\frac{\mathrm{d}f_{0\alpha}}{\mathrm{d}\boldsymbol{v}}\\ &+\text{(the same but with } 1\leftrightarrow 2\text{)}\ .\end{aligned} \tag{III.9.20}$$

The procedure for doing the integrals is as follows (see the diagram below). First, do the integral over ω' by pushing the contour up to $\mathrm{Im}(\omega')\to\infty$ and encircling only the pole at $\omega' = \omega-\omega''$ (counter-clockwise, contributing a $+2\pi\mathrm{i}$). Note that both $\exp(\mathrm{i}\omega' t_1)$

and $\exp(\mathrm{i}\omega' t_2) \to 0$, so that the horizontal parts of the contour vanish. The expression in (III.9.20) becomes

$$\frac{\mathrm{i}}{4}\frac{q_\alpha^2}{m_\alpha^2}\sum_{\boldsymbol{k}'}\frac{\mathrm{i}\boldsymbol{k}'(\boldsymbol{k}-\boldsymbol{k}')}{\omega-\boldsymbol{k}\cdot\boldsymbol{v}}:\frac{\partial}{\partial\boldsymbol{v}}\int\frac{\mathrm{d}\omega''}{2\pi}\frac{1}{\omega''-(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{v}}$$
$$\times\frac{A_1A_2\,\mathrm{e}^{\mathrm{i}[\omega t_2-\omega''(t_2-t_1)]}}{\mathcal{D}(\omega-\omega'',\boldsymbol{k}')\mathcal{D}(\omega'',\boldsymbol{k}-\boldsymbol{k}')}(\delta_{\boldsymbol{k}',\boldsymbol{k}_2}\delta_{\boldsymbol{k},\boldsymbol{k}_1+\boldsymbol{k}_2}+\delta_{\boldsymbol{k}',-\boldsymbol{k}_2}\delta_{\boldsymbol{k},-\boldsymbol{k}_1-\boldsymbol{k}_2})\frac{\mathrm{d}f_{0\alpha}}{\mathrm{d}\boldsymbol{v}}$$
$$+\ (\text{the same but with } 1\leftrightarrow 2)\ . \tag{III.9.21}$$

Next, do the integral over ω''. For the term displayed above that is explicitly written out, the contour must be pushed downwards because $t_2 > t_1$, so that $\exp[-\mathrm{i}\omega''(t_2-t_1)] \to 0$. As it goes down, it encircles the kinetic pole at $(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{v}$ (clockwise, contributing a $-2\pi\mathrm{i}$) as well as all the poles from the dielectric functions in the lower half-plane. For simplicity, we ignore all the dielectric poles, which correspond to damped eigenmodes and so will not contribute to the echo. For "(the same but with $1 \leftrightarrow 2$)", the contour must be pushed *upwards* so that $\exp[-\mathrm{i}\omega''(t_1-t_2)] \to 0$. But no poles are crossed, and so this integral vanishes. This is the Laplace-space manifestation of why one of the terms $\propto A_1A_2$ in (III.9.8) vanished by causality. Evaluating the Kronecker delta functions in $\boldsymbol{k}'$ then provides

$$-\frac{1}{4}\frac{q_\alpha^2}{m_\alpha^2}\frac{\mathrm{i}\boldsymbol{k}_1\boldsymbol{k}_2\,\delta_{\boldsymbol{k},\boldsymbol{k}_1+\boldsymbol{k}_2}}{\omega-\boldsymbol{k}\cdot\boldsymbol{v}}\,\mathrm{e}^{\mathrm{i}\omega t_2}:\frac{\partial}{\partial\boldsymbol{v}}\left[\frac{A_1A_2\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{v}(t_2-t_1)}}{\mathcal{D}(\omega-\boldsymbol{k}_1\cdot\boldsymbol{v},\boldsymbol{k}_2)\mathcal{D}(\boldsymbol{k}_1\cdot\boldsymbol{v},\boldsymbol{k}_1)}\frac{\mathrm{d}f_{0\alpha}}{\mathrm{d}\boldsymbol{v}}\right]$$
$$-\frac{1}{4}\frac{q_\alpha^2}{m_\alpha^2}\frac{\mathrm{i}\boldsymbol{k}_1\boldsymbol{k}_2\,\delta_{\boldsymbol{k},-\boldsymbol{k}_1-\boldsymbol{k}_2}}{\omega-\boldsymbol{k}\cdot\boldsymbol{v}}\,\mathrm{e}^{\mathrm{i}\omega t_2}:\frac{\partial}{\partial\boldsymbol{v}}\left[\frac{A_1A_2\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{v}(t_2-t_1)}}{\mathcal{D}(\omega+\boldsymbol{k}_1\cdot\boldsymbol{v},-\boldsymbol{k}_2)\mathcal{D}(-\boldsymbol{k}_1\cdot\boldsymbol{v},-\boldsymbol{k}_1)}\frac{\mathrm{d}f_{0\alpha}}{\mathrm{d}\boldsymbol{v}}\right]. \tag{III.9.22}$$

We now transform this expression back into the time domain by inverse-Laplace transforming, while demanding that $t > t_2$. Pushing the contour to $\mathrm{Im}(\omega) \to -\infty$ and picking up the kinetic pole at $\boldsymbol{k}\cdot\boldsymbol{v}$ (clockwise, contributing a $-2\pi\mathrm{i}$) while neglecting the dielectric poles, we finally obtain (for $t > t_2$)

$$-\frac{1}{4}\frac{q_\alpha^2}{m_\alpha^2}\boldsymbol{k}_1\boldsymbol{k}_2\,\delta_{\boldsymbol{k},\boldsymbol{k}_1+\boldsymbol{k}_2}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{v}(t-t_2)}:\frac{\partial}{\partial\boldsymbol{v}}\left[\frac{A_1A_2\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{v}(t_2-t_1)}}{\mathcal{D}(\omega-\boldsymbol{k}_1\cdot\boldsymbol{v},\boldsymbol{k}_2)\mathcal{D}(\boldsymbol{k}_1\cdot\boldsymbol{v},\boldsymbol{k}_1)}\frac{\mathrm{d}f_{0\alpha}}{\mathrm{d}\boldsymbol{v}}\right]_{\omega=\boldsymbol{k}\cdot\boldsymbol{v}}$$
$$-\frac{1}{4}\frac{q_\alpha^2}{m_\alpha^2}\boldsymbol{k}_1\boldsymbol{k}_2\,\delta_{\boldsymbol{k},-\boldsymbol{k}_1-\boldsymbol{k}_2}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{v}(t-t_2)}:\frac{\partial}{\partial\boldsymbol{v}}\left[\frac{A_1A_2\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{v}(t_2-t_1)}}{\mathcal{D}(\omega+\boldsymbol{k}_1\cdot\boldsymbol{v},-\boldsymbol{k}_2)\mathcal{D}(-\boldsymbol{k}_1\cdot\boldsymbol{v},-\boldsymbol{k}_1)}\frac{\mathrm{d}f_{0\alpha}}{\mathrm{d}\boldsymbol{v}}\right]_{\omega=\boldsymbol{k}\cdot\boldsymbol{v}}. \tag{III.9.23}$$

Setting $\mathcal{D} = 1$, distributing the $\partial/\partial\boldsymbol{v}$, and inverse-Fourier transforming nicely returns the result in (III.9.10). Good. Now for the terms associated with $\boldsymbol{k}_2-\boldsymbol{k}_1$:

$$\frac{1}{4}\frac{q_\alpha^2}{m_\alpha^2}\sum_{\boldsymbol{k}'}\frac{\mathrm{i}\boldsymbol{k}'(\boldsymbol{k}-\boldsymbol{k}')}{\omega-\boldsymbol{k}\cdot\boldsymbol{v}}:\frac{\partial}{\partial\boldsymbol{v}}\iint\frac{\mathrm{d}\omega'\mathrm{d}\omega''}{(2\pi)^2}\frac{1}{\omega-\omega'-\omega''}\frac{1}{\omega''-(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{v}}$$
$$\times\frac{A_1A_2\,\mathrm{e}^{\mathrm{i}(\omega' t_2+\omega'' t_1)}}{\mathcal{D}(\omega',\boldsymbol{k}')\mathcal{D}(\omega'',\boldsymbol{k}-\boldsymbol{k}')}(\delta_{\boldsymbol{k}',\boldsymbol{k}_2}\delta_{\boldsymbol{k},\boldsymbol{k}_2-\boldsymbol{k}_1}+\delta_{\boldsymbol{k}',-\boldsymbol{k}_2}\delta_{\boldsymbol{k},\boldsymbol{k}_1-\boldsymbol{k}_2})\frac{\mathrm{d}f_{0\alpha}}{\mathrm{d}\boldsymbol{v}}$$
$$+\ (\text{the same but with } 1\leftrightarrow 2). \tag{III.9.24}$$

The calculation proceeds exactly as above, ultimately giving (for $t > t_2$)

$$\begin{aligned} &\frac{1}{4}\frac{q_\alpha^2}{m_\alpha^2}\boldsymbol{k}_1\boldsymbol{k}_2\,\delta_{\boldsymbol{k},\boldsymbol{k}_2-\boldsymbol{k}_1}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{v}(t-t_2)}\boldsymbol{:}\frac{\partial}{\partial\boldsymbol{v}}\left[\frac{A_1A_2\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\boldsymbol{\cdot}\boldsymbol{v}(t_2-t_1)}}{\mathcal{D}(\omega+\boldsymbol{k}_1\boldsymbol{\cdot}\boldsymbol{v},\boldsymbol{k}_2)\mathcal{D}(-\boldsymbol{k}_1\boldsymbol{\cdot}\boldsymbol{v},-\boldsymbol{k}_1)}\frac{\mathrm{d}f_{0\alpha}}{\mathrm{d}\boldsymbol{v}}\right]_{\omega=\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{v}} \\ &+\frac{1}{4}\frac{q_\alpha^2}{m_\alpha^2}\boldsymbol{k}_1\boldsymbol{k}_2\,\delta_{\boldsymbol{k},\boldsymbol{k}_1-\boldsymbol{k}_2}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{v}(t-t_2)}\boldsymbol{:}\frac{\partial}{\partial\boldsymbol{v}}\left[\frac{A_1A_2\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_1\boldsymbol{\cdot}\boldsymbol{v}(t_2-t_1)}}{\mathcal{D}(\omega-\boldsymbol{k}_1\boldsymbol{\cdot}\boldsymbol{v},-\boldsymbol{k}_2)\mathcal{D}(\boldsymbol{k}_1\boldsymbol{\cdot}\boldsymbol{v},\boldsymbol{k}_1)}\frac{\mathrm{d}f_{0\alpha}}{\mathrm{d}\boldsymbol{v}}\right]_{\omega=\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{v}} \end{aligned} \tag{III.9.25}$$

as the counterpart to (III.9.23). Again, setting $\mathcal{D}=1$, distributing the $\partial/\partial\boldsymbol{v}$, and inverse-Fourier transforming returns the result in (III.9.12). Assembling everything, we have

$$\delta f_{\boldsymbol{k}}^{(\mathrm{echo})}(t,\boldsymbol{v}) = (\text{III.9.23}) + (\text{III.9.25}). \tag{III.9.26}$$

PART IV

Balescu–Lenard and Landau collision operators

> The law that entropy always increases holds, I think, the supreme position among the laws of Nature. If someone points out to you that your pet theory of the universe is in disagreement with Maxwell's equations – then so much the worse for Maxwell's equations. If it is found to be contradicted by observation – well, these experimentalists do bungle things sometimes. But if your theory is found to be against the second law of thermodynamics I can give you no hope; there is nothing for it but to collapse in deepest humiliation.
>
> Sir Arthur Stanley Eddington
> *The Nature of the Physical World* (1927)

IV.1. Derivation of the Balescu–Lenard collision operator: Method

With some knowledge of Vlasov physics behind us, let us return to the BBGKY hierarchy, closed at second order in $\Lambda^{-1} \lll 1$ by neglecting three-particle correlations ($h_{\alpha\beta\gamma} = 0$). Our equations are (II.6.2) and (II.6.3), repeated here for convenience:

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla} + \boldsymbol{a}\cdot\frac{\partial}{\partial \boldsymbol{v}}\right) f_\alpha(t,\boldsymbol{x}) = \sum_\beta \frac{q_\alpha q_\beta}{m_\alpha}\int \mathrm{d}\boldsymbol{x}'\,\frac{\partial}{\partial \boldsymbol{r}}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\cdot\frac{\partial g_{\alpha\beta}(t,\boldsymbol{x},\boldsymbol{x}')}{\partial \boldsymbol{v}}, \qquad \text{(IV.1.1)}$$

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla} + \boldsymbol{a}\cdot\frac{\partial}{\partial \boldsymbol{v}} + \boldsymbol{v}'\cdot\boldsymbol{\nabla}' + \boldsymbol{a}'\cdot\frac{\partial}{\partial \boldsymbol{v}'}\right) g_{\alpha\beta}(t,\boldsymbol{x},\boldsymbol{x}') \\
&\quad - \sum_\gamma \frac{q_\alpha q_\gamma}{m_\alpha}\int \mathrm{d}\boldsymbol{x}''\,\frac{\partial}{\partial \boldsymbol{r}}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}''|}\cdot\frac{\partial f_\alpha(t,\boldsymbol{x})}{\partial \boldsymbol{v}}\,g_{\beta\gamma}(t,\boldsymbol{x}',\boldsymbol{x}'') \\
&\quad - \sum_\gamma \frac{q_\beta q_\gamma}{m_\beta}\int \mathrm{d}\boldsymbol{x}''\,\frac{\partial}{\partial \boldsymbol{r}'}\frac{1}{|\boldsymbol{r}'-\boldsymbol{r}''|}\cdot\frac{\partial f_\beta(t,\boldsymbol{x}')}{\partial \boldsymbol{v}'}\,g_{\alpha\gamma}(t,\boldsymbol{x},\boldsymbol{x}'') \\
&\quad = \frac{\partial}{\partial \boldsymbol{r}}\frac{q_\alpha q_\beta}{|\boldsymbol{r}-\boldsymbol{r}'|}\cdot\left(\frac{1}{m_\alpha}\frac{\partial}{\partial \boldsymbol{v}} - \frac{1}{m_\beta}\frac{\partial}{\partial \boldsymbol{v}'}\right)\Big[f_\alpha(t,\boldsymbol{x})f_\beta(t,\boldsymbol{x}') + g_{\alpha\beta}(t,\boldsymbol{x},\boldsymbol{x}')\Big]. \qquad \text{(IV.1.2)}
\end{aligned}$$

The left-hand side of (IV.1.2) is homogeneous with respect to g; the right-hand side is thus a source term: particle correlations are driven by Coulomb interactions. The first part of the source term, that proportional to $f_\alpha f_\beta$, corresponds to initially uncorrelated particles becoming correlated through Coulomb scattering. (Technically, it corresponds to each of these uncorrelated particles becoming correlated due to interactions with each other's Debye clouds, but we'll come to that soon enough.) The second part of the source term, that proportional to $g_{\alpha\beta}$, corresponds to initially correlated particles becoming more or less correlated due to further interactions. This term is $\sim\Lambda^{-1} \lll 1$ smaller than the first term, and thus may be neglected in most cases. There are, however, certain situations in which the $g_{\alpha\beta}$ contribution to the source term may not be neglected; this will be discussed in §IV.6.

To proceed, note that the left-hand side of (IV.1.2) looks like a combination of linear Vlasov equations for each particle in the pair, capturing the dielectric response of the (γ) bath, with the right-hand side of the equation contribution a source term describing the two particles' mutual Coulomb interaction: $[(\partial_t + \mathcal{L} + \mathcal{L}')g]_{\alpha\beta} = \mathcal{S}_{\alpha\beta}$, with $\mathcal{L}$ given by (III.3.2) and $\mathcal{L}'$ the same but with $\boldsymbol{x} \leftrightarrow \boldsymbol{x}'$ and $\alpha \leftrightarrow \beta$. This suggests a Green's function approach, with a Vlasov Green's function for each particle used to advance the two-particle correlation forward in time (note that $\mathcal{L}$ and $\mathcal{L}'$ commute). But this will only work if we drop the $g_{\alpha\beta}$ contribution to the source term. Physically, this brute-force simplification means that, at any "initial" time from which we wish to integrate (IV.1.2), no two-particle correlations exist. We argue in favor of this idea below (§IV.2), but for now we adopt it blindly and note that this simplification allows us to write down a formal solution to (IV.1.2) using our knowledge of the Vlasov Green's function (cf. (III.3.4)):

$$g_{\alpha\beta}(t,\boldsymbol{x},\boldsymbol{x}') = \sum_{\overline{\alpha}}\sum_{\overline{\beta}} \int_0^t \mathrm{d}\overline{t} \int \mathrm{d}\overline{\boldsymbol{x}} \int \mathrm{d}\overline{\boldsymbol{x}}'\, S_{\overline{\alpha}\overline{\beta}}(\overline{t},\overline{\boldsymbol{x}},\overline{\boldsymbol{x}}') G_{\alpha\overline{\alpha}}(t-\overline{t},\boldsymbol{x};\overline{\boldsymbol{x}}) G_{\beta\overline{\beta}}(t-\overline{t},\boldsymbol{x}';\overline{\boldsymbol{x}}'), \tag{IV.1.3}$$

where $\overline{\boldsymbol{x}}$ and $\overline{\boldsymbol{x}}'$ are dummy phase-space coordinates (introduced for the integration) and $\overline{\alpha}$ and $\overline{\beta}$ are dummy species indices (introduced for the species sums); the source function

$$S_{\alpha\beta}(t,\boldsymbol{x},\boldsymbol{x}') = \frac{\partial}{\partial \boldsymbol{r}} \frac{q_\alpha q_\beta}{|\boldsymbol{r}-\boldsymbol{r}'|} \boldsymbol{\cdot} \left(\frac{1}{m_\alpha}\frac{\partial}{\partial \boldsymbol{v}} - \frac{1}{m_\beta}\frac{\partial}{\partial \boldsymbol{v}'} \right) f_\alpha(t,\boldsymbol{x}) f_\beta(t,\boldsymbol{x}'); \tag{IV.1.4}$$

and G is the appropriate Green's function (see (III.5.7)) with $G = 0$ for $t < 0$.

In writing (IV.1.3), we have expressed the two-particle correlation between α and β at time t as a superposition of the correlation effects created between $\overline{\alpha}$ and $\overline{\beta}$ at $\overline{\boldsymbol{x}}$ and $\overline{\boldsymbol{x}}'$ at time $\overline{t}$ that propagate to $\boldsymbol{x}$ and $\boldsymbol{x}'$ during the time interval $t-\overline{t}$. Substituting (IV.1.3) into (IV.1.2) and using Leibniz's rule,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{a(t)}^{b(t)} \mathrm{d}x\, f(t,x) = \int_{a(t)}^{b(t)} \mathrm{d}x\, \frac{\partial}{\partial t} f(t,x) + f(t,b(t))\frac{\mathrm{d}b}{\mathrm{d}t} - f(t,a(t))\frac{\mathrm{d}a}{\mathrm{d}t},$$

for differentiating under the integral sign, we find that

$$\begin{aligned}\left(\frac{\partial}{\partial t} + \boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla} + \boldsymbol{a}\boldsymbol{\cdot}\frac{\partial}{\partial \boldsymbol{v}}\right) & G_{\alpha\overline{\alpha}}(t-\overline{t},\boldsymbol{x};\overline{\boldsymbol{x}}) \\ & - \sum_\gamma \frac{q_\alpha q_\gamma}{m_\alpha}\int \mathrm{d}\boldsymbol{x}'\, \frac{\partial}{\partial \boldsymbol{r}}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\boldsymbol{\cdot}\frac{\partial f_\alpha}{\partial \boldsymbol{v}}\, G_{\gamma\overline{\alpha}}(t-\overline{t},\boldsymbol{x}';\overline{\boldsymbol{x}}) = \delta_{\alpha\overline{\alpha}}\delta(\boldsymbol{x}-\overline{\boldsymbol{x}})\delta(t-\overline{t}).\end{aligned} \tag{IV.1.5}$$

A similar equation holds for $G_{\beta\overline{\beta}}$ with $\alpha \to \beta$ and $\overline{\alpha} \to \overline{\beta}$ in (IV.1.5).

This is (almost) exactly the problem solved in §III.5 using the Green's function approach. Equation (IV.1.5) gives the solution to the linearized Vlasov equation. This means we already know $G_{\alpha\overline{\alpha}}$ and $G_{\beta\overline{\beta}}$ and thus how to solve the $g_{\alpha\beta}$ equation.

The "almost" is because f_α is technically time-dependent and inhomogeneous – it's evolving with $g_{\alpha\beta}$ as a source term (see (IV.1.1)). To overcome this complication (and to justify removing $g_{\alpha\beta}$ from $S_{\alpha\beta}$), we make some assumptions, which are known as...

IV.2. Bogoliubov's hypothesis

At the start of these lecture notes, we discussed Bogoliubov's timescale hierarchy (see §I.1). It concerned the hierarchy of well-separated time- and lengthscales related to the relaxation of an arbitrary perturbation. What this hierarchy means for our problem is that, on the timescale over which the two-particle correlation $g_{\alpha\beta}$ relaxes, the one-particle distribution function f is roughly constant. Thus, the time dependence of f_α and f_β in the source term (IV.1.4) can be ignored: $f_\alpha(t,\boldsymbol{x}) \to f_\alpha(\boldsymbol{x})$ and likewise for f_β. Equation (IV.1.3) is then a linear equation for $g_{\alpha\beta}$ with a temporally constant source term. The fast timescale over which f is fixed is denoted by a $\widetilde{t}$ to distinguish it from the timescale t over which f evolves due to collisions. Ultimately, we will take $\widetilde{t} \to \infty$, which amounts to a Markov assumption that the future is independent of the past (more on this in §VI.1).

A final assumption in the standard calculation is that the ensemble of plasma is spatially homogeneous. This means that $f(\boldsymbol{x}) = f(\boldsymbol{v})$ and that the acceleration $\boldsymbol{a}(t,\boldsymbol{r}) = \boldsymbol{a}(t) = 0$. Thus, any ensemble-averaged function of two spatial variables (e.g., $f_{\alpha\beta}$) can only be a function of the difference of those variables: $g_{\alpha\beta} = g_{\alpha\beta}(\widetilde{t}, \boldsymbol{r}-\boldsymbol{r}', \boldsymbol{v}, \boldsymbol{v}')$.

Under these assumptions, equations (IV.1.1) and (IV.1.2) become

$$\frac{\partial f_\alpha(t,\boldsymbol{v})}{\partial t} = \sum_\beta \frac{q_\alpha q_\beta}{m_\alpha} \int \mathrm{d}\boldsymbol{x}' \, \frac{\partial}{\partial \boldsymbol{r}} \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} \boldsymbol{\cdot} \frac{\partial}{\partial \boldsymbol{v}} g_{\alpha\beta}(t, \boldsymbol{r}-\boldsymbol{r}', \boldsymbol{v}, \boldsymbol{v}'), \tag{IV.2.1}$$

$$\begin{aligned}
&\left(\frac{\partial}{\partial \widetilde{t}} + \boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla} + \boldsymbol{v}'\boldsymbol{\cdot}\boldsymbol{\nabla}'\right) g_{\alpha\beta}(\widetilde{t}, \boldsymbol{r}-\boldsymbol{r}', \boldsymbol{v}, \boldsymbol{v}') \\
&\quad - \sum_\gamma \frac{q_\alpha q_\gamma}{m_\alpha} \int \mathrm{d}\boldsymbol{x}'' \, \frac{\partial}{\partial \boldsymbol{r}} \frac{1}{|\boldsymbol{r}-\boldsymbol{r}''|} \boldsymbol{\cdot} \frac{\partial f_\alpha}{\partial \boldsymbol{v}} \, g_{\beta\gamma}(\widetilde{t}, \boldsymbol{r}'-\boldsymbol{r}'', \boldsymbol{v}', \boldsymbol{v}'') \\
&\quad - \sum_\gamma \frac{q_\beta q_\gamma}{m_\beta} \int \mathrm{d}\boldsymbol{x}'' \, \frac{\partial}{\partial \boldsymbol{r}'} \frac{1}{|\boldsymbol{r}'-\boldsymbol{r}''|} \boldsymbol{\cdot} \frac{\partial f_\beta}{\partial \boldsymbol{v}'} \, g_{\alpha\gamma}(\widetilde{t}, \boldsymbol{r}-\boldsymbol{r}'', \boldsymbol{v}, \boldsymbol{v}'') \\
&\quad = S_{\alpha\beta}(\boldsymbol{r}-\boldsymbol{r}', \boldsymbol{v}, \boldsymbol{v}') \doteq \frac{\partial}{\partial \boldsymbol{r}} \frac{q_\alpha q_\beta}{|\boldsymbol{r}-\boldsymbol{r}'|} \boldsymbol{\cdot} \left(\frac{1}{m_\alpha}\frac{\partial}{\partial \boldsymbol{v}} - \frac{1}{m_\beta}\frac{\partial}{\partial \boldsymbol{v}'}\right) f_\alpha(\boldsymbol{v}) f_\beta(\boldsymbol{v}'),
\end{aligned} \tag{IV.2.2}$$

where $\widetilde{t}$ is the fast timescale over which f_α is temporally constant.

IV.3. Derivation of the Balescu–Lenard collision operator: Details

Following the discussion in §IV.1, equation (IV.2.2) can be solved via

$$g_{\alpha\beta}(\widetilde{t}, \boldsymbol{r}-\boldsymbol{r}', \boldsymbol{v}, \boldsymbol{v}') = \sum_{\overline{\alpha}} \sum_{\overline{\beta}} \int_0^{\widetilde{t}} \mathrm{d}\overline{t} \int \mathrm{d}\overline{\boldsymbol{x}} \int \mathrm{d}\overline{\boldsymbol{x}}' \, S_{\overline{\alpha}\overline{\beta}}(\overline{\boldsymbol{x}}, \overline{\boldsymbol{x}}') G_{\alpha\overline{\alpha}}(t-\overline{t}, \boldsymbol{x}; \overline{\boldsymbol{x}}) G_{\beta\overline{\beta}}(t-\overline{t}, \boldsymbol{x}'; \overline{\boldsymbol{x}}'). \tag{IV.3.1}$$

Fourier transforming and using (III.4.6), equation (IV.2.1) becomes

$$\begin{aligned}
\frac{\partial f_\alpha}{\partial t} &= \sum_\beta \frac{q_\alpha q_\beta}{m_\alpha} \int \mathrm{d}\boldsymbol{v}' \int \mathrm{d}\boldsymbol{r}' \, \frac{\partial}{\partial \boldsymbol{r}} \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} \boldsymbol{\cdot} \frac{\partial}{\partial \boldsymbol{v}} \int \mathrm{d}\boldsymbol{k} \, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\boldsymbol{\cdot}(\boldsymbol{r}-\boldsymbol{r}')} g_{\alpha\beta}(t, \boldsymbol{k}, \boldsymbol{v}, \boldsymbol{v}') \\
&\overset{\text{bp}}{=} -\sum_\beta \frac{q_\alpha q_\beta}{m_\alpha} \int \mathrm{d}\boldsymbol{k} \int \mathrm{d}\boldsymbol{v}' \, \mathrm{i}\boldsymbol{k}\boldsymbol{\cdot} \frac{\partial g_{\alpha\beta}(t, \boldsymbol{k}, \boldsymbol{v}, \boldsymbol{v}')}{\partial \boldsymbol{v}} \underbrace{\left(\int \mathrm{d}\boldsymbol{r}' \, \frac{\mathrm{e}^{\mathrm{i}\boldsymbol{k}\boldsymbol{\cdot}(\boldsymbol{r}-\boldsymbol{r}')}}{|\boldsymbol{r}-\boldsymbol{r}'|}\right)}_{=4\pi k^{-2}} \\
&= -\sum_\beta \frac{q_\alpha q_\beta}{m_\alpha} \int \mathrm{d}\boldsymbol{k} \, \frac{4\pi\mathrm{i}\boldsymbol{k}}{k^2} \boldsymbol{\cdot} \int \mathrm{d}\boldsymbol{v}' \, \frac{\partial g_{\alpha\beta}(t, \boldsymbol{k}, \boldsymbol{v}, \boldsymbol{v}')}{\partial \boldsymbol{v}}.
\end{aligned} \tag{IV.3.2}$$

Note that, since f_α is real, we need only compute the imaginary part of $g_{\alpha\beta}$ to close this equation. Fourier transforming (IV.3.1) gives

$$g_{\alpha\beta}(\widetilde{t},\boldsymbol{k},\boldsymbol{v},\boldsymbol{v}') = \sum_{\overline{\alpha}}\sum_{\overline{\beta}} \frac{q_{\overline{\alpha}}q_{\overline{\beta}}}{(2\pi)^3}\int_0^{\widetilde{t}}\mathrm{d}\overline{t}\int\mathrm{d}\overline{\boldsymbol{v}}\int\mathrm{d}\overline{\boldsymbol{v}}'\,\frac{4\pi\mathrm{i}\boldsymbol{k}}{k^2}\boldsymbol{\cdot}\left(\frac{1}{m_{\overline{\alpha}}}\frac{\partial}{\partial\overline{\boldsymbol{v}}}-\frac{1}{m_{\overline{\beta}}}\frac{\partial}{\partial\overline{\boldsymbol{v}}'}\right)f_{\overline{\alpha}}(\overline{\boldsymbol{v}})f_{\overline{\beta}}(\overline{\boldsymbol{v}}')$$
$$\times\, G_{\alpha\overline{\alpha}}(\widetilde{t}-\overline{t},\boldsymbol{k},\boldsymbol{v};\overline{\boldsymbol{v}})G_{\beta\overline{\beta}}(\widetilde{t}-\overline{t},-\boldsymbol{k},\boldsymbol{v}';\overline{\boldsymbol{v}}'). \tag{IV.3.3}$$

Next, Laplace transform $G_{\alpha\overline{\alpha}}$ and $G_{\beta\overline{\beta}}$ in time:

$$g_{\alpha\beta}(\widetilde{t},\boldsymbol{k},\boldsymbol{v},\boldsymbol{v}') = \sum_{\overline{\alpha}}\sum_{\overline{\beta}} \frac{q_{\overline{\alpha}}q_{\overline{\beta}}}{(2\pi)^3}\int_0^{\widetilde{t}}\mathrm{d}\overline{t}\int\mathrm{d}\overline{\boldsymbol{v}}\int\mathrm{d}\overline{\boldsymbol{v}}'\,\frac{4\pi\mathrm{i}\boldsymbol{k}}{k^2}\boldsymbol{\cdot}\left(\frac{1}{m_{\overline{\alpha}}}\frac{\partial}{\partial\overline{\boldsymbol{v}}}-\frac{1}{m_{\overline{\beta}}}\frac{\partial}{\partial\overline{\boldsymbol{v}}'}\right)f_{\overline{\alpha}}(\overline{\boldsymbol{v}})f_{\overline{\beta}}(\overline{\boldsymbol{v}}')$$
$$\times\int_L\frac{\mathrm{d}\omega}{2\pi}\,\mathrm{e}^{-\mathrm{i}\omega\overline{t}}G_{\alpha\overline{\alpha}}(\omega,\boldsymbol{k},\boldsymbol{v};\overline{\boldsymbol{v}})\int_{L'}\frac{\mathrm{d}\omega'}{2\pi}\,\mathrm{e}^{-\mathrm{i}\omega'\overline{t}}G_{\beta\overline{\beta}}(\omega',-\boldsymbol{k},\boldsymbol{v}';\overline{\boldsymbol{v}}'), \tag{IV.3.4}$$

where L and L' denote the appropriate Laplace contours. To ensure convergence, we impose the constraint $\mathrm{Im}(\omega+\omega')>0$. Then perform the time integration to obtain

$$g_{\alpha\beta}(\widetilde{t},\boldsymbol{k},\boldsymbol{v},\boldsymbol{v}') = \sum_{\overline{\alpha}}\sum_{\overline{\beta}} \frac{q_{\overline{\alpha}}q_{\overline{\beta}}}{(2\pi)^3}\int\mathrm{d}\overline{\boldsymbol{v}}\int\mathrm{d}\overline{\boldsymbol{v}}'\,\frac{4\pi\mathrm{i}\boldsymbol{k}}{k^2}\boldsymbol{\cdot}\left(\frac{1}{m_{\overline{\alpha}}}\frac{\partial}{\partial\overline{\boldsymbol{v}}}-\frac{1}{m_{\overline{\beta}}}\frac{\partial}{\partial\overline{\boldsymbol{v}}'}\right)f_{\overline{\alpha}}(\overline{\boldsymbol{v}})f_{\overline{\beta}}(\overline{\boldsymbol{v}}')$$
$$\times\int_L\frac{\mathrm{d}\omega}{2\pi}\int_{L'}\frac{\mathrm{d}\omega'}{2\pi}\,\frac{1-\mathrm{e}^{-\mathrm{i}(\omega+\omega')\widetilde{t}}}{\mathrm{i}(\omega+\omega'+\mathrm{i}0)}\,G_{\alpha\overline{\alpha}}(\omega,\boldsymbol{k},\boldsymbol{v};\overline{\boldsymbol{v}})G_{\beta\overline{\beta}}(\omega',-\boldsymbol{k},\boldsymbol{v}';\overline{\boldsymbol{v}}'). \tag{IV.3.5}$$

Recall that "0" is meant to notate an infinitesimal displacement of the pole off of the real axis (for causality reasons). The ω' integration can be carried out by closing the contour by an infinite semi-circle in the upper half-plane, since $G_{\beta\overline{\beta}}(\omega',-\boldsymbol{k})$ vanishes as $|\omega'|\to\infty$. Because $\mathrm{Im}(\omega+\omega')>0$, the only contribution to the integration arises from the pole at $\omega'=-\omega-\mathrm{i}0$. The "1" term vanishes because everything it multiplies is analytic in the upper half plane, and we are left with

$$g_{\alpha\beta}(\boldsymbol{k},\boldsymbol{v},\boldsymbol{v}') = \sum_{\overline{\alpha}}\sum_{\overline{\beta}} \frac{q_{\overline{\alpha}}q_{\overline{\beta}}}{(2\pi)^3}\int\mathrm{d}\overline{\boldsymbol{v}}\int\mathrm{d}\overline{\boldsymbol{v}}'\,\frac{4\pi\mathrm{i}\boldsymbol{k}}{k^2}\boldsymbol{\cdot}\left(\frac{1}{m_{\overline{\alpha}}}\frac{\partial}{\partial\overline{\boldsymbol{v}}}-\frac{1}{m_{\overline{\beta}}}\frac{\partial}{\partial\overline{\boldsymbol{v}}'}\right)f_{\overline{\alpha}}(\overline{\boldsymbol{v}})f_{\overline{\beta}}(\overline{\boldsymbol{v}}')$$
$$\times\int_L\frac{\mathrm{d}\omega}{2\pi}\,G_{\alpha\overline{\alpha}}(\omega,\boldsymbol{k},\boldsymbol{v};\overline{\boldsymbol{v}})G_{\beta\overline{\beta}}(-\omega,-\boldsymbol{k},\boldsymbol{v}';\overline{\boldsymbol{v}}'). \tag{IV.3.6}$$

Now we substitute in our Green's function (III.5.7), taking the extra precaution of noting what functions are analytic in upper (lower) half ω-plane by appending ω with an $+\mathrm{i}0$ ($-\mathrm{i}0$). The result is:

$$g_{\alpha\beta}(\boldsymbol{k},\boldsymbol{v},\boldsymbol{v}') = \sum_{\overline{\alpha}}\sum_{\overline{\beta}} \frac{q_{\overline{\alpha}}q_{\overline{\beta}}}{(2\pi)^3}\int\mathrm{d}\overline{\boldsymbol{v}}\int\mathrm{d}\overline{\boldsymbol{v}}'\,\frac{4\pi\mathrm{i}\boldsymbol{k}}{k^2}\boldsymbol{\cdot}\left(\frac{1}{m_{\overline{\alpha}}}\frac{\partial}{\partial\overline{\boldsymbol{v}}}-\frac{1}{m_{\overline{\beta}}}\frac{\partial}{\partial\overline{\boldsymbol{v}}'}\right)f_{\overline{\alpha}}(\overline{\boldsymbol{v}})f_{\overline{\beta}}(\overline{\boldsymbol{v}}')$$
$$\times\int_L\frac{\mathrm{d}\omega}{2\pi}\left[\frac{\mathrm{i}\delta_{\alpha\overline{\alpha}}\delta(\boldsymbol{v}-\overline{\boldsymbol{v}})}{\omega-\boldsymbol{k\cdot v}+\mathrm{i}0}-\frac{q_\alpha q_{\overline{\alpha}}}{m_\alpha}\frac{4\pi\mathrm{i}\boldsymbol{k}}{k^2}\boldsymbol{\cdot}\frac{\partial f_\alpha}{\partial\boldsymbol{v}}\frac{1}{\omega-\boldsymbol{k}\boldsymbol{\cdot}\overline{\boldsymbol{v}}+\mathrm{i}0}\frac{1}{\omega-\boldsymbol{k\cdot v}+\mathrm{i}0}\frac{1}{\mathcal{D}(\omega,\boldsymbol{k})}\right]$$
$$\times\left[-\frac{\mathrm{i}\delta_{\beta\overline{\beta}}\delta(\boldsymbol{v}'-\overline{\boldsymbol{v}}')}{\omega-\boldsymbol{k\cdot v}'-\mathrm{i}0}+\frac{q_\beta q_{\overline{\beta}}}{m_\beta}\frac{4\pi\mathrm{i}\boldsymbol{k}}{k^2}\boldsymbol{\cdot}\frac{\partial f_\beta}{\partial\boldsymbol{v}'}\frac{1}{\omega-\boldsymbol{k}\boldsymbol{\cdot}\overline{\boldsymbol{v}}'-\mathrm{i}0}\frac{1}{\omega-\boldsymbol{k\cdot v}'-\mathrm{i}0}\frac{1}{\mathcal{D}(-\omega,-\boldsymbol{k})}\right]. \tag{IV.3.7}$$

Now, this whole thing goes into the Fourier'd version of (IV.1.1), which reads (see (IV.3.2))

$$\frac{\partial f_\alpha(t,\boldsymbol{v})}{\partial t} = -\sum_\beta \frac{q_\alpha q_\beta}{m_\alpha} \int \mathrm{d}\boldsymbol{k}\, \frac{4\pi \mathrm{i}\boldsymbol{k}}{k^2} \cdot \frac{\partial}{\partial \boldsymbol{v}} \int \mathrm{d}\boldsymbol{v}'\, g_{\alpha\beta}(\boldsymbol{k},\boldsymbol{v},\boldsymbol{v}'). \tag{IV.3.8}$$

Note that the $\int \mathrm{d}\boldsymbol{v}'$ integration only touches the final term in brackets in (IV.3.7). Along with the $\sum_\beta q_\beta(\dots)$ summation in (IV.3.8), that last term of (IV.3.7), when substituted into (IV.3.8), contributes

$$\frac{-\mathrm{i}q_{\overline{\beta}}}{\omega - \boldsymbol{k}\cdot\overline{\boldsymbol{v}}' - \mathrm{i}0}\frac{1}{\mathcal{D}(-\omega,-\boldsymbol{k})}$$

(recall the definition of $\mathcal{D}(\omega,\boldsymbol{k})$ from (III.5.6)). This gives a huge simplification! Equation (IV.3.8) becomes

$$\begin{aligned}\frac{\partial f_\alpha(t,\boldsymbol{v})}{\partial t} &= -\frac{q_\alpha}{m_\alpha}\int \mathrm{d}\boldsymbol{k}\,\frac{4\pi\mathrm{i}\boldsymbol{k}}{k^2}\cdot\frac{\partial}{\partial\boldsymbol{v}}\\ &\times \sum_{\overline{\alpha}}\sum_{\overline{\beta}}\frac{q_{\overline{\alpha}}q_{\overline{\beta}}}{(2\pi)^3}\int \mathrm{d}\overline{\boldsymbol{v}}\int\mathrm{d}\overline{\boldsymbol{v}}'\,\frac{4\pi\mathrm{i}\boldsymbol{k}}{k^2}\cdot\left(\frac{1}{m_{\overline{\alpha}}}\frac{\partial}{\partial\overline{\boldsymbol{v}}}-\frac{1}{m_{\overline{\beta}}}\frac{\partial}{\partial\overline{\boldsymbol{v}}'}\right)f_{\overline{\alpha}}(\overline{\boldsymbol{v}})f_{\overline{\beta}}(\overline{\boldsymbol{v}}')\\ &\times\int_L\frac{\mathrm{d}\omega}{2\pi}\Bigg[\underbrace{\frac{\delta_{\alpha\overline{\alpha}}\delta(\boldsymbol{v}-\overline{\boldsymbol{v}})q_{\overline{\beta}}}{(\omega-\boldsymbol{k}\cdot\boldsymbol{v}+\mathrm{i}0)(\omega-\boldsymbol{k}\cdot\overline{\boldsymbol{v}}'-\mathrm{i}0)\mathcal{D}(-\omega,-\boldsymbol{k})}}_{1}\\ &\qquad\underbrace{-\frac{q_\alpha q_{\overline{\alpha}}}{m_\alpha}\frac{4\pi\boldsymbol{k}}{k^2}\cdot\frac{\partial f_\alpha}{\partial\boldsymbol{v}}\frac{q_{\overline{\beta}}}{(\omega-\boldsymbol{k}\cdot\overline{\boldsymbol{v}}+\mathrm{i}0)(\omega-\boldsymbol{k}\cdot\boldsymbol{v}+\mathrm{i}0)(\omega-\boldsymbol{k}\cdot\overline{\boldsymbol{v}}'-\mathrm{i}0)}\frac{1}{\mathcal{D}(\omega,\boldsymbol{k})\mathcal{D}(-\omega,-\boldsymbol{k})}}_{2}\Bigg].\end{aligned} \tag{IV.3.9}$$

Lots of integration to do! Let's label the terms:

$$\frac{\partial f_\alpha(t,\boldsymbol{v})}{\partial t} = -\frac{q_\alpha}{m_\alpha}\int\mathrm{d}\boldsymbol{k}\,\frac{4\pi\mathrm{i}\boldsymbol{k}}{k^2}\cdot\frac{\partial}{\partial\boldsymbol{v}}\left[1_{\overline{\alpha}}+2_{\overline{\alpha}}+1_{\overline{\beta}}+2_{\overline{\beta}}\right], \tag{IV.3.10}$$

where the subscript $\overline{\alpha}$ or $\overline{\beta}$ indicates which $m^{-1}(\partial/\partial\overline{\boldsymbol{v}})$ it multiplies. To proceed, write

$$\begin{aligned}1_{\overline{\alpha}} &\doteq \frac{q_\alpha}{m_\alpha}\frac{4\pi\mathrm{i}\boldsymbol{k}}{k^2}\cdot\frac{\partial f_\alpha}{\partial\boldsymbol{v}}\sum_{\overline{\beta}}q_{\overline{\beta}}^2\int\mathrm{d}\overline{\boldsymbol{v}}'\,f_{\overline{\beta}}(\overline{\boldsymbol{v}}')\\ &\times\int_L\frac{\mathrm{d}\omega}{2\pi}\,\frac{1}{(\omega-\boldsymbol{k}\cdot\boldsymbol{v}+\mathrm{i}0)(\omega-\boldsymbol{k}\cdot\overline{\boldsymbol{v}}'-\mathrm{i}0)\mathcal{D}(-\omega,-\boldsymbol{k})}\times\frac{\mathcal{D}(\omega,\boldsymbol{k})}{\mathcal{D}(\omega,\boldsymbol{k})}.\end{aligned} \tag{IV.3.11}$$

$$\begin{aligned}2_{\overline{\alpha}} &\doteq -\frac{q_\alpha}{m_\alpha}\frac{4\pi\mathrm{i}\boldsymbol{k}}{k^2}\cdot\frac{\partial f_\alpha}{\partial\boldsymbol{v}}\sum_{\overline{\alpha}}\frac{q_{\overline{\alpha}}^2}{m_{\overline{\alpha}}}\int\mathrm{d}\overline{\boldsymbol{v}}\,\frac{4\pi\boldsymbol{k}}{k^2}\cdot\frac{\partial f_{\overline{\alpha}}}{\partial\overline{\boldsymbol{v}}}\sum_{\overline{\beta}}q_{\overline{\beta}}^2\int\mathrm{d}\overline{\boldsymbol{v}}'\,f_{\overline{\beta}}(\overline{\boldsymbol{v}}')\\ &\times\int_L\frac{\mathrm{d}\omega}{2\pi}\,\frac{1}{\omega-\boldsymbol{k}\cdot\overline{\boldsymbol{v}}+\mathrm{i}0}\,\frac{1}{(\omega-\boldsymbol{k}\cdot\boldsymbol{v}+\mathrm{i}0)(\omega-\boldsymbol{k}\cdot\overline{\boldsymbol{v}}'-\mathrm{i}0)\mathcal{D}(-\omega,-\boldsymbol{k})\mathcal{D}(\omega,\boldsymbol{k})}\\ &= -\frac{q_\alpha}{m_\alpha}\frac{4\pi\mathrm{i}\boldsymbol{k}}{k^2}\cdot\frac{\partial f_\alpha}{\partial\boldsymbol{v}}\sum_{\overline{\beta}}q_{\overline{\beta}}^2\int\mathrm{d}\overline{\boldsymbol{v}}'\,f_{\overline{\beta}}(\overline{\boldsymbol{v}}')\\ &\times\int_L\frac{\mathrm{d}\omega}{2\pi}\,\frac{[\mathcal{D}(\omega,\boldsymbol{k})-1]}{(\omega-\boldsymbol{k}\cdot\boldsymbol{v}+\mathrm{i}0)(\omega-\boldsymbol{k}\cdot\overline{\boldsymbol{v}}'-\mathrm{i}0)\mathcal{D}(-\omega,-\boldsymbol{k})\mathcal{D}(\omega,\boldsymbol{k})},\end{aligned} \tag{IV.3.12}$$

using the definition of $\mathcal{D}(\omega, \boldsymbol{k})$ (see (III.5.6)). Adding $①_{\overline{\alpha}}$ and $②_{\overline{\alpha}}$ together eliminates the $\mathcal{D}(\omega, \boldsymbol{k})$ term:

$$①_{\overline{\alpha}} + ②_{\overline{\alpha}} = \frac{q_\alpha}{m_\alpha}\frac{4\pi\mathrm{i}\boldsymbol{k}}{k^2}\cdot\frac{\partial f_\alpha}{\partial \boldsymbol{v}}\sum_{\overline{\beta}} q_{\overline{\beta}}^2 \int \mathrm{d}\overline{\boldsymbol{v}}'\, f_{\overline{\beta}}(\overline{\boldsymbol{v}}') \times \int_L \frac{\mathrm{d}\omega}{2\pi}\frac{1}{(\omega - \boldsymbol{k}\cdot\boldsymbol{v} + \mathrm{i}0)(\omega - \boldsymbol{k}\cdot\overline{\boldsymbol{v}}' - \mathrm{i}0)\mathcal{D}(-\omega, -\boldsymbol{k})\mathcal{D}(\omega, \boldsymbol{k})}. \qquad \text{(IV.3.13)}$$

Next, let's do $②_{\overline{\beta}}$:

$$\begin{aligned} ②_{\overline{\beta}} &\doteq \frac{q_\alpha}{m_\alpha}\frac{4\pi\mathrm{i}\boldsymbol{k}}{k^2}\cdot\frac{\partial f_\alpha}{\partial \boldsymbol{v}}\sum_{\overline{\beta}}\frac{q_{\overline{\beta}}^2}{m_{\overline{\beta}}}\int \mathrm{d}\overline{\boldsymbol{v}}'\,\frac{4\pi\boldsymbol{k}}{k^2}\cdot\frac{\partial f_{\overline{\beta}}}{\partial \overline{\boldsymbol{v}}'}\sum_{\overline{\alpha}} q_{\overline{\alpha}}^2\int \mathrm{d}\overline{\boldsymbol{v}}\, f_{\overline{\alpha}}(\overline{\boldsymbol{v}}) \\ &\quad\times \int_L \frac{\mathrm{d}\omega}{2\pi}\frac{1}{\omega - \boldsymbol{k}\cdot\overline{\boldsymbol{v}}' - \mathrm{i}0}\frac{1}{(\omega - \boldsymbol{k}\cdot\boldsymbol{v} + \mathrm{i}0)(\omega - \boldsymbol{k}\cdot\overline{\boldsymbol{v}} + \mathrm{i}0)\mathcal{D}(-\omega, -\boldsymbol{k})\mathcal{D}(\omega, \boldsymbol{k})} \\ &= \frac{q_\alpha}{m_\alpha}\frac{4\pi\mathrm{i}\boldsymbol{k}}{k^2}\cdot\frac{\partial f_\alpha}{\partial \boldsymbol{v}}\sum_{\overline{\alpha}} q_{\overline{\alpha}}^2\int \mathrm{d}\overline{\boldsymbol{v}}\, f_{\overline{\alpha}}(\overline{\boldsymbol{v}}) \\ &\quad\times \int_L \frac{\mathrm{d}\omega}{2\pi}\frac{[\mathcal{D}(-\omega, -\boldsymbol{k}) - 1]}{(\omega - \boldsymbol{k}\cdot\boldsymbol{v} + \mathrm{i}0)(\omega - \boldsymbol{k}\cdot\overline{\boldsymbol{v}} + \mathrm{i}0)\mathcal{D}(-\omega, -\boldsymbol{k})\mathcal{D}(\omega, \boldsymbol{k})}. \end{aligned} \qquad \text{(IV.3.14)}$$

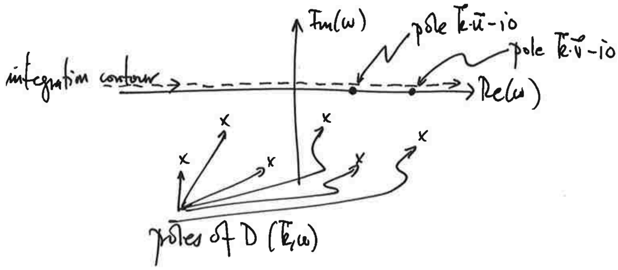
Combine this with $①_{\overline{\alpha}} + ②_{\overline{\alpha}}$ (see (IV.3.13)), noting that $\overline{\boldsymbol{v}}'$ and $\overline{\beta}$ are dummy variables and can be replaced by $\overline{\boldsymbol{v}}$ and $\overline{\alpha}$ under the integral and sum:

$$\begin{aligned} &①_{\overline{\alpha}} + ②_{\overline{\alpha}} + ②_{\overline{\beta}} = \\ &+ \frac{q_\alpha}{m_\alpha}\frac{4\pi\mathrm{i}\boldsymbol{k}}{k^2}\cdot\frac{\partial f_\alpha}{\partial \boldsymbol{v}}\sum_{\overline{\alpha}} q_{\overline{\alpha}}^2\int \mathrm{d}\overline{\boldsymbol{v}}\, f_{\overline{\alpha}}(\overline{\boldsymbol{v}})\int_L \frac{\mathrm{d}\omega}{2\pi}\frac{1}{(\omega - \boldsymbol{k}\cdot\boldsymbol{v} + \mathrm{i}0)(\omega - \boldsymbol{k}\cdot\overline{\boldsymbol{v}} + \mathrm{i}0)\mathcal{D}(\omega, \boldsymbol{k})} \\ &+ \frac{q_\alpha}{m_\alpha}\frac{4\pi\mathrm{i}\boldsymbol{k}}{k^2}\cdot\frac{\partial f_\alpha}{\partial \boldsymbol{v}}\sum_{\overline{\alpha}} q_{\overline{\alpha}}^2\int \mathrm{d}\overline{\boldsymbol{v}}\, f_{\overline{\alpha}}(\overline{\boldsymbol{v}}) \\ &\times \int_L \frac{\mathrm{d}\omega}{2\pi}\frac{1}{\omega - \boldsymbol{k}\cdot\boldsymbol{v} + \mathrm{i}0}\underbrace{\left(\frac{1}{\omega - \boldsymbol{k}\cdot\overline{\boldsymbol{v}} - \mathrm{i}0} - \frac{1}{\omega - \boldsymbol{k}\cdot\overline{\boldsymbol{v}} + \mathrm{i}0}\right)}_{\substack{\text{use Plemelj formula,} \\ \frac{1}{x \mp \mathrm{i}0} = \mathrm{PV}\left(\frac{1}{x}\right) \pm \mathrm{i}\pi\delta(x), \\ \text{for both terms to get} \\ 2\pi\mathrm{i}\delta(\omega - \boldsymbol{k}\cdot\overline{\boldsymbol{v}})}}\frac{1}{\mathcal{D}(-\omega, -\boldsymbol{k})\mathcal{D}(\omega, \boldsymbol{k})} \\ &= \frac{q_\alpha}{m_\alpha}\frac{4\pi\mathrm{i}\boldsymbol{k}}{k^2}\cdot\frac{\partial f_\alpha}{\partial \boldsymbol{v}}\sum_{\overline{\alpha}} q_{\overline{\alpha}}^2\int \mathrm{d}\overline{\boldsymbol{v}}\, f_{\overline{\alpha}}(\overline{\boldsymbol{v}})\int_L \frac{\mathrm{d}\omega}{2\pi}\frac{1}{(\omega - \boldsymbol{k}\cdot\boldsymbol{v} + \mathrm{i}0)(\omega - \boldsymbol{k}\cdot\overline{\boldsymbol{v}} + \mathrm{i}0)\mathcal{D}(\omega, \boldsymbol{k})} \\ &- \frac{q_\alpha}{m_\alpha}\frac{4\pi\boldsymbol{k}}{k^2}\cdot\frac{\partial f_\alpha}{\partial \boldsymbol{v}}\sum_{\overline{\alpha}} q_{\overline{\alpha}}^2\int \mathrm{d}\overline{\boldsymbol{v}}\, f_{\overline{\alpha}}(\overline{\boldsymbol{v}})\frac{1}{(\boldsymbol{k}\cdot\overline{\boldsymbol{v}} - \boldsymbol{k}\cdot\boldsymbol{v} + \mathrm{i}0)\mathcal{D}(-\boldsymbol{k}\cdot\overline{\boldsymbol{v}}, -\boldsymbol{k})\mathcal{D}(\boldsymbol{k}\cdot\overline{\boldsymbol{v}}, \boldsymbol{k})}. \end{aligned} \qquad \text{(IV.3.15)}$$

The first term here is actually equal to zero! Look at the ω integral:

$$\int_L \frac{\mathrm{d}\omega}{2\pi}\frac{1}{(\omega - \boldsymbol{k}\cdot\boldsymbol{v} + \mathrm{i}0)(\omega - \boldsymbol{k}\cdot\overline{\boldsymbol{v}} + \mathrm{i}0)\mathcal{D}(\omega, \boldsymbol{k})}.$$

The contour is shown below:



We can close this contour by an infinite semi-circle in the upper half ω-plane and avoid all the poles! Cauchy's theorem then gives zero. Thus,

$$\begin{aligned}&①_{\overline{\alpha}} + ②_{\overline{\alpha}} + ②_{\overline{\beta}}\\ &\quad = -\frac{q_\alpha}{m_\alpha}\frac{4\pi\boldsymbol{k}}{k^2}\cdot\frac{\partial f_\alpha}{\partial \boldsymbol{v}}\sum_{\overline{\alpha}} q_{\overline{\alpha}}^2 \int \mathrm{d}\overline{\boldsymbol{v}}\, f_{\overline{\alpha}}(\overline{\boldsymbol{v}})\frac{1}{(\boldsymbol{k}\cdot\overline{\boldsymbol{v}} - \boldsymbol{k}\cdot\boldsymbol{v} + \mathrm{i}0)|\mathcal{D}(\boldsymbol{k}\cdot\overline{\boldsymbol{v}},\boldsymbol{k})|^2}.\end{aligned} \tag{IV.3.16}$$

Looking back at (IV.3.10), we see that the right-hand side must be real, and so the above expression must be imaginary. By the Plemelj formula,

$$\mathrm{Im}\left[\frac{1}{\boldsymbol{k}\cdot(\overline{\boldsymbol{v}} - \boldsymbol{v}) + \mathrm{i}0}\right] = -\pi\delta(\boldsymbol{k}\cdot\overline{\boldsymbol{v}} - \boldsymbol{k}\cdot\boldsymbol{v}).$$

Retaining only the imaginary part of (IV.3.16) then gives

$$\mathrm{Im}\Big[①_{\overline{\alpha}} + ②_{\overline{\alpha}} + ②_{\overline{\beta}}\Big] = \frac{q_\alpha}{m_\alpha}\frac{4\pi\boldsymbol{k}}{k^2}\cdot\frac{\partial f_\alpha}{\partial \boldsymbol{v}}\sum_{\overline{\alpha}} q_{\overline{\alpha}}^2 \int \mathrm{d}\overline{\boldsymbol{v}}\, f_{\overline{\alpha}}(\overline{\boldsymbol{v}})\frac{\pi\delta(\boldsymbol{k}\cdot\overline{\boldsymbol{v}} - \boldsymbol{k}\cdot\boldsymbol{v})}{|\mathcal{D}(\boldsymbol{k}\cdot\overline{\boldsymbol{v}},\boldsymbol{k})|^2}. \tag{IV.3.17}$$

Finally, one last piece:

$$\begin{aligned}①_{\overline{\beta}} &= -q_\alpha f_\alpha(\boldsymbol{v})\frac{4\pi\mathrm{i}\boldsymbol{k}}{k^2}\cdot\sum_{\overline{\beta}}\frac{q_{\overline{\beta}}^2}{m_{\overline{\beta}}}\int \mathrm{d}\overline{\boldsymbol{v}}'\,\frac{\partial f_{\overline{\beta}}}{\partial \overline{\boldsymbol{v}}'}\\ &\qquad\times\int_L\frac{\mathrm{d}\omega}{2\pi}\frac{1}{(\omega - \boldsymbol{k}\cdot\boldsymbol{v} + \mathrm{i}0)(\omega - \boldsymbol{k}\cdot\overline{\boldsymbol{v}}' - \mathrm{i}0)\mathcal{D}(-\omega,-\boldsymbol{k})}\\ &= -\mathrm{i}q_\alpha f_\alpha(\boldsymbol{v})\int_L\frac{\mathrm{d}\omega}{2\pi}\frac{[\mathcal{D}(-\omega,-\boldsymbol{k}) - 1]}{(\omega - \boldsymbol{k}\cdot\boldsymbol{v} + \mathrm{i}0)\mathcal{D}(-\omega,-\boldsymbol{k})}\\ &= -\mathrm{i}q_\alpha f_\alpha(\boldsymbol{v})\int_L\frac{\mathrm{d}\omega}{2\pi}\frac{1}{\omega - \boldsymbol{k}\cdot\boldsymbol{v} + \mathrm{i}0}\left[1 - \frac{1}{\mathcal{D}(-\omega,-\boldsymbol{k})}\right].\end{aligned} \tag{IV.3.18}$$

Again, we need only retain the imaginary part of this:

$$\mathrm{Im}\Big[①_{\overline{\beta}}\Big] = -q_\alpha f_\alpha(\boldsymbol{v})\,\mathrm{Re}\left\{\int_L\frac{\mathrm{d}\omega}{2\pi}\frac{1}{\omega - \boldsymbol{k}\cdot\boldsymbol{v} + \mathrm{i}0}\left[1 - \frac{1}{\mathcal{D}(-\omega,-\boldsymbol{k})}\right]\right\}. \tag{IV.3.19}$$

The contour may be closed in the lower half ω-plane, since the factor in brackets only has poles in the upper half ω-plane. Thus,

$$
\begin{aligned}
\mathrm{Im}\Big[①_{\overline{\beta}}\Big] &= -q_\alpha f_\alpha(\boldsymbol{v})\,\mathrm{Re}\bigg\{(-\mathrm{i})\bigg[1-\frac{1}{\mathcal{D}(-\boldsymbol{k}\cdot\boldsymbol{v},-\boldsymbol{k})}\bigg]\bigg\} \\
&= q_\alpha f_\alpha(\boldsymbol{v})\,\mathrm{Im}\,\frac{1}{\mathcal{D}(-\boldsymbol{k}\cdot\boldsymbol{v},-\boldsymbol{k})} = q_\alpha f_\alpha(\boldsymbol{v})\frac{\mathrm{Im}\big[\mathcal{D}(\boldsymbol{k}\cdot\boldsymbol{v},\boldsymbol{v})\big]}{|\mathcal{D}(\boldsymbol{k}\cdot\boldsymbol{v},\boldsymbol{k})|^2} \\
&= \frac{q_\alpha f_\alpha(\boldsymbol{v})}{|\mathcal{D}(\boldsymbol{k}\cdot\boldsymbol{v},\boldsymbol{k})|^2}\,\mathrm{Im}\bigg[\sum_{\overline{\alpha}}\frac{q_{\overline{\alpha}}^2}{m_{\overline{\alpha}}}\frac{4\pi\boldsymbol{k}}{k^2}\cdot\int\mathrm{d}\overline{\boldsymbol{v}}\,\frac{\partial f_{\overline{\alpha}}}{\partial\overline{\boldsymbol{v}}}\frac{\delta(\omega-\boldsymbol{k}\cdot\boldsymbol{v})}{\omega-\boldsymbol{k}\cdot\overline{\boldsymbol{v}}+\mathrm{i}0}\bigg] \\
&= -\frac{q_\alpha f_\alpha(\boldsymbol{v})}{|\mathcal{D}(\boldsymbol{k}\cdot\boldsymbol{v},\boldsymbol{k})|^2}\bigg[\sum_{\overline{\alpha}}\frac{q_{\overline{\alpha}}^2}{m_{\overline{\alpha}}}\frac{4\pi\boldsymbol{k}}{k^2}\cdot\int\mathrm{d}\overline{\boldsymbol{v}}\,\frac{\partial f_{\overline{\alpha}}}{\partial\overline{\boldsymbol{v}}}\pi\delta(\omega-\boldsymbol{k}\cdot\boldsymbol{v})\delta(\omega-\boldsymbol{k}\cdot\overline{\boldsymbol{v}})\bigg] \\
&= -\frac{q_\alpha f_\alpha(\boldsymbol{v})}{|\mathcal{D}(\boldsymbol{k}\cdot\boldsymbol{v},\boldsymbol{k})|^2}\bigg[\sum_{\overline{\alpha}}\frac{q_{\overline{\alpha}}^2}{m_{\overline{\alpha}}}\frac{4\pi\boldsymbol{k}}{k^2}\cdot\int\mathrm{d}\overline{\boldsymbol{v}}\,\frac{\partial f_{\overline{\alpha}}}{\partial\overline{\boldsymbol{v}}}\pi\delta(\boldsymbol{k}\cdot\overline{\boldsymbol{v}}-\boldsymbol{k}\cdot\boldsymbol{v})\bigg],
\end{aligned}
\tag{IV.3.20}
$$

where the penultimate step follows from Plemelj's formula applied to $(\omega-\boldsymbol{k}\cdot\overline{\boldsymbol{v}}+\mathrm{i}0)^{-1}$. Putting all of the pieces together (that is, (IV.3.17) and (IV.3.20)),

$$
\mathrm{Im}\Big[①+②\Big] = q_\alpha\frac{4\pi\boldsymbol{k}}{k^2}\cdot\sum_{\overline{\alpha}}q_{\overline{\alpha}}^2\int\mathrm{d}\overline{\boldsymbol{v}}\,\frac{\pi\delta(\boldsymbol{k}\cdot\overline{\boldsymbol{v}}-\boldsymbol{k}\cdot\boldsymbol{v})}{|\mathcal{D}(\boldsymbol{k}\cdot\overline{\boldsymbol{v}},\boldsymbol{k})|^2}\left(\frac{1}{m_\alpha}\frac{\partial}{\partial\boldsymbol{v}}-\frac{1}{m_{\overline{\alpha}}}\frac{\partial}{\partial\overline{\boldsymbol{v}}}\right)f_\alpha(\boldsymbol{v})f_{\overline{\alpha}}(\overline{\boldsymbol{v}}).
\tag{IV.3.21}
$$

Inserting (IV.3.21) all the way back into (IV.3.10), freely replacing $\overline{\boldsymbol{v}}\to\boldsymbol{v}'$ and $\overline{\alpha}\to\beta$, and rearranging some terms, we finally obtain

$$
\boxed{
\begin{aligned}
\frac{\partial f_\alpha(t,\boldsymbol{v})}{\partial t} &= \sum_\beta\int\frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\,\frac{\boldsymbol{k}}{m_\alpha}\cdot\frac{\partial}{\partial\boldsymbol{v}} \\
&\times\int\mathrm{d}\boldsymbol{v}'\left|\frac{4\pi q_\alpha q_\beta}{k^2\mathcal{D}(\boldsymbol{k}\cdot\boldsymbol{v},\boldsymbol{k})}\right|^2\pi\delta(\boldsymbol{k}\cdot\boldsymbol{v}-\boldsymbol{k}\cdot\boldsymbol{v}')\left(\frac{\boldsymbol{k}}{m_\alpha}\cdot\frac{\partial}{\partial\boldsymbol{v}}-\frac{\boldsymbol{k}}{m_\beta}\cdot\frac{\partial}{\partial\boldsymbol{v}'}\right)f_\alpha(\boldsymbol{v})f_\beta(\boldsymbol{v}')
\end{aligned}
}
\tag{IV.3.22}
$$

The right-hand side of this equation is the *Balescu–Lenard collision operator*.

We'll discuss the physics in this operator shortly. But first note that, if we write the Coulomb potential in $\boldsymbol{k}$-space as

$$
\varphi_{\alpha\beta}(\boldsymbol{k}) \doteq \frac{q_\alpha q_\beta}{2\pi^2k^2},
$$

then (IV.3.22) can also be written as

$$
\begin{aligned}
\frac{\partial f_\alpha(t,\boldsymbol{v})}{\partial t} &= (2\pi)^3\sum_\beta\int\mathrm{d}\boldsymbol{k}\,\frac{\boldsymbol{k}}{m_\alpha}\cdot\frac{\partial}{\partial\boldsymbol{v}} \\
&\times\int\mathrm{d}\boldsymbol{v}'\left|\frac{\varphi_{\alpha\beta}(\boldsymbol{k})}{\mathcal{D}(\boldsymbol{k}\cdot\boldsymbol{v},\boldsymbol{k})}\right|^2\pi\delta(\boldsymbol{k}\cdot\boldsymbol{v}-\boldsymbol{k}\cdot\boldsymbol{v}')\left(\frac{\boldsymbol{k}}{m_\alpha}\cdot\frac{\partial}{\partial\boldsymbol{v}}-\frac{\boldsymbol{k}}{m_\beta}\cdot\frac{\partial}{\partial\boldsymbol{v}'}\right)f_\alpha(\boldsymbol{v})f_\beta(\boldsymbol{v}').
\end{aligned}
\tag{IV.3.23}
$$

What's nice about this form is that we could have chosen a different potential (so long as it's weak and long-range) and everything would have gone through.

Now, this $|\varphi_{\alpha\beta}(\boldsymbol{k})/\mathcal{D}(\boldsymbol{k}\cdot\boldsymbol{v},\boldsymbol{k})|^2$ term is very important. It represents the interaction of one particle (together with its shielding cloud) with the potential field of another particle (together with *its* shielding cloud). Because of the $\delta(\boldsymbol{k}\cdot\boldsymbol{v}-\boldsymbol{k}\cdot\boldsymbol{v}')$, this term handles the shielding for both particles. Thus, the Balescu–Lenard operator physically describes collisions between two shielded particles. As Krall & Trivelpiece (1973) put it, "A binary collision of charged particles in a dielectric medium is different from a collision in vacuum." This fact is manifest quite early in the calculation, where we used the fact that the solution is comprised of two Green's functions. Physically, this was a statement that, to lowest order, interacting particles move on unperturbed trajectories, carrying their shielding clouds along with them. (This shielding cloud is in the Vlasov response function.) Since the two-particle Green's function factors in this way, our solution cannot capture large-angle scatterings. This will manifest as a divergence at small scales – something we'll discuss in the next section. For now, some foreshadowing: this lowest-order factorization of the two-particle response into the product of two one-particle Green's functions, which are coupled by the source function $S_{\alpha\beta}$ describing the Coulomb interaction, underlies Rostoker's Test Particle Superposition Principle. We'll use this principle later (§V) to further understand the Balescu–Lenard collision operator and its derivatives.

IV.4. Properties of the Balescu–Lenard collision operator

Let us catalog the properties of the Balescu–Lenard collision operator. (You'll sometimes see it as the "BGL" operator, with "G" = Guernsey (1960).) But first, a reminder of what went into it:

- Three-particle correlations are negligible ($h_{\alpha\beta\gamma} = 0$);
- The ensemble of plasmas is homogeneous;
- The two-particle correlation function $g_{\alpha\beta}$ relaxes on a timescale much shorter than does the one-particle distribution function f_α.

This makes the Balescu–Lenard operator inappropriate for situations in which there is spatially inhomogeneous wave motion on relevant scales or for any phenomena that involve high frequencies like ω_p. Also, note that we took $\boldsymbol{B}_\mathrm{ext} = 0$. A version of Balescu–Lenard can be obtained when $\boldsymbol{B}_\mathrm{ext} \neq 0$ – see Klimontovich (1967), §15. In this case, the particle trajectories are helical rather than straight, and the directions parallel ($\|$) and perpendicular ($\perp$) to the magnetic-field direction $\hat{\boldsymbol{b}}_\mathrm{ext} \doteq \boldsymbol{B}_\mathrm{ext}/B_\mathrm{ext}$ are treated differently. The Balescu–Lenard collision operator (cf. IV.3.22) becomes

$$\left(\frac{\partial f_\alpha}{\partial t}\right)_\mathrm{c} = \sum_\beta \sum_{n=-\infty}^{\infty} \sum_{n'=-\infty}^{\infty} \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3} \left(\frac{\boldsymbol{k}}{m_\alpha}\cdot\frac{\partial}{\partial \boldsymbol{v}}\right)_n \int \mathrm{d}\boldsymbol{v}'\, \mathrm{J}_n^2\!\left(\frac{k_\perp v_\perp}{\Omega_\alpha}\right) \mathrm{J}_{n'}^2\!\left(\frac{k_\perp v'_\perp}{\Omega_\beta}\right)$$
$$\times \left|\frac{4\pi q_\alpha q_\beta}{k^2 \mathcal{D}(k_\| v_\| + n\Omega_\beta, \boldsymbol{k})}\right|^2 \pi\delta(k_\| v_\| + n\Omega_\alpha - k_\| v'_\| - n'\Omega_\alpha)$$
$$\times \left[\left(\frac{\boldsymbol{k}}{m_\alpha}\cdot\frac{\partial}{\partial \boldsymbol{v}}\right)_n - \left(\frac{\boldsymbol{k}}{m_\beta}\cdot\frac{\partial}{\partial \boldsymbol{v}'}\right)_{n'}\right] f_\alpha(v_\|, v_\perp, t) f_\beta(v'_\|, v'_\perp, t), \tag{IV.4.1}$$

where J_n is the n^th Bessel function, $\Omega_\alpha \doteq q_\alpha B_\mathrm{ext}/m_\alpha c$ is the Larmor frequency of species α, $\boldsymbol{k} = k_\| \hat{\boldsymbol{b}}_\mathrm{ext} + \boldsymbol{k}_\perp$, $\mathrm{d}\boldsymbol{k} = 2\pi k_\perp \mathrm{d}k_\perp \mathrm{d}k_\|$, and

$$\left(\frac{\boldsymbol{k}}{m_\alpha}\cdot\frac{\partial}{\partial \boldsymbol{v}}\right)_n = k_\| \frac{\partial}{\partial v_\|} + \frac{n\Omega_\alpha}{v_\perp}\frac{\partial}{\partial \boldsymbol{v}_\perp}.$$

If the Larmor radii of all particles $\rho_\alpha \doteq v_{\mathrm{th}\perp\alpha}/\Omega_\alpha$ are much greater than the Debye length, then trajectories are approximately straight on the scales of interest, and we recover the electrostatic Balescu–Lenard operator.

The Balescu–Lenard collision operator (IV.3.22) has a few desirable features:

- If $f_\alpha \geqslant 0$ at $t = 0$, then $f_\alpha \geqslant 0$ for all t.

Proof: If $f_\alpha > 0$ initially but becomes < 0 later, then there must have been a time at which its minimum value as a function of $\boldsymbol{v}$ first passed through zero. At that time and point in phase space, we have (i) $f_\alpha = 0$, (ii) $\partial f_\alpha/\partial \boldsymbol{v} = 0$, (iii) $\partial^2 f_\alpha/\partial\boldsymbol{v}\partial\boldsymbol{v}$ is positive-semidefinite, and (iv) $\partial f_\alpha/\partial t < 0$. Suppose the first three conditions hold. Then the right-hand side of (IV.3.22) becomes

$$\sum_\beta \frac{2\pi q_\alpha^2 q_\beta^2}{m_\alpha}\int \mathrm{d}\boldsymbol{k}\int \mathrm{d}\boldsymbol{v}'\,\frac{\boldsymbol{k}\boldsymbol{k}}{k^4} : \left[\left(\frac{\partial}{\partial\boldsymbol{v}}\frac{\delta(\boldsymbol{k\cdot v}-\boldsymbol{k\cdot v}')}{\pi|\mathcal{D}(\boldsymbol{k\cdot v},\boldsymbol{k})|^2}\right)\underbrace{\left(\frac{1}{m_\alpha}\frac{\partial}{\partial\boldsymbol{v}} - \frac{1}{m_\beta}\frac{\partial}{\partial\boldsymbol{v}'}\right) f_\alpha(\boldsymbol{v}) f_\beta(\boldsymbol{v}')}_{=\,0\text{ by }(i)\text{ and }(ii)}\right.$$
$$\left. + \frac{\delta(\boldsymbol{k\cdot v}-\boldsymbol{k\cdot v}')}{\pi|\mathcal{D}(\boldsymbol{k\cdot v},\boldsymbol{k})|^2}\left(\frac{1}{m_\alpha}\frac{\partial^2}{\partial\boldsymbol{v}\partial\boldsymbol{v}} - \underbrace{\frac{1}{m_\beta}\frac{\partial^2}{\partial\boldsymbol{v}\partial\boldsymbol{v}'}}_{=\,0\text{ by }(ii)}\right) f_\alpha(\boldsymbol{v}) f_\beta(\boldsymbol{v}')\right]$$
$$= \sum_\beta \frac{2\pi q_\alpha^2 q_\beta^2}{m_\alpha^2}\int \mathrm{d}\boldsymbol{k}\int \mathrm{d}\boldsymbol{v}'\,\frac{\boldsymbol{k}\boldsymbol{k}}{k^4} : \left[\frac{\delta(\boldsymbol{k\cdot v}-\boldsymbol{k\cdot v}')}{\pi|\mathcal{D}(\boldsymbol{k\cdot v},\boldsymbol{k})|^2}\frac{\partial^2 f_\alpha}{\partial\boldsymbol{v}\partial\boldsymbol{v}} f_\beta(\boldsymbol{v}')\right] > 0 \text{ by } (iii),$$

which contradicts (iv). *Q.E.D.*

- The collision operator is Galilean invariant.

Proof: Let $f_\alpha(\boldsymbol{v}) \to f_\alpha(\boldsymbol{v}-\boldsymbol{u})$ and $f_\beta(\boldsymbol{v}') \to f_\beta(\boldsymbol{v}'-\boldsymbol{u})$, where $\boldsymbol{u}$ is the velocity of some frame. Then define $\boldsymbol{w} \doteq \boldsymbol{v}-\boldsymbol{u}$ and $\boldsymbol{w}' \doteq \boldsymbol{v}'-\boldsymbol{u}$; and note that $\mathrm{d}\boldsymbol{v}' = \mathrm{d}\boldsymbol{w}'$, $\partial/\partial\boldsymbol{v} = \partial/\partial\boldsymbol{w}$, and $\partial/\partial\boldsymbol{v}' = \partial/\partial\boldsymbol{w}'$. Also, $\boldsymbol{v}-\boldsymbol{v}' = \boldsymbol{w}-\boldsymbol{w}'$. Then the right-hand side of (IV.3.22) becomes

$$\sum_\beta \frac{2\pi q_\alpha^2 q_\beta^2}{m_\alpha}\frac{\partial}{\partial\boldsymbol{w}}\cdot\int \mathrm{d}\boldsymbol{k}\int \mathrm{d}\boldsymbol{w}'\,\frac{\boldsymbol{k}\boldsymbol{k}}{k^4}\frac{\delta(\boldsymbol{k\cdot w}-\boldsymbol{k\cdot w}')}{\pi|\mathcal{D}(\boldsymbol{k\cdot w}+\boldsymbol{k\cdot u},\boldsymbol{k})|^2}$$
$$\cdot\left(\frac{1}{m_\alpha}\frac{\partial}{\partial\boldsymbol{w}} - \frac{1}{m_\beta}\frac{\partial}{\partial\boldsymbol{w}'}\right) f_\alpha(\boldsymbol{w}) f_\beta(\boldsymbol{w}')$$

This is the same as the original operator, but for the argument of the dielectric function $\mathcal{D}$. But $\mathcal{D}$ is Galilean invariant:

$$\begin{aligned}\mathcal{D}(\boldsymbol{k\cdot w}+\boldsymbol{k\cdot u},\boldsymbol{k}) &= 1 + \sum_\gamma \frac{q_\gamma^2}{m_\gamma}\frac{4\pi\boldsymbol{k}}{k^2}\cdot\int \mathrm{d}\boldsymbol{v}''\,\frac{\partial f_{0\gamma}/\partial\boldsymbol{v}''}{\boldsymbol{k\cdot w}+\boldsymbol{k\cdot u}-\boldsymbol{k\cdot v}''}\\ &= 1 + \sum_\gamma \frac{q_\gamma^2}{m_\gamma}\frac{4\pi\boldsymbol{k}}{k^2}\cdot\int \mathrm{d}\boldsymbol{w}''\,\frac{\partial f_{0\gamma}/\partial\boldsymbol{w}''}{\boldsymbol{k\cdot w}-\boldsymbol{k\cdot w}''}\\ &= \mathcal{D}(\boldsymbol{k\cdot w},\boldsymbol{k}).\end{aligned}$$

Q.E.D.

- Maxwell distributions for all species with equal temperatures and mean velocities are a time-independent solution:

$$f_{\mathrm{eq},\alpha} = \frac{n_\alpha}{\pi^{3/2}v_{\mathrm{th}\alpha}^3}\exp\left(-\frac{|\boldsymbol{v}-\boldsymbol{u}|^2}{v_{\mathrm{th}\alpha}^2}\right),\quad v_{\mathrm{th}\alpha}^2 \doteq \frac{2T}{m_\alpha}$$

with the same temperature T and mean velocity $\boldsymbol{u}$ for all α. The proof follows by direct substitution into (IV.3.22).

- As $t \to \infty$, any f_α satisfying $f_\alpha \geqslant 0$ approaches a Maxwell distribution with equal temperatures for all species.
- The Balescu–Lenard collision operator conserves particle number:

$$\int \mathrm{d}\boldsymbol{v} \left(\frac{\partial f_\alpha}{\partial t}\right)_{\mathrm{c}} = 0 \quad \text{for each } \alpha.$$

- The Balescu–Lenard collision operator conserves total momentum:

$$\sum_\alpha \int \mathrm{d}\boldsymbol{v}\, m_\alpha \boldsymbol{v} \left(\frac{\partial f_\alpha}{\partial t}\right)_{\mathrm{c}} = 0.$$

(NB: momentum of each individual species is *not* conserved. Newton would have a problem with that.)

- The Balescu–Lenard collision operator conserves total kinetic energy:

$$\sum_\alpha \int \mathrm{d}\boldsymbol{v}\, (1/2) m_\alpha v^2 \left(\frac{\partial f_\alpha}{\partial t}\right)_{\mathrm{c}} = 0.$$

(NB: again, this holds only for the entire plasma, not each species by itself.)

The final three properties above may be proven simultaneously as follows. First, introduce the tensor

$$\boldsymbol{Q}_{\alpha\beta}(\boldsymbol{v}, \boldsymbol{v}') \doteq \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\, \boldsymbol{k}\boldsymbol{k} \left| \frac{4\pi q_\alpha q_\beta}{k^2 \mathcal{D}(\boldsymbol{k}\cdot\boldsymbol{v}, \boldsymbol{k})} \right|^2 \pi\delta(\boldsymbol{k}\cdot\boldsymbol{v} - \boldsymbol{k}\cdot\boldsymbol{v}'). \tag{IV.4.2}$$

Then, the Balescu–Lenard operator (IV.3.22) may be written as

$$\left(\frac{\partial f_\alpha}{\partial t}\right)_{\mathrm{c}} = \sum_\beta \frac{1}{m_\alpha} \frac{\partial}{\partial \boldsymbol{v}} \cdot \int \mathrm{d}\boldsymbol{v}'\, \boldsymbol{Q}_{\alpha\beta} \cdot \left(\frac{1}{m_\alpha}\frac{\partial}{\partial \boldsymbol{v}} - \frac{1}{m_\beta}\frac{\partial}{\partial \boldsymbol{v}'}\right) f_\alpha(\boldsymbol{v}) f_\beta(\boldsymbol{v}'). \tag{IV.4.3}$$

Next, multiply (IV.4.3) by an arbitrary function of velocity, $\Phi_\alpha(\boldsymbol{v})$, sum over α, and integrate over $\boldsymbol{v}$. The result is

$$\begin{aligned}
&\sum_{\alpha,\,\beta} \int \mathrm{d}\boldsymbol{v}\, \frac{\Phi_\alpha(\boldsymbol{v})}{m_\alpha} \frac{\partial}{\partial \boldsymbol{v}} \cdot \int \mathrm{d}\boldsymbol{v}'\, \boldsymbol{Q}_{\alpha\beta} \cdot \left(\frac{1}{m_\alpha}\frac{\partial}{\partial \boldsymbol{v}} - \frac{1}{m_\beta}\frac{\partial}{\partial \boldsymbol{v}'}\right) f_\alpha(\boldsymbol{v}) f_\beta(\boldsymbol{v}') \\
&\overset{\text{bp}}{=} -\sum_{\alpha,\,\beta} \int \mathrm{d}\boldsymbol{v} \int \mathrm{d}\boldsymbol{v}'\, \frac{1}{m_\alpha} \frac{\mathrm{d}\Phi_\alpha}{\mathrm{d}\boldsymbol{v}} \cdot \boldsymbol{Q}_{\alpha\beta} \cdot \left(\frac{1}{m_\alpha}\frac{\partial}{\partial \boldsymbol{v}} - \frac{1}{m_\beta}\frac{\partial}{\partial \boldsymbol{v}'}\right) f_\alpha(\boldsymbol{v}) f_\beta(\boldsymbol{v}') \\
&= -\sum_{\alpha,\,\beta} \int \mathrm{d}\boldsymbol{v} \int \mathrm{d}\boldsymbol{v}' \left[\frac{1}{2m_\alpha} \frac{\mathrm{d}\Phi_\alpha}{\mathrm{d}\boldsymbol{v}} \cdot \boldsymbol{Q}_{\alpha\beta} \cdot \left(\frac{1}{m_\alpha}\frac{\partial}{\partial \boldsymbol{v}} - \frac{1}{m_\beta}\frac{\partial}{\partial \boldsymbol{v}'}\right) f_\alpha(\boldsymbol{v}) f_\beta(\boldsymbol{v}')\right. \\
&\qquad\qquad \left. + \frac{1}{2m_\alpha} \frac{\mathrm{d}\Phi_\alpha}{\mathrm{d}\boldsymbol{v}} \cdot \boldsymbol{Q}_{\alpha\beta} \cdot \left(\frac{1}{m_\alpha}\frac{\partial}{\partial \boldsymbol{v}} - \frac{1}{m_\beta}\frac{\partial}{\partial \boldsymbol{v}'}\right) f_\alpha(\boldsymbol{v}) f_\beta(\boldsymbol{v}')\right] \\
&= -\sum_{\alpha,\,\beta} \int \mathrm{d}\boldsymbol{v} \int \mathrm{d}\boldsymbol{v}'\, \frac{1}{2} \left(\frac{1}{m_\alpha} \frac{\mathrm{d}\Phi_\alpha}{\mathrm{d}\boldsymbol{v}} - \frac{1}{m_\beta} \frac{\mathrm{d}\Phi_\beta}{\mathrm{d}\boldsymbol{v}'}\right) \cdot \boldsymbol{Q}_{\alpha\beta} \cdot \left(\frac{1}{m_\alpha}\frac{\partial}{\partial \boldsymbol{v}} - \frac{1}{m_\beta}\frac{\partial}{\partial \boldsymbol{v}'}\right) f_\alpha(\boldsymbol{v}) f_\beta(\boldsymbol{v}'),
\end{aligned}$$

where the last step follows from $\boldsymbol{Q}_{\alpha\beta}(\boldsymbol{v}, \boldsymbol{v}') = \boldsymbol{Q}_{\beta\alpha}(\boldsymbol{v}', \boldsymbol{v})$. Finally, note that this integral only vanishes for three functions: $\Phi_\alpha(\boldsymbol{v}) = 1$, $m_\alpha \boldsymbol{v}$, and $(1/2) m_\alpha v^2$. *Q.E.D.*

Very soon, we will also see it advantageous to separate terms in (IV.4.3) by defining

$$\boldsymbol{A}_\alpha \doteq \frac{1}{m_\alpha}\sum_\beta \left(\frac{1}{m_\alpha}+\frac{1}{m_\beta}\right)\int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\,\boldsymbol{kk}\left|\frac{4\pi q_\alpha q_\beta}{k^2\mathcal{D}(\boldsymbol{k\cdot v},\boldsymbol{k})}\right|^2 \int \mathrm{d}\boldsymbol{v}'\,\pi\delta(\boldsymbol{k\cdot v}-\boldsymbol{k\cdot v}')\boldsymbol{\cdot}\frac{\partial f_\beta}{\partial \boldsymbol{v}'}, \tag{IV.4.4a}$$

$$\boldsymbol{B}_\alpha \doteq \frac{1}{m_\alpha}\sum_\beta \left(\frac{1}{m_\alpha}+\frac{1}{m_\alpha}\right)\int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\,\boldsymbol{kk}\left|\frac{4\pi q_\alpha q_\beta}{k^2\mathcal{D}(\boldsymbol{k\cdot v},\boldsymbol{k})}\right|^2 \int \mathrm{d}\boldsymbol{v}'\,\pi\delta(\boldsymbol{k\cdot v}-\boldsymbol{k\cdot v}')f_\beta(\boldsymbol{v}'), \tag{IV.4.4b}$$

so that (IV.4.3) may be written as

$$\boxed{\frac{\partial f_\alpha(t,\boldsymbol{v})}{\partial t} = -\frac{\partial}{\partial \boldsymbol{v}}\boldsymbol{\cdot}\big[\boldsymbol{A}_\alpha f_\alpha(\boldsymbol{v})\big] + \frac{1}{2}\frac{\partial}{\partial \boldsymbol{v}}\frac{\partial}{\partial \boldsymbol{v}}\boldsymbol{:}\big[\boldsymbol{B}_\alpha f_\alpha(\boldsymbol{v})\big]} \tag{IV.4.5}$$

This *Fokker-Planck* form has two pieces: the first corresponding to friction, the second corresponding to diffusion. You'll see why soon enough (§VI).

Finally, the Balescu–Lenard collision operator satisfies an "H theorem"; i.e., the entropy-like functional

$$\boxed{S(t) \doteq -\sum_\alpha \int \mathrm{d}\boldsymbol{x}\, f_\alpha(t,\boldsymbol{x})\ln f_\alpha(t,\boldsymbol{x})} \tag{IV.4.6}$$

can either increase or remain constant. It *cannot* decrease!

Proof: Take the time derivative of (IV.4.6) and use the Balescu–Lenard equation (IV.3.22) to find

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -\sum_\alpha \int \mathrm{d}\boldsymbol{x}\,\big[1+\ln f_\alpha\big]\left(\frac{\partial f_\alpha}{\partial t}\right)_{\rm c}. \tag{IV.4.7}$$

Since this operator conserves particle number, the first term vanishes. Thus, using (IV.4.3), we have

$$\begin{aligned}\frac{\mathrm{d}S}{\mathrm{d}t} &= -\sum_\alpha \int \mathrm{d}\boldsymbol{x}\,\ln f_\alpha\left(\frac{\partial f_\alpha}{\partial t}\right)_{\rm c}\\ &= -\sum_\alpha\sum_\beta \int \mathrm{d}\boldsymbol{x}\,\ln f_\alpha \frac{1}{m_\alpha}\frac{\partial}{\partial \boldsymbol{v}}\boldsymbol{\cdot}\int \mathrm{d}\boldsymbol{v}'\,\boldsymbol{Q}_{\alpha\beta}\boldsymbol{\cdot}\left(\frac{1}{m_\alpha}\frac{\partial}{\partial \boldsymbol{v}}-\frac{1}{m_\beta}\frac{\partial}{\partial \boldsymbol{v}'}\right)f_\alpha(\boldsymbol{v})f_\beta(\boldsymbol{v}')\\ &\overset{\rm bp}{=} \sum_\alpha\sum_\beta \int \mathrm{d}\boldsymbol{x}\,\frac{1}{m_\alpha}\frac{\partial \ln f_\alpha}{\partial \boldsymbol{v}}\boldsymbol{\cdot}\int \mathrm{d}\boldsymbol{v}'\,\boldsymbol{Q}_{\alpha\beta}\boldsymbol{\cdot}\left(\frac{1}{m_\alpha}\frac{\partial \ln f_\alpha}{\partial \boldsymbol{v}}-\frac{1}{m_\beta}\frac{\partial \ln f_\beta}{\partial \boldsymbol{v}'}\right)f_\alpha(\boldsymbol{v})f_\beta(\boldsymbol{v}')\\ &= \frac{1}{2}\sum_\alpha\sum_\beta \int \mathrm{d}\boldsymbol{x}\int \mathrm{d}\boldsymbol{v}'\left(\frac{1}{m_\alpha}\frac{\partial \ln f_\alpha}{\partial \boldsymbol{v}}-\frac{1}{m_\beta}\frac{\partial \ln f_\beta}{\partial \boldsymbol{v}'}\right)\boldsymbol{\cdot}\boldsymbol{Q}_{\alpha\beta}\\ &\qquad\qquad\boldsymbol{\cdot}\left(\frac{1}{m_\alpha}\frac{\partial \ln f_\alpha}{\partial \boldsymbol{v}}-\frac{1}{m_\beta}\frac{\partial \ln f_\beta}{\partial \boldsymbol{v}'}\right)f_\alpha(\boldsymbol{v})f_\beta(\boldsymbol{v}'),\end{aligned} \tag{IV.4.8}$$

where in the final step the symmetries of $\boldsymbol{Q}_{\alpha\beta}$ were exploited to introduce the extra $\partial\ln f_\beta/\partial\boldsymbol{v}'$ term. Defining the vector

$$\boldsymbol{F}_{\alpha\beta}(\boldsymbol{v},\boldsymbol{v}') \doteq \frac{1}{m_\alpha}\frac{\partial \ln f_\alpha}{\partial \boldsymbol{v}}-\frac{1}{m_\beta}\frac{\partial \ln f_\beta}{\partial \boldsymbol{v}'} \tag{IV.4.9}$$

and inserting (IV.4.2) for $\boldsymbol{Q}_{\alpha\beta}$ gives

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \sum_\alpha \sum_\beta \int \mathrm{d}\boldsymbol{x} \int \mathrm{d}\boldsymbol{v}' \int \mathrm{d}\boldsymbol{k} \left| \frac{q_\alpha q_\beta (\boldsymbol{k \cdot F}_{\alpha\beta})}{k^2 \mathcal{D}(\boldsymbol{k \cdot v}, \boldsymbol{k})} \right|^2 \delta(\boldsymbol{k \cdot v} - \boldsymbol{k \cdot v}')\, f_\alpha(\boldsymbol{v}) f_\beta(\boldsymbol{v}')$$
$$\geqslant 0. \tag{IV.4.10}$$

Q.E.D. (Question: given that the Balescu–Lenard operator cannot decrease entropy, where did the arrow of time appear in its derivation?)

Note that equality in (IV.4.10) only occurs when

$$\boldsymbol{F}_{\alpha\beta}(\boldsymbol{v}, \boldsymbol{v}') = A_{\alpha\beta}(\boldsymbol{v}, \boldsymbol{v}')(\boldsymbol{v} - \boldsymbol{v}') \tag{IV.4.11}$$

for some scalar function $A_{\alpha\beta}(\boldsymbol{v}, \boldsymbol{v}')$. We now show that the only consistent solution to this equation is for f_α and f_β to be Maxwellians. First, note that $A_{\alpha\beta}(\boldsymbol{v}, \boldsymbol{v}') = A_{\beta\alpha}(\boldsymbol{v}', \boldsymbol{v})$. Next, take the velocity-space curl of (IV.4.11) and use (IV.4.9) to find

$$\frac{\partial}{\partial \boldsymbol{v}} \times \boldsymbol{F}_{\alpha\beta} = 0 = \frac{\partial A_{\alpha\beta}}{\partial \boldsymbol{v}} \times (\boldsymbol{v} - \boldsymbol{v}'), \tag{IV.4.12}$$

which implies that $A_{\alpha\beta} = A_{\alpha\beta}(|\boldsymbol{v} - \boldsymbol{v}'|)$. It follows by setting $\boldsymbol{v}' = 0$ and then $\boldsymbol{v} = 0$ in (IV.4.11) that (see §7.2 of Montgomery & Tidman 1964)

$$\frac{1}{m_\alpha}\left(\frac{\partial \ln f_\alpha}{\partial \boldsymbol{v}}\right)_{\boldsymbol{v}=0} = \frac{1}{m_\beta}\left(\frac{\partial \ln f_\beta}{\partial \boldsymbol{v}'}\right)_{\boldsymbol{v}'=0} \doteq \boldsymbol{a}_1.$$

Using this in

$$\boldsymbol{F}_{\alpha\beta}(0, \boldsymbol{v}') + \boldsymbol{F}_{\alpha\beta}(\boldsymbol{v}, 0) = \frac{1}{m_\alpha}\left[\left(\frac{\partial \ln f_\alpha}{\partial \boldsymbol{v}}\right)_{\boldsymbol{v}=0} + \frac{\partial \ln f_\alpha}{\partial \boldsymbol{v}}\right] - \frac{1}{m_\beta}\left[\left(\frac{\partial \ln f_\beta}{\partial \boldsymbol{v}'}\right)_{\boldsymbol{v}'=0} + \frac{\partial \ln f_\beta}{\partial \boldsymbol{v}'}\right]$$

leads to the constraint

$$A_{\alpha\beta}(|\boldsymbol{v}|)\,\boldsymbol{v} - A_{\alpha\beta}(|\boldsymbol{v}'|)\,\boldsymbol{v}' = \boldsymbol{F}_{\alpha\beta}(\boldsymbol{v}, \boldsymbol{v}') = A_{\alpha\beta}(|\boldsymbol{v} - \boldsymbol{v}'|)\,(\boldsymbol{v} - \boldsymbol{v}'). \tag{IV.4.13}$$

The only continuous solution of (IV.4.13) is $A_{\alpha\beta}(|\boldsymbol{v} - \boldsymbol{v}'|) = a_2 = \text{const}$. Thus, (IV.4.11) has the solution

$$\ln f_\alpha(\boldsymbol{v}) = m_\alpha\left(\frac{1}{2}a_2 v^2 + \boldsymbol{a}_1 \boldsymbol{\cdot} \boldsymbol{v} + a_0\right) \tag{IV.4.14}$$

with $a_0 = \text{const}$. These constants can be determined from the definitions

$$n_\alpha \doteq \int \mathrm{d}\boldsymbol{v}\, f_\alpha, \quad \boldsymbol{u}_\alpha \doteq \frac{1}{n_\alpha}\int \mathrm{d}\boldsymbol{v}\, \boldsymbol{v} f_\alpha, \quad T_\alpha \doteq \frac{1}{n_\alpha}\int \mathrm{d}\boldsymbol{v}\, \frac{1}{3} m_\alpha |\boldsymbol{v} - \boldsymbol{u}_\alpha|^2 f_\alpha,$$

in which case (IV.4.14) gives

$$f_\alpha(\boldsymbol{v}) = \frac{n_\alpha}{\pi^{3/2} v_{\mathrm{th}\alpha}^3} \exp\left(-\frac{|\boldsymbol{v} - \boldsymbol{u}_\alpha|^2}{v_{\mathrm{th}\alpha}^2}\right), \quad v_{\mathrm{th}\alpha}^2 \doteq \frac{2T_\alpha}{m_\alpha}, \tag{IV.4.15}$$

a shifted Maxwellian. Substitution of this distribution back into the collision operator indicates that all species must have the same bulk velocity $\boldsymbol{u}$ and temperature T.

IV.5. Asymptotics of the Balescu–Lenard operator: The Landau operator

A notable feature of the Balescu–Lenard collision operator is the inclusion of both binary and collective processes. The term

$$\left(\frac{1}{m_\alpha}\frac{\partial}{\partial \boldsymbol{v}} - \frac{1}{m_\beta}\frac{\partial}{\partial \boldsymbol{v}'}\right) f_\alpha(\boldsymbol{v}) f_\beta(\boldsymbol{v}')$$

is associated with two-body interactions – indeed, this factor also arises in the Boltzmann collision operator (discussed further in these notes and in the homework). But the Balescu–Lenard operator also includes the dielectric function $\mathcal{D}(\omega, \boldsymbol{k})$, and so the distribution functions of all species enter non-linearly; this reflects the many-particle shielding effects of the Coulomb interaction.

An elegant consequence of this feature is that the Balescu–Lenard operator converges as $k \to 0$ (i.e., at large scales). Let's see that. Write $\mathrm{d}\boldsymbol{k} = \int \mathrm{d}\Omega_{\boldsymbol{k}} \int_0^\infty \mathrm{d}k\, k^2$. The k-dependent part of the Balescu–Lenard operator is then

$$\int \mathrm{d}\Omega_{\boldsymbol{k}} \int_0^\infty \mathrm{d}k\, \frac{\boldsymbol{k}\boldsymbol{k}}{k^2} \frac{\delta(\boldsymbol{k}\cdot\boldsymbol{v} - \boldsymbol{k}\cdot\boldsymbol{v}')}{|\mathcal{D}(\boldsymbol{k}\cdot\boldsymbol{v}, \boldsymbol{k})|^2} = \int \mathrm{d}\Omega_{\boldsymbol{k}} \int_0^\infty \frac{\mathrm{d}k}{k}\, \hat{\boldsymbol{k}}\hat{\boldsymbol{k}} \frac{\delta(\hat{\boldsymbol{k}}\cdot\boldsymbol{v} - \hat{\boldsymbol{k}}\cdot\boldsymbol{v}')}{|\mathcal{D}(\boldsymbol{k}\cdot\boldsymbol{v}, \boldsymbol{k})|^2}, \tag{IV.5.1}$$

where $\hat{\boldsymbol{k}} \doteq \boldsymbol{k}/k$. Now, $\mathcal{D}$ has a particular form (see (III.5.6)), so let's examine that:

$$\begin{aligned}
\mathcal{D}(\boldsymbol{k}\cdot\boldsymbol{v}, \boldsymbol{k}) &= 1 + \sum_\gamma \frac{q_\gamma^2}{m_\gamma} \frac{4\pi \boldsymbol{k}}{k^2} \cdot \int \mathrm{d}\boldsymbol{v}'\, \frac{\partial f_\gamma/\partial \boldsymbol{v}'}{\boldsymbol{k}\cdot(\boldsymbol{v}-\boldsymbol{v}') + \mathrm{i}0} \\
&= 1 + \sum_\gamma \frac{4\pi q_\gamma^2 n_\gamma}{k^2 T_\gamma} \int \mathrm{d}\boldsymbol{v}'\, \frac{v_{\mathrm{th}\gamma}^2}{2n_\gamma} \frac{\boldsymbol{k}\cdot\partial f_\gamma/\partial \boldsymbol{v}'}{\boldsymbol{k}\cdot(\boldsymbol{v}-\boldsymbol{v}') + \mathrm{i}0} \\
&\doteq 1 + \frac{k_\mathrm{D}^2}{k^2}\, \alpha(\boldsymbol{k}\cdot\boldsymbol{v}, \boldsymbol{v}),
\end{aligned} \tag{IV.5.2}$$

where

$$k_\mathrm{D}^2 \doteq \frac{1}{\lambda_\mathrm{D}^2} \doteq \sum_\gamma \frac{4\pi q_\gamma^2 n_\gamma}{T_\gamma}. \tag{IV.5.3}$$

Thus, writing

$$|\mathcal{D}(\boldsymbol{k}\cdot\boldsymbol{v}, \boldsymbol{k})|^2 = \left[1 + \frac{k_\mathrm{D}^2}{k^2}\,\mathrm{Re}(\alpha)\right]^2 + \left[\frac{k_\mathrm{D}^2}{k^2}\,\mathrm{Im}(\alpha)\right]^2,$$

equation (IV.5.1) is

$$\int \mathrm{d}\Omega_{\boldsymbol{k}} \int_0^\infty \frac{\mathrm{d}k}{k} \frac{\hat{\boldsymbol{k}}\hat{\boldsymbol{k}}\, \delta(\hat{\boldsymbol{k}}\cdot\boldsymbol{v} - \hat{\boldsymbol{k}}\cdot\boldsymbol{v}')}{\left[1 + \frac{k_\mathrm{D}^2}{k^2}\,\mathrm{Re}(\alpha)\right]^2 + \left[\frac{k_\mathrm{D}^2}{k^2}\,\mathrm{Im}(\alpha)\right]^2}. \tag{IV.5.4}$$

For $k_\mathrm{D}^2/k^2 \gg 1$ (i.e., long wavelengths), this becomes

$$\int \mathrm{d}\Omega_{\boldsymbol{k}} \int_0^\infty \mathrm{d}k\, k^3 \frac{\hat{\boldsymbol{k}}\hat{\boldsymbol{k}}\, \delta(\hat{\boldsymbol{k}}\cdot\boldsymbol{v} - \hat{\boldsymbol{k}}\cdot\boldsymbol{v}')}{|\alpha(\boldsymbol{k}\cdot\boldsymbol{v}, \boldsymbol{k})|^2},$$

which is nicely convergent. The physical reason is that the dielectric function works to limit the effective range of the Coulomb interaction at large distances; the Balescu–Lenard operator takes Debye shielding into account. What if we didn't account for polarization?

Set $\mathcal{D} = 1$. Then we have

$$\int \mathrm{d}\Omega_{\boldsymbol{k}} \int_0^\infty \frac{\mathrm{d}k}{k}\, \hat{\boldsymbol{k}}\hat{\boldsymbol{k}}\, \delta(\hat{\boldsymbol{k}}\cdot\boldsymbol{v} - \hat{\boldsymbol{k}}\cdot\boldsymbol{v}'),$$

which is divergent! Physically, this is because the “bare” Coulomb force has infinite range. More on this shortly…

For $k_\mathrm{D}^2/k^2 \ll 1$ (i.e., short wavelengths), note that we have a logarithmic divergence. This may be traced to our original assumption about the smallness of $g_{\alpha\beta}$ relative to f_α, and likewise for $h_{\alpha\beta\gamma}$ relative to $g_{\alpha\beta}$. (Recall that $g_{\alpha\beta}$ was neglected relative to $f_\alpha f_\beta$ in (IV.1.2)!) At short distances, the momentum transfer between particles is large, and we must worry about particle discreteness. For example, two electrons cannot get very close to one another, and so $f_{\alpha\beta} \to 0$ as $|\boldsymbol{r} - \boldsymbol{r}'| \to 0$; this implies $g_{\alpha\beta} \approx -f_\alpha f_\beta$, and so $g_{\alpha\beta}$ is *not* much smaller than f_α. HW03 explores this further.

In practice, since the divergence is logarithmic, it ain't so bad. The customary thing is to simply cut off the k integral at some large value k_0: $\int_0^\infty \mathrm{d}k \to \int_0^{k_0} \mathrm{d}k$, and choose an appropriate k_0. Being off by a little bit in this choice doesn't much affect the answer, being only logarithmic. A typical choice is the Landau length $k_0^{-1} \doteq e^2/T$, the distance of closest approach of thermal electrons. The potential isn't Coulombic on this scale anyhow, so this makes sense.

The next several pages will be devoted to understanding the physics contained within in the Balescu–Lenard equation, but first let us return briefly to the case $\mathcal{D} = 1$ – i.e., ignore the polarization effects involved in Debye screening. Then (IV.3.22) reads

$$\begin{aligned}\frac{\partial f_\alpha(t,\boldsymbol{v})}{\partial t} &= \sum_\beta \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\, \frac{\boldsymbol{k}}{m_\alpha}\cdot\frac{\partial}{\partial \boldsymbol{v}} \\ &\quad\times \int \mathrm{d}\boldsymbol{v}' \left|\frac{4\pi q_\alpha q_\beta}{k^2}\right|^2 \pi\delta(\boldsymbol{k}\cdot\boldsymbol{v} - \boldsymbol{k}\cdot\boldsymbol{v}')\left(\frac{\boldsymbol{k}}{m_\alpha}\cdot\frac{\partial}{\partial\boldsymbol{v}} - \frac{\boldsymbol{k}}{m_\beta}\cdot\frac{\partial}{\partial\boldsymbol{v}'}\right) f_\alpha(\boldsymbol{v}) f_\beta(\boldsymbol{v}') \\ &= \sum_\beta 2q_\alpha^2 q_\beta^2 \int \mathrm{d}\Omega_{\boldsymbol{k}} \int_0^\infty \frac{\mathrm{d}k}{k}\,\frac{\hat{\boldsymbol{k}}}{m_\alpha}\cdot\frac{\partial}{\partial\boldsymbol{v}} \\ &\quad\times \int \mathrm{d}\boldsymbol{v}'\, \delta(\hat{\boldsymbol{k}}\cdot\boldsymbol{v} - \hat{\boldsymbol{k}}\cdot\boldsymbol{v}')\left(\frac{\hat{\boldsymbol{k}}}{m_\alpha}\cdot\frac{\partial}{\partial\boldsymbol{v}} - \frac{\hat{\boldsymbol{k}}}{m_\beta}\cdot\frac{\partial}{\partial\boldsymbol{v}'}\right) f_\alpha(\boldsymbol{v}) f_\beta(\boldsymbol{v}').\end{aligned} \tag{IV.5.5}$$

Clearly, we have to cut off the integral over k both at large k (as with the Balescu–Lenard operator) and at small k (since we've ignored shielding and thus are allowing for long-range influence of bare particles, stripped of their Debye clouds). I've already commented on k_max. A sensible choice for k_min is $k_\mathrm{D} \doteq 1/\lambda_\mathrm{D}$. Then,

$$\int_{k_\mathrm{D}}^{k_0} \frac{\mathrm{d}k}{k} = \ln\left(\frac{k_0}{k_\mathrm{D}}\right) \doteq \ln\lambda, \tag{IV.5.6}$$

the Coulomb logarithm. Then

$$\begin{aligned}\frac{\partial f_\alpha(t,\boldsymbol{v})}{\partial t} &= \sum_\beta 2q_\alpha^2 q_\beta^2 \ln\lambda_{\alpha\beta} \int \mathrm{d}\Omega_{\boldsymbol{k}} \frac{\hat{\boldsymbol{k}}}{m_\alpha}\cdot\frac{\partial}{\partial\boldsymbol{v}} \\ &\quad\times \int \mathrm{d}\boldsymbol{v}'\, \delta(\hat{\boldsymbol{k}}\cdot\boldsymbol{v} - \hat{\boldsymbol{k}}\cdot\boldsymbol{v}')\left(\frac{\hat{\boldsymbol{k}}}{m_\alpha}\cdot\frac{\partial}{\partial\boldsymbol{v}} - \frac{\hat{\boldsymbol{k}}}{m_\beta}\cdot\frac{\partial}{\partial\boldsymbol{v}'}\right) f_\alpha(\boldsymbol{v}) f_\beta(\boldsymbol{v}').\end{aligned} \tag{IV.5.7}$$

It is possible to simplify this expression by examining the combination

$$\frac{1}{\pi}\int \mathrm{d}\Omega_{\boldsymbol{k}}\, \hat{\boldsymbol{k}}\hat{\boldsymbol{k}}\, \delta(\hat{\boldsymbol{k}}\boldsymbol{\cdot}\boldsymbol{v} - \hat{\boldsymbol{k}}\boldsymbol{\cdot}\boldsymbol{v}').$$

Without loss of generality, orient the velocity difference $\boldsymbol{u} \doteq \boldsymbol{v} - \boldsymbol{v}'$ along the z axis, and work in spherical coordinates to write $\hat{\boldsymbol{k}} = \cos\theta\,\hat{\boldsymbol{z}} + \sin\theta(\cos\phi\,\hat{\boldsymbol{x}} + \sin\phi\,\hat{\boldsymbol{y}})$. Then,

$$\begin{aligned}
\frac{1}{\pi}\int \mathrm{d}\Omega_{\boldsymbol{k}}\, \hat{\boldsymbol{k}}\hat{\boldsymbol{k}}\, \delta(\hat{\boldsymbol{k}}\boldsymbol{\cdot}\boldsymbol{u}) &= \frac{1}{\pi}\int_0^{2\pi}\mathrm{d}\phi \int_{-1}^{+1}\mathrm{d}(\cos\theta)\,\delta(u\cos\theta) \\
&\quad\times \bigl[\cos\theta\,\hat{\boldsymbol{z}} + \sin\theta(\cos\phi\,\hat{\boldsymbol{x}} + \sin\phi\,\hat{\boldsymbol{y}})\bigr]\bigl[\cos\theta\,\hat{\boldsymbol{z}} + \sin\theta(\cos\phi\,\hat{\boldsymbol{x}} + \sin\phi\,\hat{\boldsymbol{y}})\bigr] \\
&= \frac{1}{\pi}\,2\pi \int_{-1}^{+1}\mathrm{d}(\cos\theta)\;\underbrace{\delta(u\cos\theta)}_{=\,\delta(\cos\theta)/u}\left[\cos^2\theta\,\hat{\boldsymbol{z}}\hat{\boldsymbol{z}} + \sin^2\theta\left(\frac{\hat{\boldsymbol{x}}\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}}\hat{\boldsymbol{y}}}{2}\right)\right] \\
&= \frac{2}{u}\left(\frac{\hat{\boldsymbol{x}}\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}}\hat{\boldsymbol{y}}}{2}\right) \\
&= \frac{u^2\mathsf{\boldsymbol{I}} - \boldsymbol{u}\boldsymbol{u}}{u^3} \doteq \mathsf{\boldsymbol{U}}(u).
\end{aligned} \tag{IV.5.8}$$

Then (IV.5.7) becomes

$$\boxed{\frac{\partial f_\alpha(t,\boldsymbol{v})}{\partial t} = \sum_\beta \frac{2\pi q_\alpha^2 q_\beta^2 \ln\lambda_{\alpha\beta}}{m_\alpha}\frac{\partial}{\partial \boldsymbol{v}}\boldsymbol{\cdot}\int \mathrm{d}\boldsymbol{v}'\,\mathsf{\boldsymbol{U}}(\boldsymbol{v}-\boldsymbol{v}')\boldsymbol{\cdot}\left(\frac{1}{m_\alpha}\frac{\partial}{\partial\boldsymbol{v}} - \frac{1}{m_\beta}\frac{\partial}{\partial\boldsymbol{v}'}\right) f_\alpha(\boldsymbol{v})f_\beta(\boldsymbol{v}')} \tag{IV.5.9}$$

The right-hand side of this equation is the *Landau collision operator* (Landau 1937). (Note: Landau obtained this from the Boltzmann equation after many assumptions – keep reading…)

We'll revisit the Landau operator in due course, and will take a number of useful limits to obtain approximate operators of great use (§VIII). But, first, we really need to understand the content of the Balescu–Lenard and Landau equations. In the next few sections and chapters, we'll examine the BBGKY hierarchy in a plasma in thermal equilibrium (IV.6), thereby obtaining the spectrum of electrostatic fluctuations off of which particles scatter. We will also derive and use something called the test-particle superposition principle (§V), and will show how our Balescu–Lenard and Landau collision operators are Fokker-Planck operators (§VI). All of this is buried in the preceding material, masked by the mathematics. In what follows, this physics will be elucidated bit by bit. First, let us consider a plasma in thermal equilibrium…

IV.6. Equilibrium BBGKY hierarchy

In thermal equilibrium, the probability distribution $P_N(\varGamma)$ is just the familiar Gibbs distribution from your undergraduate statistical mechanics class:

$$D_N(\varGamma) \doteq \frac{1}{\mathcal{Z}}\exp\left[-\frac{\mathcal{H}(\varGamma)}{T}\right], \tag{IV.6.1}$$

where T is the (species-independent) temperature, $\mathcal{Z} \doteq \int \mathrm{d}\varGamma\, \exp(-\mathcal{H}/T)$ is the partition function, and the Hamiltonian

$$\mathcal{H} = \sum_{\alpha_i}\frac{1}{2}m_\alpha V_{\alpha_i}^2 + \sum_{\alpha_i}\sum_{\beta_j\neq\alpha_i}\frac{1}{2}\frac{q_\alpha q_\beta}{|\boldsymbol{R}_{\alpha_i} - \boldsymbol{R}_{\beta_j}|} \tag{IV.6.2}$$

contains the kinetic and potential energies of the particles. (The sum $\sum_{\alpha_i} = \sum_\alpha \sum_{i=1}^{N_\alpha}$ is a sum over all species and all particles of each species. The factor of $1/2$ in the potential energy accounts for double counting of particle pairs in the sums.) The Gibbs distribution is clearly separable in the velocity:

$$D_N = \frac{\prod_{\alpha_i} \exp(-m_\alpha V_{\alpha_i}^2/2T)}{\prod_\alpha \big[\int \mathrm{d}V_{\alpha_1}\, \exp(-m_\alpha V_{\alpha_1}^2/2T)\big]^{N_\alpha}}\, \widehat{D}_N, \tag{IV.6.3}$$

where $\widehat{D}_N$ contains all of the spatial dependence of D_N arising from the potential energy in the Hamiltonian. Thus,

$$f_\alpha(\boldsymbol{x}) = f_\alpha(\boldsymbol{v})\, \cancelto{1}{\widehat{f}_\alpha(\boldsymbol{r})}, \tag{IV.6.4a}$$

$$f_{\alpha\beta}(\boldsymbol{x}, \boldsymbol{x}') = f_\alpha(\boldsymbol{v}) f_\beta(\boldsymbol{v}')\, \cancelto{\widehat{f}_{\alpha\beta}(\boldsymbol{r}-\boldsymbol{r}')}{\widehat{f}_{\alpha\beta}(\boldsymbol{r}, \boldsymbol{r}')}, \tag{IV.6.4b}$$

$$\dots$$

with

$$f_\alpha(\boldsymbol{v}) = \frac{n_\alpha}{\pi^{3/2} v_{\mathrm{th}\alpha}^3} \exp\!\left(-\frac{v^2}{v_{\mathrm{th}\alpha}^2}\right), \qquad v_{\mathrm{th}\alpha}^2 \doteq \frac{2T}{m_\alpha}$$

for all species α. (NB: $\widehat{f}_\alpha(\boldsymbol{r}) = 1$, because integrating over all Coulomb potentials gives no net acceleration: $\partial f_\alpha / \partial \boldsymbol{r} = 0$.)

In equilibrium, the equation for the two-particle correlation (IV.2.2) becomes

$$\begin{aligned}
&\big(\boldsymbol{v}\cdot\boldsymbol{\nabla} + \boldsymbol{v}'\cdot\boldsymbol{\nabla}'\big) g_{\alpha\beta}(\boldsymbol{r}-\boldsymbol{r}', \boldsymbol{v}, \boldsymbol{v}') \\
&\qquad - \sum_\gamma \frac{q_\alpha q_\gamma}{m_\alpha} \int \mathrm{d}\boldsymbol{x}''\, \frac{\partial}{\partial \boldsymbol{r}} \frac{1}{|\boldsymbol{r}-\boldsymbol{r}''|} \cdot \frac{\mathrm{d}f_\alpha}{\mathrm{d}\boldsymbol{v}}\, g_{\beta\gamma}(\boldsymbol{r}'-\boldsymbol{r}'', \boldsymbol{v}', \boldsymbol{v}'') \\
&\qquad - \sum_\gamma \frac{q_\beta q_\gamma}{m_\beta} \int \mathrm{d}\boldsymbol{x}''\, \frac{\partial}{\partial \boldsymbol{r}'} \frac{1}{|\boldsymbol{r}'-\boldsymbol{r}''|} \cdot \frac{\mathrm{d}f_\beta}{\mathrm{d}\boldsymbol{v}'}\, g_{\alpha\gamma}(\boldsymbol{r}-\boldsymbol{r}'', \boldsymbol{v}, \boldsymbol{v}'') \\
&\qquad = \frac{\partial}{\partial \boldsymbol{r}} \frac{q_\alpha q_\beta}{|\boldsymbol{r}-\boldsymbol{r}'|} \cdot \left(\frac{1}{m_\alpha}\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{v}} - \frac{1}{m_\beta}\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{v}'}\right) f_\alpha(\boldsymbol{v}) f_\beta(\boldsymbol{v}').
\end{aligned} \tag{IV.6.5}$$

Because $g_{\alpha\beta}$ is translationally invariant and $f_\alpha(\boldsymbol{v})$ is Maxwellian, one has that

$$\boldsymbol{v}\cdot\boldsymbol{\nabla} + \boldsymbol{v}'\cdot\boldsymbol{\nabla}' = (\boldsymbol{v}-\boldsymbol{v}')\cdot\boldsymbol{\nabla},$$

$$\left(\frac{1}{m_\alpha}\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{v}} - \frac{1}{m_\beta}\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{v}'}\right) f_\alpha(\boldsymbol{v}) f_\beta(\boldsymbol{v}') = -\frac{\boldsymbol{v}-\boldsymbol{v}'}{T} f_\alpha(\boldsymbol{v}) f_\beta(\boldsymbol{v}').$$

Then (IV.6.5) becomes

$$\begin{aligned}
&\big(\boldsymbol{v}-\boldsymbol{v}'\big)\cdot\boldsymbol{\nabla} g_{\alpha\beta}(\boldsymbol{r}-\boldsymbol{r}', \boldsymbol{v}, \boldsymbol{v}') \\
&\qquad + \sum_\gamma \frac{q_\alpha q_\gamma}{T} \int \mathrm{d}\boldsymbol{x}''\, \frac{\partial}{\partial \boldsymbol{r}} \frac{1}{|\boldsymbol{r}-\boldsymbol{r}''|} \cdot \boldsymbol{v} f_\alpha(\boldsymbol{v})\, g_{\beta\gamma}(\boldsymbol{r}'-\boldsymbol{r}'', \boldsymbol{v}', \boldsymbol{v}'') \\
&\qquad + \sum_\gamma \frac{q_\beta q_\gamma}{T} \int \mathrm{d}\boldsymbol{x}''\, \frac{\partial}{\partial \boldsymbol{r}'} \frac{1}{|\boldsymbol{r}'-\boldsymbol{r}''|} \cdot \boldsymbol{v}' f_\beta(\boldsymbol{v}')\, g_{\alpha\gamma}(\boldsymbol{r}-\boldsymbol{r}'', \boldsymbol{v}, \boldsymbol{v}'') \\
&\qquad = -\frac{\partial}{\partial \boldsymbol{r}} \frac{q_\alpha q_\beta}{|\boldsymbol{r}-\boldsymbol{r}'|} \cdot \frac{\boldsymbol{v}-\boldsymbol{v}'}{T} f_\alpha(\boldsymbol{v}) f_\beta(\boldsymbol{v}').
\end{aligned} \tag{IV.6.6}$$

Writing $\widehat{g}_{\alpha\beta} \doteq g_{\alpha\beta}/(f_\alpha(\boldsymbol{v}) f_\beta(\boldsymbol{v}'))$ and introducing $\boldsymbol{s} = \boldsymbol{r}-\boldsymbol{r}'$, the middle two (shielding) terms are given by

$$\sum_\gamma \frac{q_\alpha q_\gamma}{T}\int \mathrm{d}\boldsymbol{x}''\,\frac{\partial}{\partial \boldsymbol{s}}\frac{1}{|\boldsymbol{s}-\boldsymbol{s}''|}\cdot\boldsymbol{v}f_\alpha(\boldsymbol{v})\,\widehat{g}_{\beta\gamma}(\boldsymbol{s}'')f_\beta(\boldsymbol{v}')f_\gamma(\boldsymbol{v}'')$$
$$= \sum_\gamma \frac{q_\alpha q_\gamma n_\gamma}{T}\int \mathrm{d}\boldsymbol{s}''\,\frac{\partial}{\partial \boldsymbol{s}}\frac{1}{|\boldsymbol{s}-\boldsymbol{s}''|}\cdot\boldsymbol{v}f_\alpha(\boldsymbol{v})\,\widehat{g}_{\beta\gamma}(\boldsymbol{s}'')f_\beta(\boldsymbol{v}'), \tag{IV.6.7a}$$

$$\sum_\gamma \frac{q_\beta q_\gamma}{T}\int \mathrm{d}\boldsymbol{x}''\,\frac{\partial}{\partial \boldsymbol{s}'}\frac{1}{|\boldsymbol{s}'-\boldsymbol{s}''|}\cdot\boldsymbol{v}'f_\beta(\boldsymbol{v}')\,\widehat{g}_{\alpha\gamma}(\boldsymbol{s}'')f_\alpha(\boldsymbol{v})f_\gamma(\boldsymbol{v}'')$$
$$= -\sum_\gamma \frac{q_\beta q_\gamma n_\gamma}{T}\int \mathrm{d}\boldsymbol{s}''\,\frac{\partial}{\partial \boldsymbol{s}}\frac{1}{|\boldsymbol{s}-\boldsymbol{s}''|}\cdot\boldsymbol{v}'f_\beta(\boldsymbol{v}')\,\widehat{g}_{\alpha\gamma}(\boldsymbol{s}'')f_\alpha(\boldsymbol{v}), \tag{IV.6.7b}$$

respectively, so that their sum may be written as

$$\sum_\gamma \frac{q_\gamma n_\gamma}{T}\int \mathrm{d}\boldsymbol{s}''\,\frac{\partial}{\partial \boldsymbol{s}}\frac{1}{|\boldsymbol{s}-\boldsymbol{s}''|}\cdot\big(q_\alpha\boldsymbol{v}\widehat{g}_{\beta\gamma}(\boldsymbol{s}'')-q_\beta\boldsymbol{v}'\widehat{g}_{\alpha\gamma}(\boldsymbol{s}'')\big)f_\alpha(\boldsymbol{v})f_\beta(\boldsymbol{v}'). \tag{IV.6.8}$$

Equation (IV.6.6) then reads

$$(\boldsymbol{v}-\boldsymbol{v}')\cdot\boldsymbol{\nabla}\left(\widehat{g}_{\alpha\beta}(\boldsymbol{s})+\frac{q_\alpha q_\beta}{T}\frac{1}{|\boldsymbol{s}|}\right)f_\alpha(\boldsymbol{v})f_\beta(\boldsymbol{v}')$$
$$+\sum_\gamma \frac{q_\gamma n_\gamma}{T}\int \mathrm{d}\boldsymbol{s}''\,\frac{\partial}{\partial \boldsymbol{s}}\frac{1}{|\boldsymbol{s}-\boldsymbol{s}''|}\cdot\big(q_\alpha\boldsymbol{v}\widehat{g}_{\beta\gamma}(\boldsymbol{s}'')-q_\beta\boldsymbol{v}'\widehat{g}_{\alpha\gamma}(\boldsymbol{s}'')\big)f_\alpha(\boldsymbol{v})f_\beta(\boldsymbol{v}')=0. \tag{IV.6.9}$$

Because this equation must be satisfied for each $\boldsymbol{v}$ and $\boldsymbol{v}'$, we must have that

$$\sum_\gamma q_\alpha q_\gamma n_\gamma\widehat{g}_{\beta\gamma}(\boldsymbol{s}'')=\sum_\gamma q_\beta q_\gamma n_\gamma\widehat{g}_{\alpha\gamma}(\boldsymbol{s}'').$$

Then (IV.6.9) may be rearranged to obtain

$$(\boldsymbol{v}-\boldsymbol{v}')\cdot\boldsymbol{\nabla}\left(\widehat{g}_{\alpha\beta}(\boldsymbol{s})+\frac{q_\alpha q_\beta}{T}\frac{1}{|\boldsymbol{s}|}+\sum_\gamma\frac{q_\alpha q_\gamma n_\gamma}{T}\int \mathrm{d}\boldsymbol{s}''\,\frac{\widehat{g}_{\beta\gamma}(\boldsymbol{s}'')}{|\boldsymbol{s}-\boldsymbol{s}''|}\right)f_\alpha(\boldsymbol{v})f_\beta(\boldsymbol{v}')=0. \tag{IV.6.10}$$

Fourier transforming in $\boldsymbol{s}$ and requiring that the solution be valid for arbitrary $\boldsymbol{v}-\boldsymbol{v}'$ leads to

$$\widehat{g}_{\alpha\beta}(\boldsymbol{k})+\frac{q_\alpha q_\beta}{2\pi^2k^2T}+\sum_\gamma\frac{4\pi q_\alpha q_\gamma n_\gamma}{k^2T}\widehat{g}_{\beta\gamma}(\boldsymbol{k})=0. \tag{IV.6.11}$$

This is our equation for the equilibrium two-particle correlation function.

To solve (IV.6.11), first multiply by $q_\alpha n_\alpha$ and sum over α:

$$\sum_\alpha q_\alpha n_\alpha\widehat{g}_{\alpha\beta}(\boldsymbol{k})+\sum_\alpha\frac{q_\alpha^2q_\beta n_\alpha}{2\pi^2k^2T}+\left(\sum_\alpha\frac{4\pi q_\alpha^2n_\alpha}{k^2T}\right)\left(\sum_\gamma q_\gamma n_\gamma\widehat{g}_{\beta\gamma}(\boldsymbol{k})\right)=0.$$

With α and γ being dummy summation indices, this equation may be rearranged to obtain

$$\sum_\gamma q_\gamma n_\gamma\widehat{g}_{\beta\gamma}(\boldsymbol{k})=\frac{-\displaystyle\sum_\alpha\frac{q_\alpha^2q_\beta n_\alpha}{2\pi^2k^2T}}{1+\displaystyle\sum_\alpha\frac{4\pi q_\alpha^2n_\alpha}{k^2T}}, \tag{IV.6.12}$$

which may be substituted back into (IV.6.11) to find that

$$\boxed{\widehat{g}_{\alpha\beta}(\boldsymbol{k}) \doteq \frac{g_{\alpha\beta}(\boldsymbol{k},\boldsymbol{v},\boldsymbol{v}')}{f_\alpha(\boldsymbol{v})f_\beta(\boldsymbol{v}')} = -\frac{q_\alpha q_\beta}{2\pi^2 T}\frac{1}{k^2+k_\mathrm{D}^2}} \tag{IV.6.13}$$

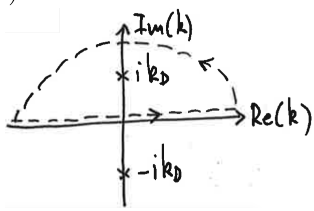
where $k_\mathrm{D}^2 \doteq \sum_\alpha 4\pi q_\alpha^2 n_\alpha/T$. Noting that $\mathcal{D}(0,\boldsymbol{k}) = 1+(k_\mathrm{D}/k)^2$, this expression may also be written as

$$\widehat{g}_{\alpha\beta}(\boldsymbol{k}) = -\frac{q_\alpha q_\beta}{2\pi^2 k^2 T}\frac{1}{\mathcal{D}(0,\boldsymbol{k})}. \tag{IV.6.14}$$

Now we take the inverse Fourier transform of (IV.6.13):

$$\begin{aligned}
\widehat{g}_{\alpha\beta}(\boldsymbol{\mathscr{r}}) &= -\frac{q_\alpha q_\beta}{2\pi^2 T}\int \mathrm{d}\boldsymbol{k}\,\frac{\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\mathscr{r}}}}{k^2+k_\mathrm{D}^2} \quad \text{with} \quad \boldsymbol{\mathscr{r}} \doteq \boldsymbol{r}-\boldsymbol{r}' \\
&= -\frac{q_\alpha q_\beta}{2\pi^2 T}\int_0^{2\pi}\mathrm{d}\phi_k \int_{-1}^{+1}\mathrm{d}\cos\theta_k \int_0^\infty \mathrm{d}k\,\frac{\mathrm{e}^{\mathrm{i}k\mathscr{r}\cos\theta_k}}{1+(k_\mathrm{D}/k)^2} \\
&= \frac{q_\alpha q_\beta}{2\pi^2 T}\frac{2\pi\mathrm{i}}{\mathscr{r}}\int_0^\infty \frac{\mathrm{d}k}{k}\,\frac{\mathrm{e}^{\mathrm{i}k\mathscr{r}}-\mathrm{e}^{-\mathrm{i}k\mathscr{r}}}{1+(k_\mathrm{D}/k)^2} \\
&= \frac{q_\alpha q_\beta}{2\pi^2 T}\frac{2\pi\mathrm{i}}{\mathscr{r}}\int_{-\infty}^\infty \mathrm{d}k\,\frac{k\,\mathrm{e}^{\mathrm{i}k\mathscr{r}}}{(k+\mathrm{i}k_\mathrm{D})(k-\mathrm{i}k_\mathrm{D})}.
\end{aligned} \tag{IV.6.15}$$

This integral can be done via contour integration. There are simple poles at $\pm\mathrm{i}k_\mathrm{D}$; the contour runs along the real axis from $-\infty$ to $+\infty$ and closes in the upper half k-plane (since $\exp(\mathrm{i}k\mathscr{r}) \to 0$ as $k \to \mathrm{i}\infty$):



Picking up the residue from the $+\mathrm{i}k_\mathrm{D}$ pole, we have

$$\boxed{\widehat{g}_{\alpha\beta}(\mathscr{r}) = -\frac{q_\alpha q_\beta}{T}\frac{\mathrm{e}^{-k_\mathrm{D}\mathscr{r}}}{\mathscr{r}}} \tag{IV.6.16}$$

Thus, the *equilibrium two-particle distribution function* is

$$\boxed{f_{\alpha\beta}(\boldsymbol{x},\boldsymbol{x}') = f_\alpha(\boldsymbol{v})f_\beta(\boldsymbol{v}')\left(1-\frac{q_\alpha q_\beta}{T}\frac{\mathrm{e}^{-k_\mathrm{D}|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}\right)} \tag{IV.6.17}$$

There are three things to note about (IV.6.17):

(1) The equilibrium two-particle correlation

$$\widehat{g}_{\alpha\beta}(\mathscr{r}) = \frac{q_\alpha q_\beta}{T}\frac{\mathrm{e}^{-k_\mathrm{D}\mathscr{r}}}{\mathscr{r}} = \frac{q_\alpha q_\beta n}{T k_\mathrm{D}^2}\frac{k_\mathrm{D}^3}{n}\frac{\mathrm{e}^{-k_\mathrm{D}\mathscr{r}}}{k_\mathrm{D}\mathscr{r}} = \frac{1}{4\pi\Lambda}\frac{q_\alpha q_\beta n}{\sum_\gamma q_\gamma^2 n_\gamma}\frac{\mathrm{e}^{-k_\mathrm{D}\mathscr{r}}}{k_\mathrm{D}\mathscr{r}} \sim \Lambda^{-1} \lll 1.$$

This is consistent with our expectation that $g_{\alpha\beta} \sim \mathcal{O}(\Lambda^{-1})$; i.e., that particle correlations are weak when Λ is large.

(2) The joint probability of finding a particle of species α at phase-space location $\boldsymbol{x}$ and a particle of species β at phase-space location $\boldsymbol{x}'$ is modified substantially

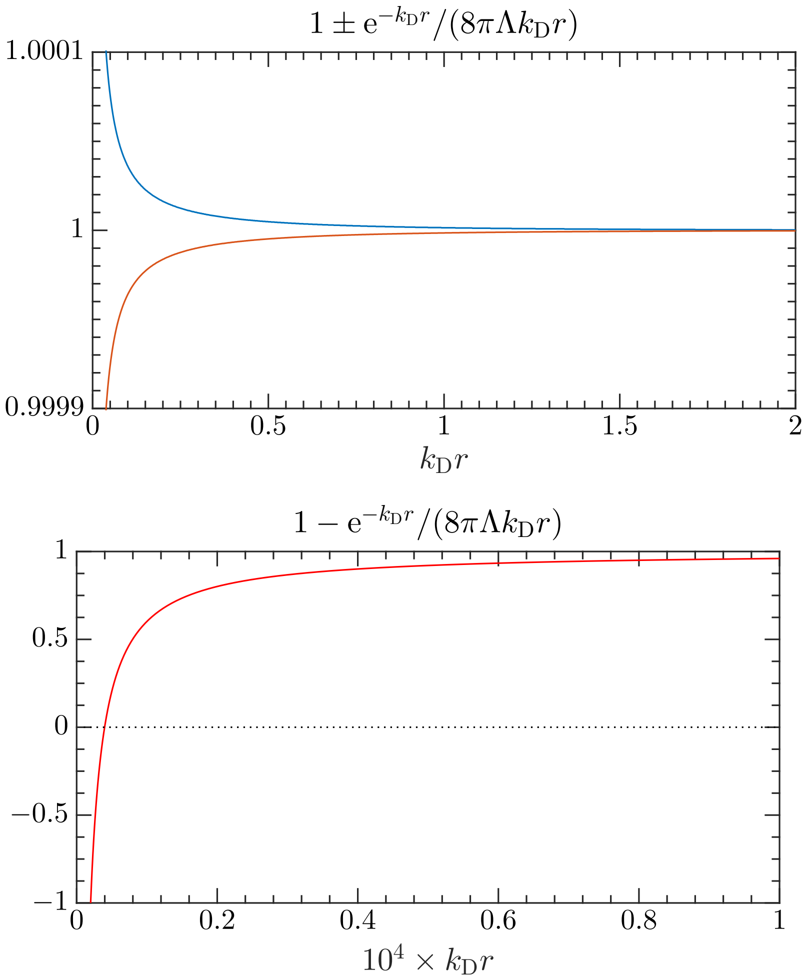


FIGURE 5. Equilibrium two-particle distribution function $\widehat{f}_{\alpha\beta}$ for a hydrogenic plasma with plasma parameter $\Lambda = 10^4$. Plus (minus) sign corresponds to unlike-(like-)signed charges. Note that the probability goes negative for like-signed charges (red), a consequence of neglecting $g_{\alpha\beta}$ in the BBGKY source term for $g_{\alpha\beta}$.

at separations $|\boldsymbol{r} - \boldsymbol{r}'| \sim \lambda_\mathrm{D}$. This is the effect of Debye shielding. Much beyond λ_D, particle β is distributed independently of particle α. This is because all other particles are correlated with particle α, which is surrounded by a cloud of radius $\sim\lambda_\mathrm{D}$ with net charge $Q = -q_\alpha$. Oppositely signed charges are statistically more likely to be found close to one another than like-signed charges.

(3) Figure 5 shows a plot of $\widehat{f}_{\alpha\beta}$ for a hydrogenic plasma,

$$1 \pm \frac{1}{8\pi\Lambda} \frac{\mathrm{e}^{-k_\mathrm{D} \mathscr{r}}}{k_\mathrm{D} \mathscr{r}},$$

with $\Lambda = 10^4$. Note that the probability shown here goes negative for like-signed charges (proton-proton or electron-electron), which is obviously not good! Where could we have gone wrong? Remember that $g_{\alpha\beta}$ we threw away in the source term because we argued it to be small relative to $f_\alpha f_\beta$? Yeah… can't do that. As $f_{\alpha\beta} \to 0$, $g_{\alpha\beta} \to -f_\alpha f_\beta$. One fix, in the spirit of Chapters 11.2.2 and 11.2.3

of Krommes (2018), is to consider a Yukawa-type potential $\varphi = (q/r)\exp(-k_{\mathrm{D}}r)$. This gives

$$\widehat{g}_{\alpha\beta}(r) = -1 + \exp\left[-\frac{q_\alpha q_\beta}{Tr}\exp(-k_{\mathrm{D}}r)\right],$$

which safely goes to -1 as $r \to 0$ for $q_\alpha q_\beta > 0$. The problem then is for oppositely signed charges ($q_\alpha q_\beta < 0$) – it blows up! As Greg Hammett put it in his lecture notes: "nothing classical can prevent electrons from collapsing onto ions with infinite negative potential energy. Only quantum effects can prevent this collapse." Read the end of Krommes (2018) Chapter 11.2.3 for interesting applications. The point here is that $\Lambda \ggg 1$ implies that the mean kinetic energy is much much greater than the mean potential energy, and that's just not true as $r \to 0$. (Recall that KE/PE $\sim \Lambda^{2/3}$.)

PART V
Discrete particle effects and the test-particle superposition principle

Collision operators capture the discrete nature of the particles comprising a plasma. Thus, to understand our collision operators, it would be prudent to investigate some consequences of this discreteness. I've already touched on some of these things, but in this part we'll dig a little deeper. The results will ultimately allow us to unpack all the formalism involved in calculating the Balescu–Lenard operator and understand fully its physical contents.

V.1. Moving test charges

Let us proceed by considering the motion of a single particle – a "test particle" – through a plasma, which is taken to be otherwise uniform and field-free. We've already said much about Debye shielding, particularly in the case of a static particle surrounded by a screening cloud. Now, we determine the properties of that cloud as the particle is moving. Most texts at this point launch into a calculation of particle motion and Vlasov dynamics, but, by virtue of our Green's function solution to the Vlasov equation, we already have everything we need. Recall (III.5.7):

$$\begin{aligned} G_{\alpha\beta}(\omega, \boldsymbol{k}, \boldsymbol{v}; \boldsymbol{v}') &= \frac{\delta_{\alpha\beta}\delta(\boldsymbol{v}-\boldsymbol{v}')}{-\mathrm{i}\omega + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{v}} \\ &\quad + \frac{q_\alpha q_\beta}{m_\alpha}\frac{4\pi\mathrm{i}\boldsymbol{k}}{k^2}\cdot\frac{\partial f_{0\alpha}}{\partial \boldsymbol{v}}\frac{1}{\mathcal{D}(\omega,\boldsymbol{k})}\frac{1}{-\mathrm{i}\omega+\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{v}'}\frac{1}{-\mathrm{i}\omega+\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{v}}. \end{aligned} \tag{V.1.1}$$

The solution for $\delta f_\alpha(t, \boldsymbol{k}, \boldsymbol{v})$ that follows from applying this Green's function is (cf. (III.5.14))

$$\begin{aligned} \delta f_\alpha(t,\boldsymbol{k},\boldsymbol{v}) &= \left[\delta f_\alpha(t=0,\boldsymbol{k},\boldsymbol{v}) - \frac{q_\alpha}{m_\alpha}\mathrm{i}\boldsymbol{k}\cdot\frac{\partial f_{0\alpha}}{\partial\boldsymbol{v}}\sum_j\frac{c_j}{-\mathrm{i}\omega_j+\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{v}}\right]\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{v}t} \\ &\quad + \frac{q_\alpha}{m_\alpha}\mathrm{i}\boldsymbol{k}\cdot\frac{\partial f_{0\alpha}}{\partial\boldsymbol{v}}\sum_j\frac{c_j\mathrm{e}^{-\mathrm{i}\omega_j t}}{-\mathrm{i}\omega_j+\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{v}}, \end{aligned} \tag{V.1.2}$$

where the c_j are the residues of ω_j determined from the solution for the potential:

$$\begin{aligned}\varphi(\omega,\boldsymbol{k}) &= \frac{4\pi}{k^2}\frac{1}{\mathcal{D}(\omega,\boldsymbol{k})}\sum_\beta q_\beta \int \mathrm{d}\boldsymbol{v}'\,\frac{\delta f_\beta(t=0,\boldsymbol{k},\boldsymbol{v}')}{-\mathrm{i}\omega+\mathrm{i}\boldsymbol{k\cdot v}'}\\ &= \sum_j \frac{c_j}{-\mathrm{i}(\omega-\omega_j)} + A(\omega),\end{aligned} \tag{V.1.3}$$

where $A(\omega)$ is the analytic piece (with $A(\boldsymbol{k\cdot v})=0$). Inverse-Laplace transforming the latter gave (cf. (III.5.12))

$$\varphi(t,\boldsymbol{k}) = \sum_j c_j \mathrm{e}^{-\mathrm{i}\omega_j t}. \tag{V.1.4}$$

For our test-particle problem, $\delta f_\alpha(t=0,\boldsymbol{k},\boldsymbol{v})=0$; i.e., the initial plasma is in equilibrium and is not yet disturbed. The distribution of the test particle at $t=0$ is

$$\delta f_{\mathrm{T}}(t=0,\boldsymbol{x}) = \delta(\boldsymbol{r}-\boldsymbol{R}_0)\delta(\boldsymbol{v}-\boldsymbol{V}_0), \tag{V.1.5}$$

where $(\boldsymbol{R}_0,\boldsymbol{V}_0)$ are the initial phase-space coordinates of the test particle. (The subscript "T" denotes "test".) The Fourier transform of (V.1.5) is

$$\delta f_{\mathrm{T}}(t=0,\boldsymbol{k},\boldsymbol{v}) = \frac{1}{(2\pi)^3}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k\cdot R}_0}\delta(\boldsymbol{v}-\boldsymbol{V}_0); \tag{V.1.6}$$

then

$$\begin{aligned}\varphi(\omega,\boldsymbol{k}) &= \frac{q_{\mathrm{T}}}{2\pi^2k^2}\frac{\mathrm{e}^{-\mathrm{i}\boldsymbol{k\cdot R}_0}}{\mathcal{D}(\omega,\boldsymbol{k})}\int\mathrm{d}\boldsymbol{v}'\,\frac{\delta(\boldsymbol{v}'-\boldsymbol{V}_0)}{-\mathrm{i}\omega+\mathrm{i}\boldsymbol{k\cdot v}'}\\ &= \frac{q_{\mathrm{T}}}{2\pi^2k^2}\frac{\mathrm{e}^{-\mathrm{i}\boldsymbol{k\cdot R}_0}}{\mathcal{D}(\omega,\boldsymbol{k})}\frac{1}{-\mathrm{i}(\omega-\boldsymbol{k\cdot V}_0)}\\ \Longrightarrow c_{\mathrm{T}} &= \frac{q_{\mathrm{T}}}{2\pi^2k^2}\frac{\mathrm{e}^{-\mathrm{i}\boldsymbol{k\cdot R}_0}}{\mathcal{D}(\boldsymbol{k\cdot V}_0,\boldsymbol{k})}\end{aligned} \tag{V.1.7}$$

for the kinetic pole $\omega=\boldsymbol{k\cdot V}_0$. Other poles give damping from $\mathcal{D}(\omega,\boldsymbol{k})=0$ eigenmodes, which don't survive for long. Thus, equation (V.1.4) becomes

$$\boxed{\varphi(t,\boldsymbol{k}) \approx \frac{q_{\mathrm{T}}}{2\pi^2k^2}\frac{\mathrm{e}^{-\mathrm{i}\boldsymbol{k\cdot R}(t)}}{\mathcal{D}(\boldsymbol{k\cdot V}_0,\boldsymbol{k})}} \tag{V.1.8}$$

at late times, where $\boldsymbol{R}(t)\doteq\boldsymbol{R}_0+\boldsymbol{V}_0t$ is the (straight) test-particle trajectory. Equation (V.1.2) for the perturbed distribution function then gives

$$\begin{aligned}\delta f_\alpha(t,\boldsymbol{k},\boldsymbol{v}) &\approx \mathrm{e}^{-\mathrm{i}\boldsymbol{k\cdot v}t}\int^t\mathrm{d}t'\,\mathrm{e}^{\mathrm{i}\boldsymbol{k\cdot v}t'}\,\frac{\mathrm{i}q_\alpha q_{\mathrm{T}}}{2\pi^2k^2}\frac{\boldsymbol{k}}{m_\alpha}\boldsymbol{\cdot}\frac{\partial f_{0\alpha}}{\partial\boldsymbol{v}}\frac{\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\boldsymbol{\cdot}(\boldsymbol{R}_0+\boldsymbol{V}_0t')}}{\mathcal{D}(\boldsymbol{k\cdot V}_0,\boldsymbol{k})}\\ &\approx \frac{q_\alpha q_{\mathrm{T}}}{2\pi^2k^2}\frac{\boldsymbol{k}}{m_\alpha}\boldsymbol{\cdot}\frac{\partial f_{0\alpha}}{\partial\boldsymbol{v}}\frac{\mathrm{e}^{-\mathrm{i}\boldsymbol{k\cdot R}(t)}}{[\boldsymbol{k}\boldsymbol{\cdot}(\boldsymbol{v}-\boldsymbol{V}_0)]\,\mathcal{D}(\boldsymbol{k\cdot V}_0,\boldsymbol{k})}.\end{aligned} \tag{V.1.9}$$

The test particle clearly generates a response in the plasma! Physically, what's going on is that the sudden presence of a moving test charge in the plasma excites a spectrum of Landau-damped plasma oscillations due to the local charge imbalance. On timescales longer than the inverse of the Landau-damping rate, what remains is an equilibrated Debye cloud that surrounds the test charge. Indeed, the argument of the dielectric function is $\boldsymbol{k\cdot V}_0$, which is the remaining zero-frequency mode Doppler-shifted to the frame of the moving test charge. We'll see that this corresponds to an equilibrated, but spatially distorted, Debye cloud.

Let us take some limiting cases by varying $\boldsymbol{V}_0$. For that, it'll help to recall the definition of $\mathcal{D}(\omega, \boldsymbol{k})$ (cf. (III.5.6)):

$$\mathcal{D}(\omega, \boldsymbol{k}) \doteq 1 + \sum_\gamma \frac{q_\gamma^2}{m_\gamma} \frac{4\pi \boldsymbol{k}}{k^2} \cdot \int \mathrm{d}\boldsymbol{v}' \, \frac{\partial f_{0\gamma}/\partial \boldsymbol{v}'}{\omega - \boldsymbol{k} \cdot \boldsymbol{v}' + \mathrm{i}0}.$$

Proceeding...

(1) Test particle in vacuum ($\mathcal{D} \to 1$):

$$\varphi(t, \boldsymbol{k}) \to \frac{q_\mathrm{T}}{2\pi^2 k^2} \, \mathrm{e}^{-\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{R}(t)}$$

$$\begin{aligned}
\Longrightarrow \varphi(t, \boldsymbol{r}) &= \int_{-1}^{+1} \mathrm{d}(\cos\theta_k) \int_0^\infty \mathrm{d}k \, 2\pi k^2 \, \frac{q_\mathrm{T}}{2\pi^2 k^2} \, \mathrm{e}^{\mathrm{i}k|\boldsymbol{r} - \boldsymbol{R}(t)| \cos\theta_k} \\
&= \frac{q_\mathrm{T}}{2\pi^2} \frac{2\pi \mathrm{i}}{|\boldsymbol{r} - \boldsymbol{R}(t)|} \int_0^\infty \frac{\mathrm{d}k}{k} \left(\mathrm{e}^{-\mathrm{i}k|\boldsymbol{r} - \boldsymbol{R}(t)|} - \mathrm{e}^{\mathrm{i}k|\boldsymbol{r} - \boldsymbol{R}(t)|} \right) \\
&= \frac{q_\mathrm{T}}{|\boldsymbol{r} - \boldsymbol{R}(t)|} \underbrace{\frac{2}{\pi} \int_0^\infty \frac{\mathrm{d}k}{k} \, \sin(k|\boldsymbol{r} - \boldsymbol{R}(t)|)}_{=1} \\
&= \frac{q_\mathrm{T}}{|\boldsymbol{r} - \boldsymbol{R}(t)|}.
\end{aligned}$$

Good! The potential of a moving charge.

(2) Slowly moving test charge ($V_0 \ll v_{\mathrm{th}\alpha}$ for all α). In this case, the dielectric function

$$\mathcal{D}(\boldsymbol{k} \cdot \boldsymbol{V}_0, \boldsymbol{k}) \to 1 - \sum_\gamma \frac{4\pi q_\gamma^2}{m_\gamma k^2} \int \mathrm{d}u \, \frac{1}{u} \frac{\partial F_{0\gamma}}{\partial u},$$

where $F_{0\gamma}(u) \doteq \int \mathrm{d}\boldsymbol{v} \, f_{0\gamma}(\boldsymbol{v}) \delta(u - \hat{\boldsymbol{k}} \cdot \boldsymbol{v})$. Defining a generic Debye wavenumber via

$$K_\mathrm{D}^2 \doteq - \sum_\gamma \frac{4\pi q_\gamma^2}{m_\gamma} \int \mathrm{d}u \, \frac{1}{u} \frac{\partial F_{0\gamma}}{\partial u},$$

which equals k_D^2 for a Maxwellian plasma, we have

$$\varphi(t, \boldsymbol{k}) \to \frac{q_\mathrm{T}}{2\pi^2 k^2} \frac{\mathrm{e}^{-\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{R}(t)}}{1 + (K_\mathrm{D}/k)^2}.$$

We've already inverse-Fourier'd such a function before (see (IV.6.16)), so we know that

$$\varphi(t, \boldsymbol{r}) \approx \frac{q_\mathrm{T}}{|\boldsymbol{r} - \boldsymbol{R}(t)|} \, \mathrm{e}^{-K_\mathrm{D}|\boldsymbol{r} - \boldsymbol{R}(t)|}, \tag{V.1.10}$$

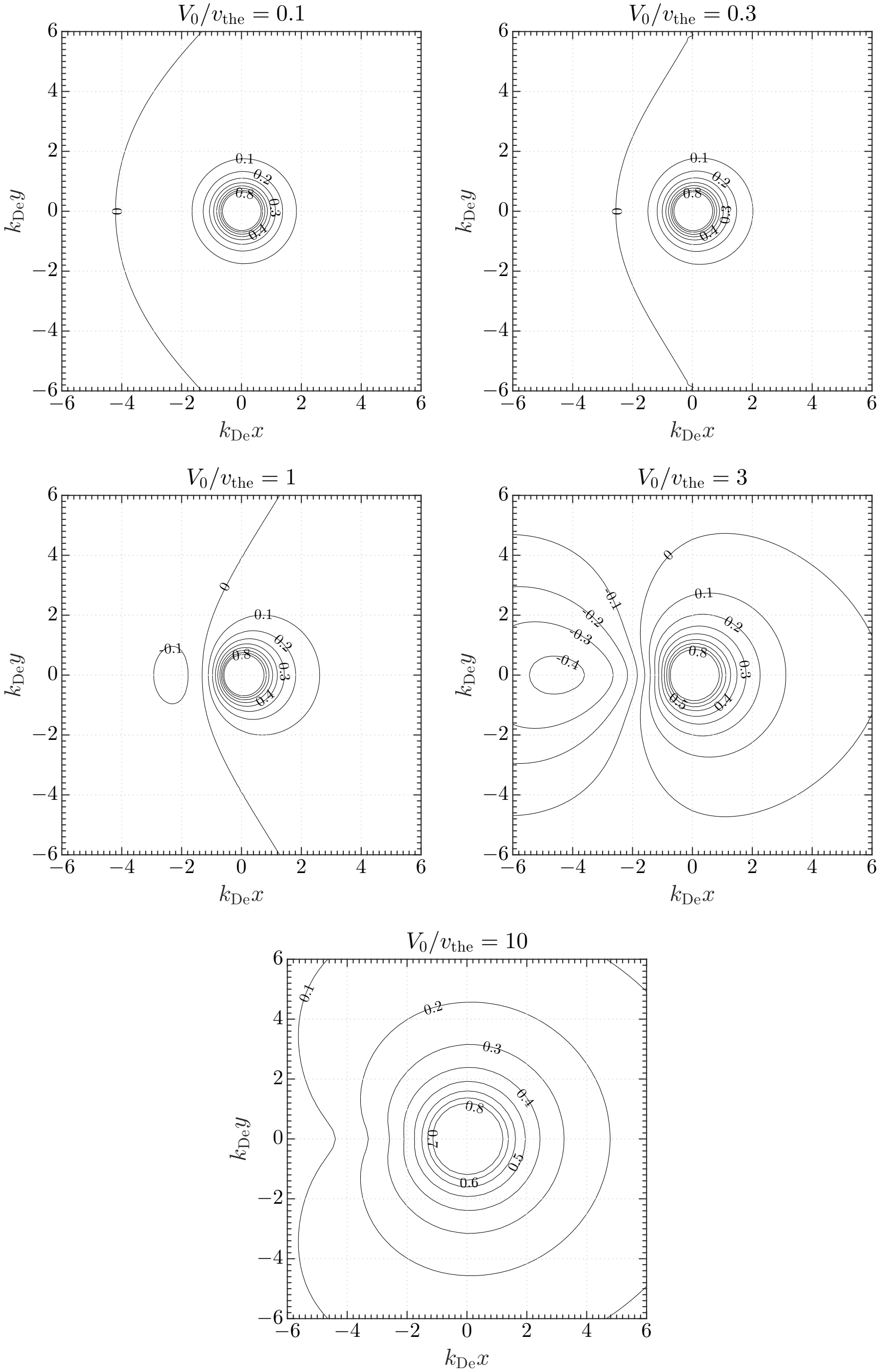
which is just a Debye-shielded moving test particle.

(3) Fast moving charge ($V_0 \gg v_{\mathrm{th}\alpha}$ for all α). In this case, $\mathcal{D} \to 1$, and so

$$\varphi(t, \boldsymbol{r}) \approx \frac{q_\mathrm{T}}{|\boldsymbol{r} - \boldsymbol{R}(t)|}, \tag{V.1.11}$$

which is just a moving charge in vacuum (to leading order). This is because the plasma cannot set up a Debye cloud fast enough.

What's going on here is that the plasma is trying to shield a charge at a location from which it has already departed. In effect, the plasma feels a retarded potential. In HW04, you will show that this potential causes a spatially distorted Debye cloud, and you will compute the first-order corrections to (V.1.10) and (V.1.11) that capture this distortion. You will also interpret the following plots of equipotential surfaces for $V_0/v_{\rm the} = 0.1$, 0.3, 1, 3, and 10, which were obtained by numerically inverse-Fourier transforming (V.1.8):



The density response of the plasma to these potentials – so called "Cerenkov wakes" – may be computed as follows. Note that Poisson's equations in Fourier space is

$$-k^2\varphi(t,\boldsymbol{k}) = -4\pi\sum_\alpha q_\alpha\delta n_\alpha(t,\boldsymbol{k}) - \frac{q_\mathrm{T}}{2\pi^2}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{R}(t)},$$

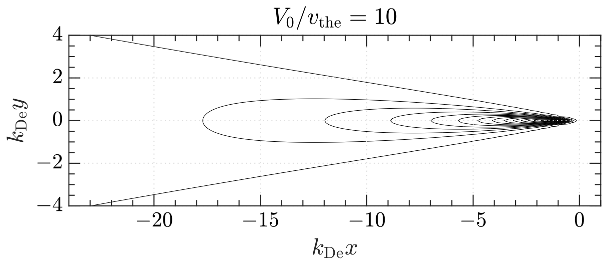
where $\boldsymbol{R}(t) = \boldsymbol{R}_0 + \boldsymbol{V}_0 t$ is the position of the test charge and $\delta n_\alpha \doteq \int \mathrm{d}\boldsymbol{v}\,\delta f_\alpha$. Using (V.1.8), the density response may be written as

$$\sum_\alpha q_\alpha\delta n_\alpha(t,\boldsymbol{k}) = \frac{q_\mathrm{T}}{(2\pi)^3}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{R}(t)}\left[\frac{1}{\mathcal{D}(\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{V}_0,\boldsymbol{k})} - 1\right]. \tag{V.1.12}$$

Inverse-Fourier transforming then provides

$$\sum_\alpha q_\alpha\delta n_\alpha(t,\boldsymbol{\mathscr{r}}) = \frac{q_\mathrm{T}}{(2\pi)^3}\int \mathrm{d}\boldsymbol{k}\,\exp(\mathrm{i}\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{\mathscr{r}})\left[\frac{1}{\mathcal{D}(\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{V}_0,\boldsymbol{k})} - 1\right], \tag{V.1.13}$$

where $\boldsymbol{\mathscr{r}} = \boldsymbol{r} - \boldsymbol{R}(t)$. Numerically evaluating this integral for $V_0/v_\mathrm{the} = 10$ shows that the density fluctuations in the plasma form a conic structure reminiscent of a Mach cone from a supersonic aircraft, with the density fluctuations confined inside $|y/x| \approx C/V_0 \ll 1$ with $C = \sqrt{3}v_\mathrm{the}$:



Some additional references you can check out on this topic are Perkins (1965), Wang *et al.* (1981), and Dewar (2010).

V.2. Electric-field fluctuations

Here we calculate the electric field generated by a moving test charge as it sweeps through a (responsive) plasma. We have seen in the last section that a moving test charge excites waves at $\omega = \boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{V}_0$ (see (V.1.8)), and that these waves are subject to Landau damping when there is an appreciable fraction of the particles in the plasma with velocities $\boldsymbol{v}$ satisfying $\hat{\boldsymbol{k}}\boldsymbol{\cdot}(\boldsymbol{v} - \boldsymbol{V}_0) \approx 0$. Thus, waves are being emitted and absorbed. A *detailed balance* emerges between emission and absorption, which in turn implies a steady-state level of electric-field fluctuations. We will compute this level.

From (V.1.8), we have

$$\boldsymbol{E}(t,\boldsymbol{k}) = -\mathrm{i}\boldsymbol{k}\varphi(t,\boldsymbol{k}) = -\mathrm{i}\boldsymbol{k}\,\frac{q_\mathrm{T}}{2\pi^2k^2}\,\frac{\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{R}(t)}}{\mathcal{D}(\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{V}_0,\boldsymbol{k})}, \tag{V.2.1}$$

so that

$$\boldsymbol{E}(t,\boldsymbol{r}) = \int \mathrm{d}\boldsymbol{k}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{r}}\boldsymbol{E}(t,\boldsymbol{k}) = -\mathrm{i}\frac{q_\mathrm{T}}{2\pi^2}\int \mathrm{d}\boldsymbol{k}\,\frac{\boldsymbol{k}}{k^2}\,\frac{\mathrm{e}^{\mathrm{i}\boldsymbol{k}\boldsymbol{\cdot}(\boldsymbol{r}-\boldsymbol{R}(t))}}{\mathcal{D}(\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{V}_0,\boldsymbol{k})}. \tag{V.2.2}$$

The total energy density in this field is given by

$$\frac{\boldsymbol{E}(t,\boldsymbol{r})\boldsymbol{\cdot}\boldsymbol{E}^*(t,\boldsymbol{r})}{8\pi} = \frac{q_\mathrm{T}^2}{(2\pi)^5}\left[\int \mathrm{d}\boldsymbol{k}\,\frac{\boldsymbol{k}}{k^2}\,\frac{\mathrm{e}^{\mathrm{i}\boldsymbol{k}\boldsymbol{\cdot}(\boldsymbol{r}-\boldsymbol{R}(t))}}{\mathcal{D}(\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{V}_0,\boldsymbol{k})}\right]\boldsymbol{\cdot}\left[\int \mathrm{d}\boldsymbol{k}'\,\frac{\boldsymbol{k}'}{k'^2}\,\frac{\mathrm{e}^{\mathrm{i}\boldsymbol{k}'\boldsymbol{\cdot}(\boldsymbol{r}-\boldsymbol{R}(t))}}{\mathcal{D}(\boldsymbol{k}'\boldsymbol{\cdot}\boldsymbol{V}_0,\boldsymbol{k}')}\right]^*, \tag{V.2.3}$$

where $*$ denotes the complex conjugate. We want the average energy density in the plasma, and for that we do something special: we consider each and every particle as a test charge. These "test" charges are completely uncorrelated, uniformly distributed in space, and distributed in velocity according to $f_{0\alpha}(\boldsymbol{v})$. This *test-particle superposition principle* (TPSP) works because the leading-order effects in the particle correlations have already been taken into account in the construction of each shielded ("dressed") test particle. Ichimaru puts it this way (with his equation numbers and notation changed to conform to mine):

> In terms of the Bogoliubov hierarchy (§I.1) on the characteristic timescales, we may describe the superposition principle in the following way. Let us again fix on a charged particle in the plasma, which we regard as a moving test charge. It will act to polarize the medium and so carry a screening cloud with it. This action corresponds to that establishment of a pair correlation; the characteristic time associated with such a process has been denoted by $\sim\omega_{\rm p}^{-1}$. In the course of its motion, however, the test charge collides with other field particles; its energy and momentum change abruptly. The polarization cloud originally associated with the test charge will no longer represent a screening cloud appropriate to the new situation; the polarization cloud must adjust itself to these revised circumstances. If the mean free time $\sim\nu^{-1}$ between such collisions is much longer than $\sim\omega_{\rm p}^{-1}$ in such an event, the adjustment will take place quickly so that, for most of the time between two successive short-range collisions, the test charge can be considered as carrying a well-established cloud. It is clear that the *superposition calculation ... amounts to assuming that each particle, not almost always, but always carries such an equilibrated screening cloud* [emphasis added]. We thus argue that the superposition principle represents a good approximation as long as $\omega_{\rm p} \gg \nu$. For a stable plasma in weak coupling, such a condition is well satisfied.

Thus, we may compute the ensemble-averaged electric energy density by performing

$$\begin{aligned}
\left\langle \frac{\boldsymbol{E}(t,\boldsymbol{r})\cdot\boldsymbol{E}^*(t,\boldsymbol{r})}{8\pi} \right\rangle &\doteq \sum_\alpha \int \mathrm{d}\boldsymbol{V}_0 \int \mathrm{d}\boldsymbol{R}_0\, f_{0\alpha}(\boldsymbol{R}_0,\boldsymbol{V}_0)\, \frac{\boldsymbol{E}(t,\boldsymbol{r})\cdot\boldsymbol{E}^*(t,\boldsymbol{r})}{8\pi} \\
&= \sum_\alpha \frac{q_\alpha^2}{(2\pi)^5} \int \mathrm{d}\boldsymbol{V}_0\, f_{0\alpha}(\boldsymbol{V}_0) \int \mathrm{d}\boldsymbol{R}_0 \left[\int \mathrm{d}\boldsymbol{k}\, \frac{\boldsymbol{k}}{k^2} \frac{\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{R}(t))}}{\mathcal{D}(\boldsymbol{k}\cdot\boldsymbol{V}_0,\boldsymbol{k})} \right] \\
&\quad \cdot \left[\int \mathrm{d}\boldsymbol{k}'\, \frac{\boldsymbol{k}'}{k'^2} \frac{\mathrm{e}^{\mathrm{i}\boldsymbol{k}'\cdot(\boldsymbol{r}-\boldsymbol{R}(t))}}{\mathcal{D}(\boldsymbol{k}'\cdot\boldsymbol{V}_0,\boldsymbol{k}')} \right]^* \\
&= \sum_\alpha \frac{q_\alpha^2}{(2\pi)^5} \int \mathrm{d}\boldsymbol{V}_0\, f_{0\alpha}(\boldsymbol{V}_0) \int \mathrm{d}\boldsymbol{k} \int \mathrm{d}\boldsymbol{k}'\, \frac{\boldsymbol{k}\cdot\boldsymbol{k}'}{k^2 k'^2} \\
&\quad \times \frac{\mathrm{e}^{\mathrm{i}(\boldsymbol{k}-\boldsymbol{k}')\cdot(\boldsymbol{r}-\boldsymbol{V}_0 t)}}{\mathcal{D}(\boldsymbol{k}\cdot\boldsymbol{V}_0,\boldsymbol{k})\mathcal{D}^*(\boldsymbol{k}'\cdot\boldsymbol{V}_0,\boldsymbol{k}')} \underbrace{\int \mathrm{d}\boldsymbol{R}_0\, \mathrm{e}^{-\mathrm{i}(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{R}_0}}_{=\,(2\pi)^3\,\delta(\boldsymbol{k}-\boldsymbol{k}')} \\
&= \sum_\alpha \frac{q_\alpha^2}{(2\pi)^2} \int \mathrm{d}\boldsymbol{V}_0\, f_{0\alpha}(\boldsymbol{V}_0) \int \mathrm{d}\boldsymbol{k}\, \frac{1}{k^2} \frac{1}{|\mathcal{D}(\boldsymbol{k}\cdot\boldsymbol{V}_0,\boldsymbol{k})|^2}.
\end{aligned} \tag{V.2.4}$$

Defining, as usual, the one-dimensional distribution function $F_{0\alpha}(u) \doteq \int \mathrm{d}\boldsymbol{V}_0\, f_{0\alpha}(\boldsymbol{V}_0)\delta(u - \hat{\boldsymbol{k}}\cdot\boldsymbol{V}_0)$ and introducing $\omega = ku$,

$$(\text{V.2.4}) = \sum_\alpha \frac{q_\alpha^2}{2\pi} \int \frac{\mathrm{d}\omega}{2\pi} \int \mathrm{d}\boldsymbol{k}\, \frac{1}{k^3} \frac{F_{0\alpha}(\omega/k)}{|\mathcal{D}(\omega,k)|^2} \doteq \int \frac{\mathrm{d}\omega}{2\pi} \int \mathrm{d}\boldsymbol{k}\, W_{\omega,k}, \tag{V.2.5}$$

where

$$W_{\omega,k} \doteq \sum_\alpha \frac{q_\alpha^2}{2\pi k^3} \frac{F_{0\alpha}(\omega/k)}{|\mathcal{D}(\omega,k)|^2} \tag{V.2.6}$$

is the spectral density of the field fluctuations. Thus,

$$\boxed{W \doteq \left\langle \frac{\boldsymbol{E}(t,\boldsymbol{r}) \cdot \boldsymbol{E}^*(t,\boldsymbol{r})}{8\pi} \right\rangle = \int \frac{\mathrm{d}\omega}{2\pi} \int \mathrm{d}\boldsymbol{k}\, W_{\omega,k}} \tag{V.2.7}$$

is the ensemble-averaged electric energy density, derived by considering the plasma as an ensemble of dressed test particles.

Let us evaluate (V.2.7) for a Maxwellian background, for which

$$F_{0\alpha}\left(\frac{\omega}{k}\right) = \frac{n_\alpha}{\sqrt{\pi} v_{\mathrm{th}\alpha}} \exp\left(-\frac{\omega^2}{k^2 v_{\mathrm{th}\alpha}^2}\right).$$

Equation (V.2.7) becomes

$$W = \sum_\alpha \frac{q_\alpha^2 n_\alpha}{2\pi^{3/2} v_{\mathrm{th}\alpha}} \int \frac{\mathrm{d}\omega}{2\pi} \int \mathrm{d}\boldsymbol{k}\, \frac{1}{k^3} \frac{\mathrm{e}^{-(\omega/k v_{\mathrm{th}\alpha})^2}}{|\mathcal{D}(\omega,k)|^2}, \tag{V.2.8}$$

with

$$\mathcal{D}(\omega,k) = 1 + \sum_\alpha \frac{k_{\mathrm{D}\alpha}^2}{k^2}\left[1 + \zeta_\alpha Z(\zeta_\alpha)\right]. \tag{V.2.9}$$

Here, $k_{\mathrm{D}\alpha}^2 \doteq 4\pi q_\alpha^2 n_\alpha / T$, $\zeta_\alpha \doteq \omega / |k| v_{\mathrm{th}\alpha}$, and

$$Z(\zeta) \doteq \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \mathrm{d}x\, \frac{\mathrm{e}^{-x^2}}{x - \zeta} \tag{V.2.10}$$

is the plasma dispersion function. Equation (V.2.8) looks like a nasty integral. But there's a trick! Note that

$$\mathrm{Im}\left[\mathcal{D}(\omega,k)\right] = \sum_\alpha \frac{k_{\mathrm{D}\alpha}^2}{k^2} \zeta_\alpha \mathrm{Im}\left[Z(\zeta_\alpha)\right] = \sum_\alpha \frac{k_{\mathrm{D}\alpha}^2}{k^2} \zeta_\alpha \sqrt{\pi} \mathrm{e}^{-\zeta_\alpha^2}.$$

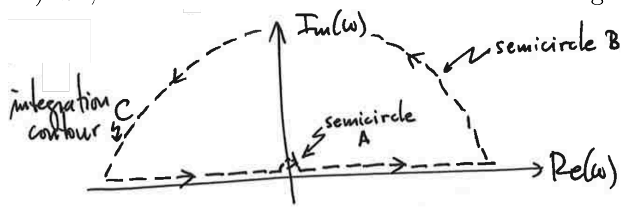
Then, (V.2.8) becomes

$$W = \int \frac{\mathrm{d}\omega}{2\pi} \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3} \frac{T}{\omega} \frac{\mathrm{Im}\left[\mathcal{D}(\omega,k)\right]}{|\mathcal{D}(\omega,k)|^2} = -\int \frac{\mathrm{d}\omega}{2\pi} \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3} \frac{T}{\omega} \mathrm{Im}\left[\frac{1}{\mathcal{D}(\omega,k)}\right]. \tag{V.2.11}$$

Believe it or not, this integral can be done. Recalling that ω is real, move it inside the Im operator and consider the following integral:

$$\mathrm{PV} \int \frac{\mathrm{d}\omega}{2\pi} \mathrm{Im}\left[\frac{1}{\omega \mathcal{D}(\omega,k)}\right], \tag{V.2.12}$$

where PV denotes the principal value. All of its poles are in the lower half ω-plane (since the plasma is stable). So, let's take a contour that does the following:



Write the integral (V.2.12) as

$$\int_C \frac{\mathrm{d}\omega}{2\pi}\,\mathrm{Im}\left[\frac{1}{\omega\mathcal{D}(\omega,k)}\right] = \int_{-\infty}^{-\epsilon} + \int_A + \int_\epsilon^\infty + \int_B$$

and take $\epsilon \to 0$ and the B part of the contour to ∞. Rearranging, equation (V.2.12) becomes

$$\lim_{\epsilon\to 0}\left\{\left(\int_{-\infty}^{-\epsilon} + \int_\epsilon^\infty\right)\frac{\mathrm{d}\omega}{2\pi}\,\mathrm{Im}\left[\frac{1}{\omega\mathcal{D}(\omega,k)}\right]\right\} = \left(\int_C - \int_A - \int_B\right)\frac{\mathrm{d}\omega}{2\pi}\,\mathrm{Im}\left[\frac{1}{\omega\mathcal{D}(\omega,k)}\right].$$

The C contour integration yields zero, since there are no poles enclosed. To do the A and B contour integrals, write $\omega = r\exp(\mathrm{i}\theta)$; note furthermore that the Im operator can be taken outside of these integrals. Each of these are then:

$$\int_A \frac{\mathrm{d}\omega}{2\pi}\,\mathrm{Im}\left[\frac{1}{\omega\mathcal{D}(\omega,k)}\right] = \mathrm{Im}\int_\pi^0 \frac{\mathrm{i}\mathrm{d}\theta}{2\pi}\,\frac{1}{\mathcal{D}(\epsilon,k)} = -\frac{1}{2}\frac{1}{\mathcal{D}(0,k)},$$

$$\int_B \frac{\mathrm{d}\omega}{2\pi}\,\mathrm{Im}\left[\frac{1}{\omega\mathcal{D}(\omega,k)}\right] = \mathrm{Im}\int_0^\pi \frac{\mathrm{i}\mathrm{d}\theta}{2\pi}\,\frac{1}{\mathcal{D}(\infty,k)} = +\frac{1}{2}\frac{1}{\mathcal{D}(\infty,k)},$$

and so (V.2.12) becomes

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\,\mathrm{Im}\left[\frac{1}{\omega\mathcal{D}(\omega,k)}\right] = \frac{1}{2}\left[\frac{1}{\mathcal{D}(0,k)} - \frac{1}{\mathcal{D}(\infty,k)}\right].$$

Now, $\mathcal{D}(\infty,k) = 1$ and $\mathcal{D}(0,k) = 1 + (k_\mathrm{D}/k)^2$. So, equation (V.2.8) is equal to

$$\boxed{W = \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\,\frac{T}{2}\,\frac{1}{1+(k/k_\mathrm{D})^2}} \tag{V.2.13}$$

The manipulations above are related to the Kramers–Kronig relations. Namely, consider a function $S(\omega)$ that is analytic in the upper half ω plane and vanishes sufficiently rapidly at infinity. Then by Cauchy's residue theorem,

$$\int_{-\infty}^{\infty} \mathrm{d}\omega'\,\frac{S(\omega')}{\omega'-\omega+\mathrm{i}\epsilon} = 0, \quad \epsilon > 0.$$

Use Plemelj's formula

$$\frac{1}{\omega'-\omega\pm\mathrm{i}\epsilon} = \mathrm{PV}\,\frac{1}{\omega'-\omega} \mp \mathrm{i}\pi\delta(\omega'-\omega)$$

to get

$$\mathrm{i}\pi S(\omega) = \mathrm{PV}\int \mathrm{d}\omega'\,\frac{S(\omega')}{\omega'-\omega}.$$

Take the imaginary part of this to obtain

$$\mathrm{Re}\,S(\omega) = \frac{1}{\pi}\mathrm{PV}\int \mathrm{d}\omega'\,\frac{\mathrm{Im}\,S(\omega')}{\omega'-\omega}.$$

(This is one of the Kramers-Kronig relations.) In our case, we have $S = 1/\mathcal{D} - 1$ where $\mathcal{D} = \mathcal{D}(\omega)$ is the dielectric function; note that S vanishes at infinity, since $\mathcal{D}(\infty) = 1$. Now set $\omega = 0$. Then the above expression becomes

$$\mathrm{Re}\left[\frac{1}{\mathcal{D}(0)} - 1\right] = \frac{1}{\pi}\mathrm{PV}\int \mathrm{d}\omega'\,\frac{1}{\omega'}\,\mathrm{Im}\left[\frac{1}{\mathcal{D}(\omega')}\right] = \frac{1}{\pi}\mathrm{PV}\int \mathrm{d}\omega'\,\mathrm{Im}\left[\frac{1}{\omega'\mathcal{D}(\omega')}\right].$$

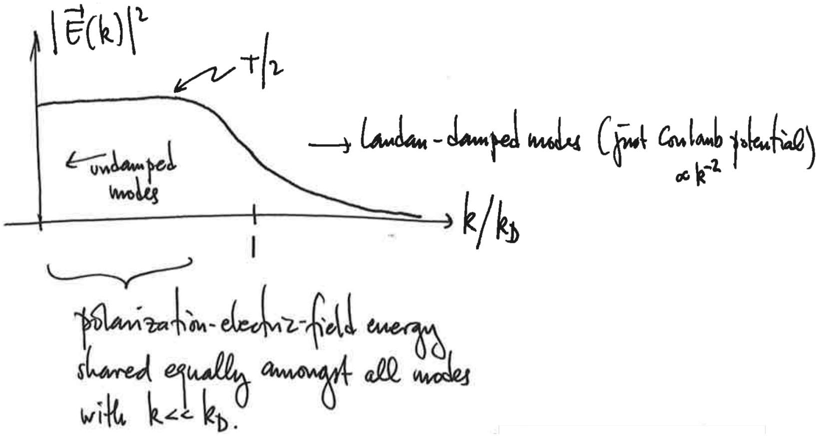
But $\mathcal{D}(0)$ is already real, so

$$\frac{1}{\mathcal{D}(0)} - 1 = \frac{1}{\pi}\mathrm{PV}\int \mathrm{d}\omega'\,\mathrm{Im}\left[\frac{1}{\omega'\mathcal{D}(\omega')}\right]$$

This leads to the same result as above, namely (V.2.13).

Three important notes regarding (V.2.13):

(1) All modes with $k \ll k_\mathrm{D}$ are equally probable (i.e., equally excited with the same energy – that energy is $T/2$, half of the thermal energy). That the energy is $T/2$ agrees with the equipartition theorem in statistical mechanics that each degree of freedom acquires $T/2$ worth of average energy. See the figure below:



(2) This implies

$$W_{\omega,k} = -\frac{T}{\omega}\mathrm{Im}\left[\frac{1}{\mathcal{D}(\omega,k)}\right],$$

which is a consequence of the *fluctuation-dissipation theorem*. The fluctuation level and the damping rate are related in such a way that there is $T/2$ of energy per degree of freedom (a standard result from equilibrium statistical mechanics). This is worth memorizing.

(3) Let's try to do the k integral in (V.2.13):

$$W = \int_0^\infty \frac{\mathrm{d}k\,4\pi k^2}{(2\pi)^3}\,\frac{T}{2}\,\frac{1}{1+(k/k_\mathrm{D})^2} \Longrightarrow \frac{W}{nT} = \frac{1}{4\pi^2\Lambda}\int_0^\infty \mathrm{d}x\,\frac{x^2}{1+x^2} \to \infty.$$

Ah! It diverges! Why? To find out, write the integral as

$$\begin{aligned}\int_0^\infty \mathrm{d}x\,\frac{-1+1+x^2}{1+x^2} &= -\int_0^\infty \mathrm{d}x\,\frac{1}{1+x^2} + \int_0^\infty \mathrm{d}x \\ &= \underbrace{-\frac{\pi}{2}}_{\substack{\text{energy due to}\\ \text{correlations}\\ \text{(ask: why the}\\ \text{minus sign?)}}} + \underbrace{\int_0^\infty \mathrm{d}x}_{\substack{\text{energy due to}\\ \text{uncorrelated}\\ \text{particles (infinite}\\ \text{self-energy of}\\ \text{an unshielded}\\ \text{charged particle)}}}\end{aligned}$$

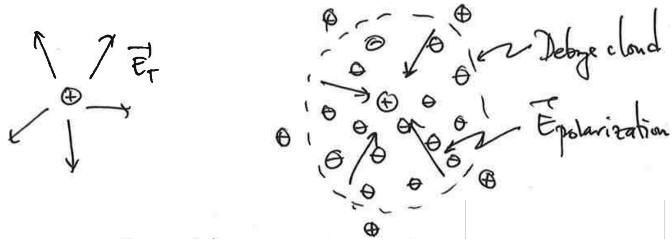
So,

$$\frac{\delta W}{nT} \doteq \frac{W - W_{\text{self}}}{nT} = -\frac{1}{8\pi\Lambda} \sim \mathcal{O}(\Lambda^{-1})$$

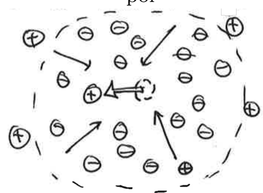
$\Longrightarrow$ more particles, less thermal noise, less scattering $\to$ Vlasov

V.3. Polarization drag

In the last two sections, we saw that moving (test) charges excite electric-field fluctuations, which enter into a balance between generation and dissipation from Landau damping. But from where did the power to support these waves come? Here we show that such particles lose energy due to polarization drag. This "drag" comes from the fact that the velocity of a particle affects its ability to acquire an effective Debye cloud. This Debye cloud, in turn, affects the (test) particle:



Debye shielding takes $\sim\omega_{\text{p}}^{-1}$ to set up. This generates $\boldsymbol{E}_{\text{pol}}$ due to charge redistribution. If, in that time, the test charge has moved, then the charge will be off-center to the cloud, and the charge will see a headwind from $\boldsymbol{E}_{\text{pol}}$:



Thus, there will be a component of $\boldsymbol{E}_{\text{pol}}$ directed against the test-particle's motion. This is called *polarization drag.*

The drag force is simply

$$\begin{aligned}
\boldsymbol{F}_{\text{pol}} &= q_{\text{T}}\boldsymbol{E}_{\text{pol}} \\
&= q_{\text{T}}\left[\begin{array}{c}\text{total electric field generated by a moving test charge as it sweeps} \\ \text{through a plasma minus the electric field from the bare test charge}\end{array}\right] \\
&= -\frac{\mathrm{i}q_{\text{T}}^2}{2\pi^2}\int \mathrm{d}\boldsymbol{k}\,\frac{\boldsymbol{k}}{k^2}\left[\frac{1}{\mathcal{D}(\boldsymbol{k\cdot V}_0,\boldsymbol{k})} - 1\right],
\end{aligned} \tag{V.3.1}$$

using (V.2.2). Writing the integrand as $(1/2)[\text{original} + (\boldsymbol{k}\to-\boldsymbol{k})]$ gives

$$\begin{aligned}
\boldsymbol{F}_{\text{pol}} &= -\frac{\mathrm{i}q_{\text{T}}^2}{4\pi^2}\int \mathrm{d}\boldsymbol{k}\,\frac{\boldsymbol{k}}{k^2}\left[\frac{1}{\mathcal{D}(\boldsymbol{k\cdot V}_0,\boldsymbol{k})} - \frac{1}{\mathcal{D}^*(\boldsymbol{k\cdot V}_0,\boldsymbol{k})}\right] \\
&= -\frac{\mathrm{i}q_{\text{T}}^2}{4\pi^2}\int \mathrm{d}\boldsymbol{k}\,\frac{\boldsymbol{k}}{k^2}\left[\frac{\mathcal{D}^* - \mathcal{D}}{|\mathcal{D}|^2}\right] = -\frac{\mathrm{i}q_{\text{T}}^2}{4\pi^2}\int \mathrm{d}\boldsymbol{k}\,\frac{\boldsymbol{k}}{k^2}\left[\frac{-2\mathrm{i}\,\mathrm{Im}\mathcal{D}}{|\mathcal{D}|^2}\right] \\
&= \frac{q_{\text{T}}^2}{2\pi^2}\int \mathrm{d}\boldsymbol{k}\,\frac{\boldsymbol{k}}{k^2}\,\mathrm{Im}\left[\frac{1}{\mathcal{D}(\boldsymbol{k\cdot V}_0,\boldsymbol{k})}\right].
\end{aligned} \tag{V.3.2}$$

The particle will lose energy due to this drag force at a rate

$$\frac{\mathrm{d}W_{\mathrm{T}}}{\mathrm{d}t} = \boldsymbol{F}_{\mathrm{pol}} \cdot \boldsymbol{V}_0 = \frac{q_{\mathrm{T}}^2}{2\pi^2} \int \mathrm{d}\boldsymbol{k}\, \frac{\boldsymbol{k} \cdot \boldsymbol{V}_0}{k^2} \operatorname{Im}\left[\frac{1}{\mathcal{D}(\boldsymbol{k} \cdot \boldsymbol{V}_0, \boldsymbol{k})}\right]. \tag{V.3.3}$$

Next, we adopt the TPSP introduced in the previous section and average this power over the distribution of "test" charges:

$$P = \left\langle \frac{\mathrm{d}W_{\mathrm{T}}}{\mathrm{d}t} \right\rangle = \sum_\alpha \frac{q_\alpha^2}{2\pi^2} \int \mathrm{d}\boldsymbol{V}_0\, f_{0\alpha}(\boldsymbol{V}_0) \int \mathrm{d}\boldsymbol{k}\, \frac{\boldsymbol{k} \cdot \boldsymbol{V}_0}{k^2} \operatorname{Im}\left[\frac{1}{\mathcal{D}(\boldsymbol{k} \cdot \boldsymbol{V}_0, \boldsymbol{k})}\right]. \tag{V.3.4}$$

Recalling the definition $F_{0\alpha}(u) \doteq \int \mathrm{d}\boldsymbol{v}\, f_{0\alpha}(\boldsymbol{v}) \delta(u - \hat{\boldsymbol{k}} \cdot \boldsymbol{v})$, equation (V.3.4) becomes (with $\omega = ku$)

$$P = \left\langle \frac{\mathrm{d}W_{\mathrm{T}}}{\mathrm{d}t} \right\rangle = \int \frac{\mathrm{d}\omega}{2\pi} \int \mathrm{d}\boldsymbol{k}\, P_{\omega,k} \quad \text{with} \quad P_{\omega,k} \doteq \sum_\alpha \frac{q_\alpha^2}{\pi} \frac{\omega}{k^3} \operatorname{Im}\left[\frac{F_{0\alpha}(\omega/k)}{\mathcal{D}(\omega, \boldsymbol{k})}\right]. \tag{V.3.5}$$

As in the last section, consider a Maxwellian background. We have

$$\sum_\alpha \frac{q_\alpha^2}{\pi} \frac{\omega}{k^3} F_{0\alpha}\left(\frac{\omega}{k}\right) = \frac{T}{4\pi^3} \operatorname{Im}\big[D(\omega, k)\big],$$

and so

$$P = \int \frac{\mathrm{d}\omega}{2\pi} \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\, 2T \operatorname{Im}\big[\mathcal{D}(\omega, k)\big] \operatorname{Im}\left[\frac{1}{\mathcal{D}(\omega, k)}\right]. \tag{V.3.6}$$

Comparing this with (V.2.11), we see that

$$\boxed{P_{\omega,k} + 2\omega W_{\omega,k} \operatorname{Im}\big[\mathcal{D}(\omega, k)\big] = 0} \tag{V.3.7}$$

This states that the power lost by the particles due to polarization drag is gained by the emitted (and Landau-damped) waves. This kind of balance between emission and absorption is called Kirchoff's law.

We can check this explicitly by recalling the discussion at the end of the Vlasov section (§III.6): for a weakly damped, stable mode, the decay rate is (see (III.7.20))

$$\gamma_{\boldsymbol{k}} = -\frac{\operatorname{Im} \mathcal{D}(\omega, \boldsymbol{k})}{\dfrac{\partial}{\partial \omega} \operatorname{Re} \mathcal{D}(\omega, \boldsymbol{k})} = -\frac{\left[-\displaystyle\sum_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{k^2} \frac{\pi}{n_\alpha} F'_{0\alpha}\left(\frac{\omega}{k}\right)\right]}{\left(-\dfrac{2}{\omega}\right)}.$$

Using this in (V.3.7) gives

$$\begin{aligned} P_{\omega,k} &= -2\omega W_{\omega,k} \operatorname{Im}\big[\mathcal{D}(\omega, k)\big] \\ &= -2\omega W_{\omega,k} \left[-\gamma_{\boldsymbol{k}} \left(-\frac{2}{\omega}\right)\right] \\ &= -4\gamma_{\boldsymbol{k}} W_{\omega,k} \\ &= -2\frac{\mathrm{d}W_{\omega,k}}{\mathrm{d}t} = -\frac{\mathrm{d}W_{\omega,k;\mathrm{wave}}}{\mathrm{d}t}. \end{aligned} \tag{V.3.8}$$

[The factor of 2 in the last line is there because the total wave energy (= electrostatic energy + mechanical energy) is divided evenly amongst the electrostatic and mechanical energies.]

Another way of computing the power is to calculate

$$W = 2(2\pi)^4 \,\mathrm{Im} \int \frac{\mathrm{d}\omega}{2\pi} \int \mathrm{d}\boldsymbol{k}\, \frac{|\hat{\boldsymbol{k}} \cdot \boldsymbol{J}_{\mathrm{T}}(\omega, \boldsymbol{k})|^2}{\omega \mathcal{D}(\omega, \boldsymbol{k})},$$

which is the radiation emitted by a test current (i.e., the current density of the test particle) over a time $\mathcal{T}$. (This will be derived by you in HW04.) Then, with

$$\boldsymbol{J}_{\mathrm{T}}(t, \boldsymbol{r}) = q_{\mathrm{T}} \boldsymbol{V}_0 \delta(\boldsymbol{r} - \boldsymbol{R}_0 - \boldsymbol{V}_0 t),$$

we have

$$\begin{aligned}
\boldsymbol{J}_{\mathrm{T}}(\omega, \boldsymbol{k}) &= \int \frac{\mathrm{d}\boldsymbol{r}}{(2\pi)^3} \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{r} + \mathrm{i}\omega t} \boldsymbol{J}_{\mathrm{T}}(t, \boldsymbol{r}) \\
&= \frac{q_{\mathrm{T}} \boldsymbol{V}_0}{(2\pi)^3} \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{R}_0} \mathrm{e}^{-\mathrm{i}(\omega - \boldsymbol{k} \cdot \boldsymbol{V}_0)t} \\
&= \frac{q_{\mathrm{T}} \boldsymbol{V}_0}{(2\pi)^2} \mathrm{e}^{-\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{R}_0} \delta(\omega - \boldsymbol{k} \cdot \boldsymbol{V}_0).
\end{aligned}$$

The rate of radiation – the power – is $W/\mathcal{T}$, where $\mathcal{T}$ is the time ($\to \infty$) over which the radiation is emitted. Then the power from a single test particle is

$$\lim_{\mathcal{T} \to \infty} \frac{2}{\mathcal{T}} \,\mathrm{Im} \int \frac{\mathrm{d}\omega}{2\pi} \int \mathrm{d}\boldsymbol{k}\, (\hat{\boldsymbol{k}} \cdot \boldsymbol{V}_0)^2 \frac{q_{\mathrm{T}}^2 |\delta(\omega - \boldsymbol{k} \cdot \boldsymbol{V}_0)|^2}{\omega \mathcal{D}(\omega, \boldsymbol{k})}.$$

We learned how to square a delta function in §III.6 (see (III.7.15), which applies when the time interval is $\mathcal{T}$): $|\delta(\omega)|^2 = (\mathcal{T}/2\pi)\delta(\omega)$. So the power is finite:

$$\begin{aligned}
P &= \mathrm{Im} \int \frac{\mathrm{d}\omega}{2\pi} \int \mathrm{d}\boldsymbol{k}\, (\hat{\boldsymbol{k}} \cdot \boldsymbol{V}_0)^2 \frac{q_{\mathrm{T}}^2}{\pi} \frac{\delta(\omega - \boldsymbol{k} \cdot \boldsymbol{V}_0)}{\omega \mathcal{D}(\omega, \boldsymbol{k})} \\
&= \frac{q_{\mathrm{T}}^2}{2\pi^2} \,\mathrm{Im} \int \mathrm{d}\boldsymbol{k}\, \frac{\boldsymbol{k} \cdot \boldsymbol{V}_0}{k^2} \frac{1}{\mathcal{D}(\boldsymbol{k} \cdot \boldsymbol{V}_0, \boldsymbol{k})},
\end{aligned} \tag{V.3.9}$$

which is precisely what we have from (V.3.3).

Let's explore an example application of (V.3.2): the drag on a fast test ion in a Maxwellian plasma. Assume $v_{\mathrm{th}i} \ll V_0 \ll v_{\mathrm{th}e}$ for the test ion. Then $\mathcal{D}(\boldsymbol{k} \cdot \boldsymbol{V}_0, \boldsymbol{k}) \approx 1 + (k_{\mathrm{D}e}/k)^2$ and

$$\begin{aligned}
\mathrm{Im}\, \mathcal{D}(\boldsymbol{k} \cdot \boldsymbol{V}_0, \boldsymbol{k}) &= -\sum_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{k^2} \frac{\pi}{n_\alpha} F_{0\alpha}'(\hat{\boldsymbol{k}} \cdot \boldsymbol{V}_0) \\
&= -\sum_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{k^2} \frac{\pi}{n_\alpha} \frac{n_\alpha}{\sqrt{\pi} v_{\mathrm{th}\alpha}} \left(-\frac{2\hat{\boldsymbol{k}} \cdot \boldsymbol{V}_0}{v_{\mathrm{th}\alpha}^2} \right) \mathrm{e}^{-(\hat{\boldsymbol{k}} \cdot \boldsymbol{V}_0 / v_{\mathrm{th}\alpha})^2} \\
&= \sum_\alpha \frac{k_{\mathrm{D}\alpha}^2}{k^2} \frac{\pi}{n_\alpha} \hat{\boldsymbol{k}} \cdot \boldsymbol{V}_0 F_{\mathrm{M}\alpha}(\hat{\boldsymbol{k}} \cdot \boldsymbol{V}_0) \\
&\approx \sqrt{\pi} \frac{k_{\mathrm{D}e}^2}{k^2} \frac{\hat{\boldsymbol{k}} \cdot \boldsymbol{V}_0}{v_{\mathrm{th}e}} \quad \text{using } v_{\mathrm{th}i} \ll V_0 \ll v_{\mathrm{th}e}.
\end{aligned}$$

Then the acceleration of an ion (charge $q_i = Ze$) due to polarization drag is (see (V.3.2))

$$\begin{aligned}
\boldsymbol{a}_{\mathrm{pol}} = \frac{\boldsymbol{F}_{\mathrm{pol}}}{m_i} &= -\frac{Z^2 e^2}{2\pi^2 m_i} \int \mathrm{d}\boldsymbol{k}\, \frac{\boldsymbol{k}}{k^2} \sqrt{\pi} \frac{k_{\mathrm{D}e}^2}{k^2} \frac{\hat{\boldsymbol{k}} \cdot \boldsymbol{V}_0}{v_{\mathrm{th}e}} \left(1 + \frac{k_{\mathrm{D}e}^2}{k^2} \right)^{-2} \\
&= -\frac{Z^2 e^2 \sqrt{\pi} k_{\mathrm{D}e}^2}{2\pi^2 m_i v_{\mathrm{th}e}} \int \mathrm{d}\boldsymbol{k}\, \boldsymbol{k} \frac{\hat{\boldsymbol{k}} \cdot \boldsymbol{V}_0}{k^4} \left(1 + \frac{k_{\mathrm{D}e}^2}{k^2} \right)^{-2}.
\end{aligned}$$

Dotting this expression into $\widehat{\boldsymbol{V}_0} \doteq \boldsymbol{V}_0/V_0$ yields

$$\begin{aligned}
\boldsymbol{a}_{\rm pol} \boldsymbol{\cdot} \widehat{\boldsymbol{V}_0} &= -\frac{Z^2 e^2 \sqrt{\pi}}{2\pi^2 m_i v_{\rm the}} \frac{4\pi e^2 n_e}{T_e} \times 2\pi V_0 \int_{-1}^{+1} {\rm d}\mu\, \mu^2 \int_0^\infty {\rm d}k\, k^2 \frac{1}{k^3} \left(1 + \frac{k_{\rm De}^2}{k^2}\right)^{-2} \\
&= -V_0 \frac{4\sqrt{\pi} Z^2 e^4 n_e}{m_i v_{\rm the} T_e} \times \frac{2}{3} \underbrace{\int_0^\infty \frac{{\rm d}k}{k} \left(1 + \frac{k_{\rm De}^2}{k^2}\right)^{-2}}_{\substack{= -\frac{1}{2} + \frac{1}{2}\ln\left(\frac{k_{\rm De}^2 + \infty^2}{k_{\rm De}^2}\right) \\ \text{must cut off wavenumber} \\ \text{integration! Set } \infty \to k_{\rm max}, \\ \text{so that this is } \simeq \ln \lambda_{ie}.}} \\
&= -V_0 \frac{8\sqrt{\pi} Z^2 e^4 n_e \ln \lambda_{ie}}{3 m_i v_{\rm the} T_e} = -V_0 \frac{4\sqrt{2\pi} m_e^{1/2} Z^2 e^4 n_e \ln \lambda_{ie}}{3 m_i T_e^{3/2}}
\end{aligned}$$

$$\Longrightarrow \boxed{\boldsymbol{a}_{\rm pol} = -\boldsymbol{V}_0 \left(\frac{4\sqrt{2\pi} m_e^{1/2} Z^2 e^4 n_e \ln \lambda_{ie}}{3 m_i T_e^{3/2}}\right)} \tag{V.3.10}$$

Look in your NRL formulary, page 37... this is

$$\boldsymbol{a}_{\rm pol} = -\nu_{ie} \boldsymbol{V}_0 \quad \text{with} \quad \nu_{ie} \doteq \frac{m_e n_e}{m_i n_i} \frac{1}{\tau_{ei}}, \ \tau_{ei} \doteq \frac{3\sqrt{m_e} T_e^{3/2}}{4\sqrt{2\pi} n_i Z^2 e^4 \ln \lambda_{ie}}. \tag{V.3.11}$$

This is our first glimpse of transport coefficients! Also, a strong suggestion that polarization drag is related to collisions. Stay tuned.

V.4. Summary: Recovery of Balescu–Lenard

Before proceeding, let us recapitulate the last few sections. We learned that moving "test" charges in a plasma radiate plasma waves, and that this is because the asymmetry of a Debye cloud around a moving charge pulls back on the charge. We learned that these radiated waves are Landau damped by the plasma, and that a balance emerges between emission and absorption. This provides a minimum amount of thermal "noise" in the plasma. That this noise is precisely that which is responsible for "collisions" in a plasma will soon be shown. We also invoked the test-particle superposition principle (TPSP) to treat all the particles in a plasma as simultaneously being in the bath and being test charges. This led to a picture of a plasma in which Debye clouds – rather than Vlasov particles – are statistically independent.

I must admit that this is hard to picture. Each "test" particle serves as the nucleus of a Debye cloud, and each particle serves as a member of another "test" particle's Debye cloud. This means that the "dressed" particles can (and do) overlap, even though such clouds do not interact. This is really a statement about Λ being large, but not so large that Vlasov is an adequate description.

From this picture, we found that each "test particle" with charge q_α and velocity $\boldsymbol{v}$ feels a velocity-dependent polarization electric field given by (see (V.3.2))

$$\boldsymbol{E}_{\rm pol}(\boldsymbol{v}) = -\frac{q_\alpha}{2\pi^2} \int {\rm d}\boldsymbol{k}\, \frac{\boldsymbol{k}}{k^2} \frac{{\rm Im}\big[\mathcal{D}(\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{v}, \boldsymbol{k})\big]}{|\mathcal{D}(\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{v}, \boldsymbol{k})|^2}, \tag{V.4.1}$$

where

$$\mathrm{Im}\big[\mathcal{D}(\boldsymbol{k}\cdot\boldsymbol{v},\boldsymbol{k})\big] = -\sum_\beta \frac{4\pi^2 q_\beta^2}{m_\beta}\frac{\boldsymbol{k}}{k^2}\cdot\frac{\partial}{\partial\boldsymbol{v}}\int \mathrm{d}\boldsymbol{v}'\, f_\beta(\boldsymbol{v}')\delta(\boldsymbol{k}\cdot\boldsymbol{v}'-\boldsymbol{k}\cdot\boldsymbol{v}). \tag{V.4.2}$$

Combining (V.4.1) and (V.4.2) and exploiting the symmetries of the delta function in the latter, we see that each particle in the plasma feels a force

$$q_\alpha \boldsymbol{E}_{\mathrm{pol}}(\boldsymbol{v}) = \sum_\beta \frac{1}{m_\beta}\int\frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\,\boldsymbol{k}\boldsymbol{k}\left|\frac{4\pi q_\alpha q_\beta}{k^2\mathcal{D}(\boldsymbol{k}\cdot\boldsymbol{v},\boldsymbol{k})}\right|^2\int\mathrm{d}\boldsymbol{v}'\,\pi\delta(\boldsymbol{k}\cdot\boldsymbol{v}-\boldsymbol{k}\cdot\boldsymbol{v}')\cdot\frac{\partial f_\beta}{\partial\boldsymbol{v}'}. \tag{V.4.3}$$

Store this in your short-term memory; you'll need it in about 5 minutes.

We also found that each of these test particles excites a fluctuating electric field in the responsive background plasma, which is given by (see (V.2.2))

$$\begin{aligned}
\delta\boldsymbol{E}(t,\boldsymbol{r}) &= -\mathrm{i}\frac{q_\alpha}{2\pi^2}\int\mathrm{d}\boldsymbol{k}\,\frac{\boldsymbol{k}}{k^2}\frac{\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{R}_0-\boldsymbol{v}t)}}{\mathcal{D}(\boldsymbol{k}\cdot\boldsymbol{v},\boldsymbol{k})} \\
&= -\mathrm{i}\frac{q_\alpha}{2\pi^2}\int\frac{\mathrm{d}\omega}{2\pi}\int\mathrm{d}\boldsymbol{k}\,\frac{\boldsymbol{k}}{k^2}\frac{\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{R}_0)-\mathrm{i}\omega t}}{\mathcal{D}(\omega,\boldsymbol{k})}2\pi\delta(\omega-\boldsymbol{k}\cdot\boldsymbol{v}) && \text{(V.4.4}a\text{)} \\
&\doteq \int\frac{\mathrm{d}\omega}{2\pi}\int\mathrm{d}\boldsymbol{k}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}-\mathrm{i}\omega t}\delta\boldsymbol{E}(\omega,\boldsymbol{k}), && \text{(V.4.4}b\text{)}
\end{aligned}$$

where $\boldsymbol{R}_0$ is the initial position of the test charge. Each of these test particles must in turn navigate through the resulting sea of such electric-field fluctuations, which have been generated by all the other dressed "test particles" (whose initial positions are randomly distributed). Now, at this point in the course, you might not understand why I'm about to calculate what I'm about to calculate, but there is a surprise waiting for you at the end, so hold tight. Ensemble-average $\delta\boldsymbol{E}(t,\boldsymbol{r})\delta\boldsymbol{E}^*(0,\boldsymbol{r})$ over the distribution of particle initial positions $\boldsymbol{R}_0$ and velocities $\boldsymbol{v}$:

$$
\begin{aligned}
\left\langle \delta\boldsymbol{E}(t,\boldsymbol{r})\delta\boldsymbol{E}^*(0,\boldsymbol{r})\right\rangle &\doteq \sum_\beta \int \mathrm{d}\boldsymbol{v}' \int \mathrm{d}\boldsymbol{R}_0\, f_\beta(\boldsymbol{R}_0,\boldsymbol{v}')\, \delta\boldsymbol{E}(t,\boldsymbol{r})\delta\boldsymbol{E}^*(0,\boldsymbol{r}) \\
&= \sum_\beta \int \mathrm{d}\boldsymbol{v}' \int \mathrm{d}\boldsymbol{R}_0\, f_\beta(\boldsymbol{R}_0,\boldsymbol{v}') \\
&\quad \times \frac{q_\beta^2}{(2\pi^2)^2} \int \mathrm{d}\boldsymbol{k} \int \mathrm{d}\boldsymbol{k}'\, \frac{\boldsymbol{k}\boldsymbol{k}'}{k^2k'^2}\, \frac{\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{R}_0-\boldsymbol{v}'t)}}{\mathcal{D}(\boldsymbol{k}\cdot\boldsymbol{v}',\boldsymbol{k})}\, \frac{\mathrm{e}^{-\mathrm{i}\boldsymbol{k}'\cdot(\boldsymbol{r}-\boldsymbol{R}_0)}}{\mathcal{D}^*(\boldsymbol{k}'\cdot\boldsymbol{v}',\boldsymbol{k}')} \\
&= \sum_\beta \int \mathrm{d}\boldsymbol{v}'\, f_\beta(\boldsymbol{v}')\, \frac{q_\beta^2}{(2\pi^2)^2} \int \mathrm{d}\boldsymbol{k} \int \mathrm{d}\boldsymbol{k}'\, \frac{\boldsymbol{k}\boldsymbol{k}'}{k^2k'^2}\, \frac{\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{v}'t}}{\mathcal{D}(\boldsymbol{k}\cdot\boldsymbol{v}',\boldsymbol{k})\mathcal{D}^*(\boldsymbol{k}'\cdot\boldsymbol{v}',\boldsymbol{k}')} \\
&\quad \times \underbrace{\int \mathrm{d}\boldsymbol{R}_0\, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}-\boldsymbol{k}')\cdot(\boldsymbol{r}-\boldsymbol{R}_0)}}_{=\,(2\pi)^3\delta(\boldsymbol{k}-\boldsymbol{k}')} \\
&= \sum_\beta \int \mathrm{d}\boldsymbol{v}'\, f_\beta(\boldsymbol{v}') \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\, \boldsymbol{k}\boldsymbol{k} \left| \frac{4\pi q_\beta}{k^2\mathcal{D}(\boldsymbol{k}\cdot\boldsymbol{v}',\boldsymbol{k})} \right|^2 \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{v}'t} \\
&= \int \frac{\mathrm{d}\omega}{2\pi}\, \mathrm{e}^{-\mathrm{i}\omega t} \sum_\beta \int \mathrm{d}\boldsymbol{v}'\, f_\beta(\boldsymbol{v}') \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\, \boldsymbol{k}\boldsymbol{k} \left| \frac{4\pi q_\beta}{k^2\mathcal{D}(\omega,\boldsymbol{k})} \right|^2 2\pi\delta(\omega-\boldsymbol{k}\cdot\boldsymbol{v}') \qquad \text{(V.4.5}a\text{)} \\
&\doteq \int \frac{\mathrm{d}\omega}{2\pi}\, \mathrm{e}^{-\mathrm{i}\omega t} \left\langle \delta\boldsymbol{E}\delta\boldsymbol{E}^*\right\rangle_\omega, \qquad \text{(V.4.5}b\text{)}
\end{aligned}
$$

where $\left\langle \delta\boldsymbol{E}\delta\boldsymbol{E}^*\right\rangle_\omega$ is the *fluctuation spectrum* of the fluctuating electric field. It satisfies

$$
\left\langle \delta\boldsymbol{E}\delta\boldsymbol{E}^*\right\rangle_{\omega=\boldsymbol{k}\cdot\boldsymbol{v}} = \sum_\beta \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\, \boldsymbol{k}\boldsymbol{k} \left| \frac{4\pi q_\beta}{k^2\mathcal{D}(\boldsymbol{k}\cdot\boldsymbol{v},\boldsymbol{k})} \right|^2 \int \mathrm{d}\boldsymbol{v}'\, 2\pi\delta(\boldsymbol{k}\cdot\boldsymbol{v}-\boldsymbol{k}\cdot\boldsymbol{v}') f_\beta(\boldsymbol{v}'). \qquad \text{(V.4.6)}
$$

You'll soon see (§VI.3) why I calculated this particular quantity.

Now, dig (V.4.3) out of your short-term memory and consider it alongside (V.4.6). These formulae should look eerily familiar. Go *all the way back* to our derivation of the Balescu–Lenard equation. Comparing these to (IV.4.4), it becomes apparent that

$$
\boldsymbol{\mathsf{B}}_\alpha = \left(\frac{q_\alpha}{m_\alpha}\right)^2 \left\langle \delta\boldsymbol{E}\delta\boldsymbol{E}^*\right\rangle_{\omega=\boldsymbol{k}\cdot\boldsymbol{v}} \qquad \text{(V.4.7)}
$$

and

$$
\boldsymbol{A}_\alpha = \frac{q_\alpha}{m_\alpha}\boldsymbol{E}_{\mathrm{pol}} + \frac{1}{2}\frac{\partial}{\partial\boldsymbol{v}}\cdot\boldsymbol{\mathsf{B}}_\alpha(\boldsymbol{v}). \qquad \text{(V.4.8)}
$$

Ah! So, we have discovered that the Balescu-Lenard operator (IV.4.5) accounts for polarization drag, radiation of plasma waves, and the effects of particle motion on the efficacy of Debye shielding. Now, what is the meaning of $\boldsymbol{A}_\alpha$ and $\boldsymbol{\mathsf{B}}_\alpha$? For that, we make a foray into Fokker-Planck theory. . .

PART VI

The Master equation and Fokker-Planck theory

Take therefore no thought for the morrow:
for the morrow shall take thought for the things of itself.

Matthew 6:34

A common thread running through all we have done thus far – dropping three-particle correlations, obtaining the Balescu–Lenard collision operator, the test-particle superposition principle – can be traced back to HW01 and the fact that multiple small-angle Coulomb scatterings are more important than a single large-angle scattering (provided $\Lambda \ggg 1$). In fact, this idea was hidden in the calculation of discrete-particle effects; notice that the trajectory of the test charge was a straight line (a consequence of using the Vlasov response function). Another important theme, borne out in the calculation of Balescu–Lenard from the BBGKY hierarchy, was that the two-particle correlation function $g_{\alpha\beta}$ is sourced by uncorrelated particles (recall $S_{\alpha\beta} \propto f_\alpha f_\beta$) and that, on the timescale on which $g_{\alpha\beta}$ relaxes, f_α and f_β are assumed to be temporally constant.

In this part, we will show that the latter theme is equivalent to the Markov assumption and that the former theme leads to the Fokker-Planck equation. Not surprisingly, then, we will be able to recover the Balescu–Lenard operator via such a Fokker-Planck equation using the lessons learned in using the test-particle superposition principle.

VI.1. The Chapman–Kolmogorov equation

As a result of these scattering events, particle orbits acquire a random, or probabilistic, nature. A given phase-space point does not have a unique mapping through time, but rather has a probability of arriving in a certain phase-space volume at some later time. Let us then define the transition probability density $\Delta t\, W(t+\Delta t, \boldsymbol{x}; t, \boldsymbol{x}-\Delta\boldsymbol{x})$ to be the probability that a particle changes its phase-space coordinate $\boldsymbol{x}-\Delta\boldsymbol{x}$ at time t by an amount $\Delta\boldsymbol{x}$ (not necessarily small) so that it arrives at the phase-space coordinate $\boldsymbol{x}$ after a time interval Δt. The distribution function at time $t+\Delta t$ is then obtained by summing up all possible origins of $f(t+\Delta t, \boldsymbol{x})$, weighted by their transition probabilities; the result is the *Chapman–Kolmogorov equation*

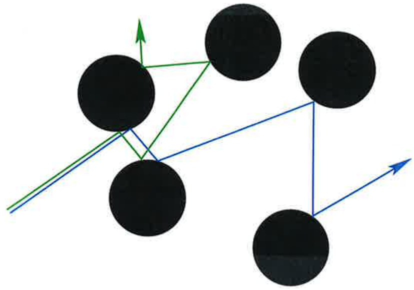
$$\boxed{f(t+\Delta t, \boldsymbol{x}) = \Delta t \int \mathrm{d}(\Delta\boldsymbol{x})\, W(t+\Delta t, \boldsymbol{x}; t, \boldsymbol{x}-\Delta\boldsymbol{x})\, f(t, \boldsymbol{x}-\Delta\boldsymbol{x})} \tag{VI.1.1}$$

All particles end up somewhere, of course, and so

$$\Delta t \int \mathrm{d}(\Delta\boldsymbol{x})\, W(t+\Delta t, \boldsymbol{x}+\Delta\boldsymbol{x}; t, \boldsymbol{x}) = 1. \tag{VI.1.2}$$

(Note that things are scaled such that W is a *rate*.)

The assumption here is that the future only depends on now, and not the past. This is the *Markov assumption*. That this should hold true is not at all obvious, but can be argued for as follows. Consider two billiard balls that somehow have become correlated in their motion (perhaps due to a recent collision), so that their trajectories are aligned with one another. Put a bunch of randomly distributed obstacles (say, other billiard balls) on the pool table. Here is one possible outcome (after Krommes (2018, figure 4.2)):



Obviously the particles are correlated on a collision timescale (not Markovian). But after many collision times, the trajectories diverge wildly, and the situation is very nearly Markovian. This is also a demonstration of extreme sensitivity to initial conditions, and that even completely deterministic trajectories can wander randomly over the entire available phase space. (A classic example of this is Brownian motion, to which we will return shortly.) With $\Lambda \ggg 1$, it is easy to imagine a plasma being close to Markovian, since there are so many particles affecting the two-particle correlations; things become chaotic pretty quickly!

This is how irreversibility enters the picture. But Chapman–Kolmogorov does not require irreversibility. Indeed, we didn't discount some "sneaky conspiracy in the motions" of the individual particles (Carroll 2010). Note that, if the transition probability is taken to be a delta function along the self-consistent Vlasov characteristic $\boldsymbol{x}' = \boldsymbol{x}'(t, \boldsymbol{x})$, i.e.,

$$\Delta t \, W(t + \Delta t, \boldsymbol{x}; t, \boldsymbol{x} - \Delta\boldsymbol{x}) = \delta(\boldsymbol{x} - \Delta\boldsymbol{x} - \boldsymbol{x}'(t; t + \Delta t, \boldsymbol{x})),$$

then

$$f(t + \Delta t, \boldsymbol{x}) = f(t, \boldsymbol{x}'(t; t + \Delta t, \boldsymbol{x})),$$

which is just the Vlasov equation (f = constant along characteristics). Only phase-space coordinates along the characteristic feed into the future distribution function. The differential form of the Vlasov equation can be obtained by taking the $\Delta t \to 0$ limit (see below). This is important to note, since nothing irreversible is necessarily built into the Chapman–Kolmogorov equation. But, for smooth, non-singular transition probabilities (i.e., no sneaky conspiracies), irreversibility is implied by (VI.1.1).

A differential version of (VI.1.1) can be obtained in the $\Delta t \to$ "0" limit as follows:[9]

$$\begin{aligned} \lim_{\Delta t \to \text{“0”}} & \frac{f(t + \Delta t, \boldsymbol{x}) - f(t, \boldsymbol{x})}{\Delta t} \\ &= \lim_{\Delta t \to \text{“0”}} \int \mathrm{d}(\Delta\boldsymbol{x}) \, \Big[\underbrace{W(t + \Delta t, \boldsymbol{x}; t, \boldsymbol{x} - \Delta\boldsymbol{x}) f(t, \boldsymbol{x} - \Delta\boldsymbol{x})}_{\text{“in”}} \\ &\qquad - \underbrace{W(t + \Delta t, \boldsymbol{x} + \Delta\boldsymbol{x}; t, \boldsymbol{x}) f(t, \boldsymbol{x})}_{\text{“out”}} \Big], \end{aligned} \tag{VI.1.3}$$

or

$$\boxed{\frac{\partial f(t, \boldsymbol{x})}{\partial t} = \int \mathrm{d}\overline{\boldsymbol{x}} \, \big[W(\boldsymbol{x}; t, \overline{\boldsymbol{x}}) f(t, \overline{\boldsymbol{x}}) - W(\overline{\boldsymbol{x}}; t, \boldsymbol{x}) f(t, \boldsymbol{x}) \big]} \tag{VI.1.4}$$

This is the *master equation*. It describes the evolution of $f(t, \boldsymbol{x})$ due to inward transition

[9] $\Delta t \to$ "0" means to take t to 0, but not smaller than ω_{p}^{-1}; the process must be nearly Markovian.

from all other occupied states $\overline{\boldsymbol{x}}$ at time t into $\boldsymbol{x}$ (first term) and due to outward transitions from $\boldsymbol{x}$ at time t into all possible states $\overline{\boldsymbol{x}}$ (second term).

A bit of kinetic theory history... The Boltzmann equation,

$$\frac{\mathrm{D}f_\alpha}{\mathrm{D}t} = \sum_\beta \int \mathrm{d}\boldsymbol{v}' \int \mathrm{d}\Omega \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right) |\boldsymbol{v}-\boldsymbol{v}'| \bigl[f_\alpha(t,\boldsymbol{r},\boldsymbol{w}) f_\beta(t,\boldsymbol{r},\boldsymbol{w}') - f_\alpha(t,\boldsymbol{r},\boldsymbol{v}) f_\beta(t,\boldsymbol{r},\boldsymbol{v}')\bigr], \tag{VI.1.5}$$

where $\boldsymbol{w}$ and $\boldsymbol{w}'$ are the velocities of the α and β particles after a collision and $\mathrm{d}\sigma/\mathrm{d}\Omega$ is the differential cross section (see HW01), is an example of a master equation (perhaps *the* example). The Markov assumption entered through what Boltzmann called the *Stosszahlansatz* = "collision number hypothesis" or, less literal but perhaps more descriptive, "molecular chaos hypothesis"; this is an assumption that all particles enter into a collision uncorrelated (and so prior collisions do not affect future collisions). We have already remarked on this in the context of Bogoliubov's hypothesis.

VI.2. The Fokker-Planck equation

Now we leverage our knowledge about collisions in a weakly coupled Coulombic plasma – that they are predominantly *small-angle* collisions. While not particularly necessary, we also follow the assumption, used in the derivation of the Balescu–Lenard equation, of a homogeneous plasma. Then (VI.1.4) can be Taylor expanded about $\boldsymbol{v}$ in small $\Delta\boldsymbol{v}$ using

$$\begin{aligned} W(t+\Delta t, \boldsymbol{v}; t, \boldsymbol{v}-\Delta\boldsymbol{v}) &= W(t+\Delta t, \boldsymbol{v}+\Delta\boldsymbol{v}-\Delta\boldsymbol{v}; t, \boldsymbol{v}-\Delta\boldsymbol{v}) \\ &\approx W - \Delta\boldsymbol{v}\boldsymbol{\cdot}\frac{\partial W}{\partial \boldsymbol{v}} + \frac{1}{2}\Delta\boldsymbol{v}\Delta\boldsymbol{v} \boldsymbol{:} \frac{\partial^2 W}{\partial\boldsymbol{v}\partial\boldsymbol{v}} \end{aligned}$$

and

$$f(t,\boldsymbol{v}-\Delta\boldsymbol{v}) \approx f - \Delta\boldsymbol{v}\boldsymbol{\cdot}\frac{\partial f}{\partial\boldsymbol{v}} + \frac{1}{2}\Delta\boldsymbol{v}\Delta\boldsymbol{v}\boldsymbol{:}\frac{\partial^2 f}{\partial\boldsymbol{v}\partial\boldsymbol{v}}$$

to find

$$\frac{\partial f}{\partial t} \approx \int \mathrm{d}(\Delta\boldsymbol{v}) \left[-\Delta\boldsymbol{v}\boldsymbol{\cdot}\frac{\partial}{\partial\boldsymbol{v}}(Wf) + \frac{1}{2}\Delta\boldsymbol{v}\Delta\boldsymbol{v}\boldsymbol{:}\frac{\partial}{\partial\boldsymbol{v}}\frac{\partial}{\partial\boldsymbol{v}}(Wf)\right], \tag{VI.2.1}$$

where $f = f(t,\boldsymbol{v})$ and $W = W(t+\Delta t, \boldsymbol{v}+\Delta\boldsymbol{v}; t, \boldsymbol{v})$.[10] Adopting the shorthand notation

$$\boldsymbol{A} \doteq \int \mathrm{d}(\Delta\boldsymbol{v})\, \Delta\boldsymbol{v}\, W, \tag{VI.2.2a}$$

$$\boldsymbol{B} \doteq \int \mathrm{d}(\Delta\boldsymbol{v})\, \Delta\boldsymbol{v}\Delta\boldsymbol{v}\, W, \tag{VI.2.2b}$$

equation (VI.2.1) may be written as

$$\boxed{\frac{\partial f}{\partial t} \approx -\frac{\partial}{\partial\boldsymbol{v}}\boldsymbol{\cdot}(\boldsymbol{A}f) + \frac{1}{2}\frac{\partial}{\partial\boldsymbol{v}}\frac{\partial}{\partial\boldsymbol{v}}\boldsymbol{:}(\boldsymbol{B}f)} \tag{VI.2.3}$$

This should look familiar! (Recall (IV.4.5).) This is the *Fokker-Planck equation*.

[10] $\boldsymbol{AB}\boldsymbol{:}\boldsymbol{CD} = \sum_{ij} A_i B_j C_i D_j$.

Sometimes the Fokker-Planck coefficients (VI.2.2) are written as

$$\boldsymbol{A} = \lim_{\Delta t \to \text{“}0\text{”}} \frac{\langle \Delta \boldsymbol{v} \rangle}{\Delta t}, \tag{VI.2.4a}$$

$$\mathsf{\boldsymbol{B}} = \lim_{\Delta t \to \text{“}0\text{”}} \frac{\langle \Delta \boldsymbol{v} \Delta \boldsymbol{v} \rangle}{\Delta t}, \tag{VI.2.4b}$$

where

$$\langle \Delta \boldsymbol{v} \rangle = \Delta t \int \mathrm{d}(\Delta \boldsymbol{v})\, \Delta \boldsymbol{v}\, W, \tag{VI.2.5a}$$

$$\langle \Delta \boldsymbol{v} \Delta \boldsymbol{v} \rangle = \Delta t \int \mathrm{d}(\Delta \boldsymbol{v})\, \Delta \boldsymbol{v} \Delta \boldsymbol{v}\, W \tag{VI.2.5b}$$

are the *jump moments*. We need to calculate them.

VI.3. Calculating the jump moments

Start by writing the equations of motion for a particle:

$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \boldsymbol{v}(t), \tag{VI.3.1a}$$

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \frac{1}{m} \boldsymbol{F}(t, \boldsymbol{x}(t)) \doteq \underbrace{\frac{q}{m} \boldsymbol{E}_{\mathrm{pol}}(\boldsymbol{v}(t))}_{\substack{\text{force on charge due} \\ \text{to polarization drag}}} + \underbrace{\boldsymbol{a}(t, \boldsymbol{x}(t))}_{\substack{\text{force on charge due to inter-} \\ \text{actions with all other particles} \\ \text{(plus external fields, if present;} \\ \text{in HW05, you'll include } \boldsymbol{B}_{\mathrm{ext}})}} . \tag{VI.3.1b}$$

Integrating (VI.3.1b), we find

$$\boldsymbol{v}(t) = \boldsymbol{v}(0) + \frac{1}{m} \int_0^t \mathrm{d}t'\, \boldsymbol{F}(t', \boldsymbol{x}(t')). \tag{VI.3.2}$$

This may be substituted for the right-hand side of (VI.3.1a) and the result integrated from $t' = 0$ to t to obtain

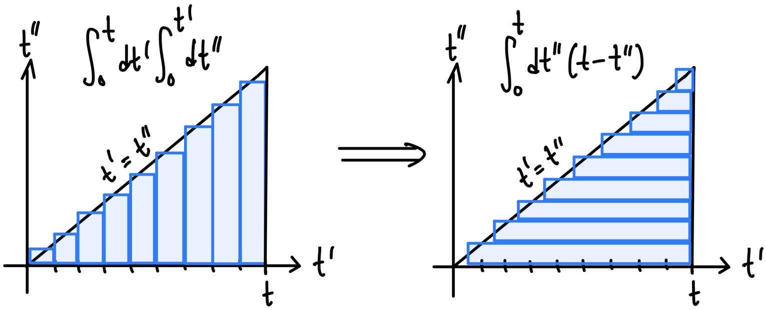
$$\boldsymbol{r}(t) = \boldsymbol{r}(0) + \boldsymbol{v}(0)t + \frac{1}{m} \int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t''\, \boldsymbol{F}(t'', \boldsymbol{x}(t'')), \tag{VI.3.3a}$$

$$= \boldsymbol{r}(0) + \boldsymbol{v}(0)t + \frac{1}{m} \int_0^t \mathrm{d}t''\, (t - t'') \boldsymbol{F}(t'', \boldsymbol{x}(t'')). \tag{VI.3.3b}$$

To obtain the second equality above, consider the following diagram:[11]

---

[11] Alternatively, write $\int_0^t \mathrm{d}t' \int_0^t \mathrm{d}t''\, H(t' - t'') \boldsymbol{F}(t'', \boldsymbol{x}(t''))$ and use $\int_0^t \mathrm{d}t'\, H(t' - t'') = t - t''$ for $t'' < t$.



Using (VI.3.2) and (VI.3.3b), the increments $\Delta \boldsymbol{r}(\Delta t)$ and $\Delta \boldsymbol{v}(\Delta t)$ are then

$$\Delta \boldsymbol{r}(\Delta t) \doteq \boldsymbol{r}(\Delta t) - \boldsymbol{r}(0) = \boldsymbol{v}(0)\Delta t + \frac{1}{m}\int_0^{\Delta t} \mathrm{d}s\, s\boldsymbol{F}(\Delta t - s, \boldsymbol{x}(\Delta t - s)), \tag{VI.3.4a}$$

$$\Delta \boldsymbol{v}(\Delta t) \doteq \boldsymbol{v}(\Delta t) - \boldsymbol{v}(0) = \frac{1}{m}\int_0^{\Delta t} \mathrm{d}s\, \boldsymbol{F}(s, \boldsymbol{x}(s)). \tag{VI.3.4b}$$

These can solved perturbatively in small Δt:

$$\begin{aligned}\Delta \boldsymbol{r}(\Delta t) = \boldsymbol{v}_0\Delta t &+ \frac{1}{m}\int_0^{\Delta t} \mathrm{d}s\, s\boldsymbol{F}(\Delta t - s, \boldsymbol{x}_0(\Delta t - s)) \\ &+ \frac{1}{m}\int_0^{\Delta t} \mathrm{d}s\, s\big[\boldsymbol{x}(\Delta t - s) - \boldsymbol{x}_0(\Delta t - s)\big]\cdot\frac{\partial}{\partial \boldsymbol{x}_0}\boldsymbol{F}(\Delta t - s, \boldsymbol{x}_0(\Delta t - s)) + \dots\end{aligned} \tag{VI.3.5a}$$

$$\Delta \boldsymbol{v}(\Delta t) = \frac{1}{m}\int_0^{\Delta t} \mathrm{d}s\, \boldsymbol{F}(s, \boldsymbol{x}_0(s)) + \frac{1}{m}\int_0^{\Delta t} \mathrm{d}s\, \big[\boldsymbol{x}(s) - \boldsymbol{x}_0(s)\big]\cdot\frac{\partial}{\partial \boldsymbol{x}_0}\boldsymbol{F}(s, \boldsymbol{x}_0(s)) + \dots, \tag{VI.3.5b}$$

where $\boldsymbol{x}_0(t) \doteq [\boldsymbol{r}(0) + \boldsymbol{v}(0)t, \boldsymbol{v}(0)]$ is the unperturbed phase-space trajectory. (Foreshadowing: these are *Lagrangian* increments, measured with the moving particle, rather than *Eulerian* increments, measured at a fixed position.) Henceforth, $\boldsymbol{F} = q\boldsymbol{E}_{\rm pol}(\boldsymbol{v}(t)) + m\boldsymbol{a}(t, \boldsymbol{r}(t))$ with $\langle \boldsymbol{a}\rangle = 0$; i.e., no magnetic field and a homogeneous background. With

$$\boldsymbol{r}(\Delta t - s) - \boldsymbol{r}_0(\Delta t - s) \simeq \frac{1}{m}\int_0^{\Delta t - s} \mathrm{d}s'\, s'\boldsymbol{F}(\Delta t - s - s', \boldsymbol{x}_0(\Delta t - s - s')), \tag{VI.3.6a}$$

$$\boldsymbol{v}(\Delta t - s) - \boldsymbol{v}_0(\Delta t - s) \simeq \frac{1}{m}\int_0^{\Delta t - s} \mathrm{d}s'\boldsymbol{F}(s', \boldsymbol{x}_0(s')), \tag{VI.3.6b}$$

$$\boldsymbol{r}(s) - \boldsymbol{r}_0(s) \simeq \frac{1}{m}\int_0^{s} \mathrm{d}s'\, s'\boldsymbol{F}(s - s', \boldsymbol{x}_0(s - s')), \tag{VI.3.6c}$$

$$\boldsymbol{v}(s) - \boldsymbol{v}_0(s) \simeq \frac{1}{m}\int_0^{s} \mathrm{d}s'\, \boldsymbol{F}(s', \boldsymbol{x}_0(s')) \tag{VI.3.6d}$$

inserted for the bracketed terms in (VI.3.5), we can proceed.

It turns out that it's easiest to compute $\boldsymbol{B}$ first (see (VI.2.4*b*)), so let's do that:

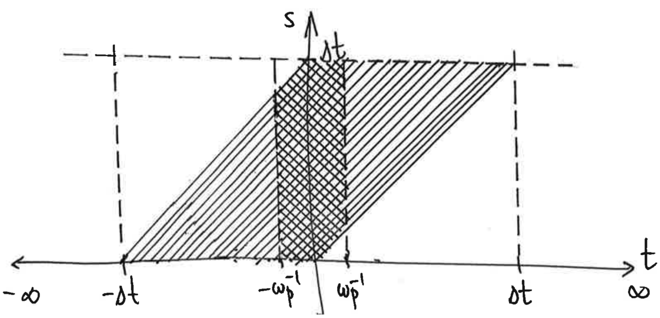
$$
\begin{aligned}
\boldsymbol{B} &\doteq \lim_{\Delta t\to\text{“0”}} \frac{\langle \Delta\boldsymbol{v}\Delta\boldsymbol{v}\rangle}{\Delta t} \\
&= \lim_{\Delta t\to\text{“0”}} \left\{ \frac{1}{\Delta t}\left\langle \frac{1}{m^2}\int_0^{\Delta t} \mathrm{d}s\, \boldsymbol{F}(s,\boldsymbol{x}_0(s)) \int_0^{\Delta t} \mathrm{d}s'\, \boldsymbol{F}(s',\boldsymbol{x}_0(s'))\right\rangle + \dots \right\} \\
&\simeq \lim_{\Delta t\to\text{“0”}} \left\{ \frac{1}{\Delta t}\frac{1}{m^2}\int_0^{\Delta t} \mathrm{d}s \int_0^{\Delta t} \mathrm{d}s' \Big\langle \big[q\boldsymbol{E}_{\mathrm{pol}}(\boldsymbol{v}_0) + m\boldsymbol{a}(s,\boldsymbol{r}_0(s))\big] \right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad \left. \times \big[q\boldsymbol{E}_{\mathrm{pol}}(\boldsymbol{v}_0) + m\boldsymbol{a}(s',\boldsymbol{r}_0(s'))\big]\Big\rangle \right\} \\
&= \lim_{\Delta t\to\text{“0”}} \left\{ \Delta t\left(\frac{q}{m}\right)^2 \boldsymbol{E}_{\mathrm{pol}}(\boldsymbol{v}_0)\boldsymbol{E}_{\mathrm{pol}}(\boldsymbol{v}_0) \right. \\
&\qquad\qquad \left. + \frac{1}{\Delta t}\int_0^{\Delta t}\mathrm{d}s\int_0^{\Delta t}\mathrm{d}s' \big\langle \boldsymbol{a}(s,\boldsymbol{r}_0(s))\,\boldsymbol{a}(s',\boldsymbol{r}_0(s'))\big\rangle \right\}.
\end{aligned}
\tag{VI.3.7}
$$

Recall that we've assumed translational invariance so that $\langle \boldsymbol{a}(t,\boldsymbol{r}_0(t))\rangle = 0$ along the unperturbed orbit of the particle.

Before proceeding any further, note that the first term in this equation is smaller than the second by a factor of $\nu\Delta t \ll 1$ if the latter is associated with the (Markovian) relaxation time ν^{-1}, e.g., if $\boldsymbol{a} = (q/m)\delta\boldsymbol{E}$, with $\delta\boldsymbol{E}$ being the fluctuating electric field generated by the thermal bath of charged particles. (Physically, drag and diffusion are of the same order, and so $(\text{drag})^2 \ll$ diffusion.) Thus, we may drop the sub-dominant $\boldsymbol{E}_{\mathrm{pol}}\boldsymbol{E}_{\mathrm{pol}}$ term in (VI.3.7), a simplification that is verified *a posteriori* at the end of this section. With $\boldsymbol{a} = (q/m)\delta\boldsymbol{E}$, equation (VI.3.7) then becomes

$$
\begin{aligned}
\boldsymbol{B} &\simeq \lim_{\Delta t\to\text{“0”}} \frac{1}{\Delta t}\left(\frac{q}{m}\right)^2 \int_0^{\Delta t}\mathrm{d}s\int_0^{\Delta t}\mathrm{d}s' \big\langle \delta\boldsymbol{E}(s,\boldsymbol{r}_0(s))\,\delta\boldsymbol{E}(s',\boldsymbol{r}_0(s'))\big\rangle \\
&= \lim_{\Delta t\to\text{“0”}} \frac{1}{\Delta t}\left(\frac{q}{m}\right)^2 \int_0^{\Delta t}\mathrm{d}s\int_{s-\Delta t}^{s}\mathrm{d}t \big\langle \delta\boldsymbol{E}(s,\boldsymbol{r}_0(s))\,\delta\boldsymbol{E}(s-t,\boldsymbol{r}_0(s-t))\big\rangle,
\end{aligned}
\tag{VI.3.8}
$$

where one of the integration variables has been changed to t using $t = s - s'$. To perform the integrals in (VI.3.8), examine the domain of integration (the region):



The correlation $\langle \delta\boldsymbol{E}(s,\boldsymbol{r}_0(s))\,\delta\boldsymbol{E}(s-t,\boldsymbol{r}_0(s-t))\rangle$ is only important when $\omega_{\mathrm{p}}t \sim 1$, which is denoted by the region. Outside of there, contributions to the integral are $\sim(\omega_{\mathrm{p}}\Delta t)^{-1} \ll$

1. Thus we may freely extend the limits on the t integration in (VI.3.8) to $\pm\infty$:

$$\boldsymbol{B} \simeq \lim_{\Delta t \to \text{“}0\text{”}} \frac{1}{\Delta t}\left(\frac{q}{m}\right)^2 \int_0^{\Delta t} \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}t \left\langle \delta\boldsymbol{E}(s,\boldsymbol{r}_0(s))\, \delta\boldsymbol{E}(s-t,\boldsymbol{r}_0(s-t))\right\rangle. \tag{VI.3.9}$$

Now, write

$$\delta\boldsymbol{E}(s,\boldsymbol{r}_0(s)) = \int \frac{\mathrm{d}\omega}{2\pi} \int \mathrm{d}\boldsymbol{k}\, \mathrm{e}^{-\mathrm{i}\omega s + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}_0(s)}\, \delta\boldsymbol{E}_{\omega,\boldsymbol{k}},$$

$$\delta\boldsymbol{E}(s-t,\boldsymbol{r}_0(s-t)) = \int \frac{\mathrm{d}\omega'}{2\pi} \int \mathrm{d}\boldsymbol{k}'\, \mathrm{e}^{-\mathrm{i}\omega'(s-t) + \mathrm{i}\boldsymbol{k}'\cdot\boldsymbol{r}_0(s-t)}\, \delta\boldsymbol{E}_{\omega',\boldsymbol{k}'},$$

with $\boldsymbol{r}_0(s) = \boldsymbol{r}(0) + \boldsymbol{v}_0 s$ and $\boldsymbol{r}_0(s-t) = \boldsymbol{r}(0) + \boldsymbol{v}_0(s-t) = \boldsymbol{r}_0(s) - \boldsymbol{v}_0 t$. Then (VI.3.9) becomes[12]

$$\begin{aligned}
\boldsymbol{B} &= \lim_{\Delta t \to \text{“}0\text{”}} \frac{1}{\Delta t}\left(\frac{q}{m}\right)^2 \int_0^{\Delta t} \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}t \int \frac{\mathrm{d}\omega}{2\pi} \int \frac{\mathrm{d}\omega'}{2\pi} \int \mathrm{d}\boldsymbol{k} \int \mathrm{d}\boldsymbol{k}' \\
&\qquad\qquad \times \left\langle \mathrm{e}^{-\mathrm{i}(\omega+\omega')s} \mathrm{e}^{\mathrm{i}(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{r}_0(s)} \mathrm{e}^{\mathrm{i}(\omega' - \boldsymbol{k}'\cdot\boldsymbol{v}_0)t}\, \delta\boldsymbol{E}_{\omega,\boldsymbol{k}} \delta\boldsymbol{E}_{\omega',\boldsymbol{k}'} \right\rangle \\
&= \lim_{\Delta t \to \text{“}0\text{”}} \frac{1}{\Delta t}\left(\frac{q}{m}\right)^2 \int_0^{\Delta t} \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}t \int \frac{\mathrm{d}\omega}{2\pi} \int \frac{\mathrm{d}\omega'}{2\pi} \int \mathrm{d}\boldsymbol{k} \int \mathrm{d}\boldsymbol{k}' \\
&\qquad\qquad \times \mathrm{e}^{-\mathrm{i}(\omega+\omega')s} (2\pi)^3 \delta(\boldsymbol{k}+\boldsymbol{k}') \left\langle \mathrm{e}^{\mathrm{i}(\omega' - \boldsymbol{k}'\cdot\boldsymbol{v}_0)t}\, \delta\boldsymbol{E}_{\omega,\boldsymbol{k}} \delta\boldsymbol{E}_{\omega',\boldsymbol{k}'} \right\rangle \\
&= \lim_{\Delta t \to \text{“}0\text{”}} \frac{1}{\Delta t}\left(\frac{q}{m}\right)^2 \int_0^{\Delta t} \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}t \int \frac{\mathrm{d}\omega}{2\pi} \int \frac{\mathrm{d}\omega'}{2\pi} \int \mathrm{d}\boldsymbol{k} \\
&\qquad\qquad \times \mathrm{e}^{-\mathrm{i}(\omega+\omega')s} (2\pi)^3 \left\langle \mathrm{e}^{\mathrm{i}(\omega' + \boldsymbol{k}\cdot\boldsymbol{v}_0)t}\, \delta\boldsymbol{E}_{\omega,\boldsymbol{k}} \delta\boldsymbol{E}_{\omega',-\boldsymbol{k}} \right\rangle \\
&= \lim_{\Delta t \to \text{“}0\text{”}} \frac{1}{\Delta t}\left(\frac{q}{m}\right)^2 \int_0^{\Delta t} \mathrm{d}s \int \frac{\mathrm{d}\omega}{2\pi} \int \frac{\mathrm{d}\omega'}{2\pi} \int \mathrm{d}\boldsymbol{k} \\
&\qquad\qquad \times \mathrm{e}^{-\mathrm{i}(\omega+\omega')s} (2\pi)^3 \left\langle 2\pi\delta(\omega' + \boldsymbol{k}\cdot\boldsymbol{v}_0)\, \delta\boldsymbol{E}_{\omega,\boldsymbol{k}} \delta\boldsymbol{E}_{\omega',-\boldsymbol{k}} \right\rangle \\
&= \lim_{\Delta t \to \text{“}0\text{”}} \frac{1}{\Delta t}\left(\frac{q}{m}\right)^2 \int_0^{\Delta t} \mathrm{d}s \int \frac{\mathrm{d}\omega}{2\pi} \int \mathrm{d}\boldsymbol{k}\, (2\pi)^3 \left\langle \mathrm{e}^{-\mathrm{i}(\omega - \boldsymbol{k}\cdot\boldsymbol{v}_0)s}\, \delta\boldsymbol{E}_{\omega,\boldsymbol{k}} \delta\boldsymbol{E}_{-\boldsymbol{k}\cdot\boldsymbol{v}_0,-\boldsymbol{k}} \right\rangle \\
&= \lim_{\Delta t \to \text{“}0\text{”}} \frac{1}{\Delta t}\left(\frac{q}{m}\right)^2 \int \frac{\mathrm{d}\omega}{2\pi} \int \mathrm{d}\boldsymbol{k}\, (2\pi)^3 \left\langle \frac{\mathrm{e}^{-\mathrm{i}(\omega - \boldsymbol{k}\cdot\boldsymbol{v}_0)\Delta t} - 1}{-\mathrm{i}(\omega - \boldsymbol{k}\cdot\boldsymbol{v}_0)}\, \delta\boldsymbol{E}_{\omega,\boldsymbol{k}} \delta\boldsymbol{E}_{-\boldsymbol{k}\cdot\boldsymbol{v}_0,-\boldsymbol{k}} \right\rangle \\
&= \left(\frac{q}{m}\right)^2 \int \frac{\mathrm{d}\omega}{2\pi} \int \mathrm{d}\boldsymbol{k}\, (2\pi)^3 \left\langle \delta\boldsymbol{E}_{\omega,\boldsymbol{k}} \delta\boldsymbol{E}_{-\boldsymbol{k}\cdot\boldsymbol{v}_0,-\boldsymbol{k}} \right\rangle \\
&= \left(\frac{q}{m}\right)^2 \int \mathrm{d}\boldsymbol{k}\, \left\langle \delta\boldsymbol{E}\delta\boldsymbol{E} \right\rangle_{\omega = \boldsymbol{k}\cdot\boldsymbol{v}_0, \boldsymbol{k}},
\end{aligned} \tag{VI.3.10}$$

which is precisely (V.4.7)![13] Therefore, the Fokker–Planck coefficient $\boldsymbol{B}_\alpha$ in the Balescu–Lenard collision operator corresponds to diffusion in velocity space due to many small-angle scatterings of particles off of correlated electric-field fluctuations. Because $\omega = \boldsymbol{k}\cdot\boldsymbol{v}_0$, equation (VI.3.10) says that diffusion is a *resonant phenomena*; the only compo-

[12] The following steps are related to the relationship between Lagrangian and Eulerian correlations – that is, those measured in the frame of the particle versus those measured at a fixed position in the lab frame. See §5.3 of Krommes (2018), as well as later in these notes, for more.

[13] To obtain the final equality in (VI.3.10), we have used $(2\pi)^2\langle\delta\boldsymbol{E}_\omega\delta\boldsymbol{E}_{\omega'}\rangle = \langle\delta\boldsymbol{E}\delta\boldsymbol{E}\rangle_\omega\,\delta(\omega+\omega')$.

nent of $\delta \boldsymbol{E}_{\omega,\boldsymbol{k}}$ that matters is the one with $\omega = 0$ in the frame of the particle. This makes sense – the other frequency components just wiggle the particle back and forth, and that oscillation averages out in the long-time limit. (This is related to quasi-linear diffusion.)

Now let's compute $\boldsymbol{A}$ (see (VI.2.4a)):

$$\begin{aligned}
\boldsymbol{A} &\doteq \lim_{\Delta t \to \text{``0''}} \frac{\langle \Delta \boldsymbol{v} \rangle}{\Delta t} \\
&= \lim_{\Delta t \to \text{``0''}} \left\{ \frac{1}{\Delta t} \left\langle \frac{1}{m} \int_0^{\Delta t} \mathrm{d}s\, \boldsymbol{F}(s, \boldsymbol{x}_0(s)) \right.\right. \\
&\quad \left.\left. + \frac{1}{m} \int_0^{\Delta t} \mathrm{d}s \left[\boldsymbol{x}(s) - \boldsymbol{x}_0(s) \right] \cdot \frac{\partial}{\partial \boldsymbol{x}_0} \boldsymbol{F}(s, \boldsymbol{x}_0(s)) + \ldots \right\rangle \right\} \\
&= \lim_{\Delta t \to \text{``0''}} \left\{ \frac{1}{\Delta t} \frac{q}{m} \int_0^{\Delta t} \mathrm{d}s\, \boldsymbol{E}_{\text{pol}}(\boldsymbol{v}_0) + \frac{1}{\Delta t} \int_0^{\Delta t} \mathrm{d}s\, \underbrace{\langle \boldsymbol{a}(s, \boldsymbol{r}_0(s)) \rangle}_{=0} \right. \\
&\quad + \frac{1}{\Delta t} \frac{q}{m} \int_0^{\Delta t} \mathrm{d}s\, \underbrace{\langle \boldsymbol{v}(s) - \boldsymbol{v}_0(s) \rangle}_{\text{use (VI.3.6}d\text{)}} \cdot \frac{\partial}{\partial \boldsymbol{v}_0} \boldsymbol{E}_{\text{pol}}(\boldsymbol{v}_0) \\
&\quad \left. + \frac{1}{\Delta t} \int_0^{\Delta t} \mathrm{d}s \left\langle \underbrace{\left[\boldsymbol{r}(s) - \boldsymbol{r}_0(s) \right]}_{\text{use (VI.3.6}c\text{)}} \cdot \frac{\partial}{\partial \boldsymbol{r}_0} \boldsymbol{a}(s, \boldsymbol{r}_0(s)) \right\rangle + \ldots \right\} \\
&= \lim_{\Delta t \to \text{``0''}} \left\{ \frac{1}{\Delta t} \frac{q}{m} \int_0^{\Delta t} \mathrm{d}s\, \boldsymbol{E}_{\text{pol}}(\boldsymbol{v}_0) \right. \\
&\quad + \frac{1}{\Delta t} \left(\frac{q}{m} \right)^2 \int_0^{\Delta t} \mathrm{d}s \int_0^s \mathrm{d}s'\, \boldsymbol{E}_{\text{pol}}(\boldsymbol{v}_0) \cdot \frac{\partial}{\partial \boldsymbol{v}_0} \boldsymbol{E}_{\text{pol}}(\boldsymbol{v}_0) \\
&\quad \left. + \frac{1}{\Delta t} \int_0^{\Delta t} \mathrm{d}s \int_0^s \mathrm{d}s'\, s' \left\langle \boldsymbol{a}(s - s', \boldsymbol{r}_0(s - s')) \cdot \frac{\partial}{\partial \boldsymbol{r}_0} \boldsymbol{a}(s, \boldsymbol{r}_0(s)) \right\rangle + \ldots \right\}. \quad \text{(VI.3.11)}
\end{aligned}$$

Again, with $\boldsymbol{a} = (q/m)\delta \boldsymbol{E}$, the second term is small compared to the last term by a factor $\nu \Delta t \ll 1$; equation (VI.3.11) then becomes

$$\begin{aligned}
\boldsymbol{A} = \lim_{\Delta t \to \text{``0''}} &\left\{ \frac{q}{m} \boldsymbol{E}_{\text{pol}}(\boldsymbol{v}_0) \right. \\
&\left. + \left\langle \frac{1}{\Delta t} \left(\frac{q}{m} \right)^2 \int_0^{\Delta t} \mathrm{d}s \int_0^s \mathrm{d}s'\, s'\, \delta \boldsymbol{E}(s - s', \boldsymbol{r}_0(s - s')) \cdot \frac{\partial}{\partial \boldsymbol{r}_0} \delta \boldsymbol{E}(s, \boldsymbol{r}_0(s)) \right\rangle + \ldots \right\}.
\end{aligned} \quad \text{(VI.3.12)}$$

Because Δt, while small, is nevertheless $\gg \omega_{\text{p}}^{-1}$ (remember the Markov assumption?), the integration limit on s' in (VI.3.12) can be extended to $+\infty$:

$$\boldsymbol{A} \simeq \frac{q}{m}\boldsymbol{E}_{\mathrm{pol}}(\boldsymbol{v}_0) + \lim_{\Delta t\to \text{“}0\text{”}} \frac{1}{\Delta t}\left(\frac{q}{m}\right)^2 \int_0^{\Delta t} \mathrm{d}s \int_0^\infty \mathrm{d}s'\, s' \left\langle \delta\boldsymbol{E}(s-s', \boldsymbol{r}_0(s-s')) \cdot \frac{\partial}{\partial \boldsymbol{r}_0} \delta\boldsymbol{E}(s, \boldsymbol{r}_0(s)) \right\rangle. \tag{VI.3.13}$$

As before, write

$$\delta\boldsymbol{E}(s, \boldsymbol{r}_0(s)) = \int \frac{\mathrm{d}\omega}{2\pi} \int \mathrm{d}\boldsymbol{k}\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}_0(s) - \mathrm{i}\omega s}\, \delta\boldsymbol{E}_{\omega,\boldsymbol{k}},$$

$$\delta\boldsymbol{E}(s-s', \boldsymbol{r}_0(s-s')) = \int \frac{\mathrm{d}\omega'}{2\pi} \int \mathrm{d}\boldsymbol{k}'\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}'\cdot\boldsymbol{r}_0(s) - \mathrm{i}\omega' s + \mathrm{i}(\omega' - \boldsymbol{k}'\cdot\boldsymbol{v}_0)s'}\, \delta\boldsymbol{E}_{\omega',\boldsymbol{k}'}.$$

Then (VI.3.13) becomes

$$\begin{aligned}
\boldsymbol{A} &\simeq \frac{q}{m}\boldsymbol{E}_{\mathrm{pol}}(\boldsymbol{v}_0) + \lim_{\Delta t\to \text{“}0\text{”}} \frac{1}{\Delta t}\left(\frac{q}{m}\right)^2 \int_0^{\Delta t} \mathrm{d}s \int_0^\infty \mathrm{d}s'\, s' \int \frac{\mathrm{d}\omega}{2\pi} \int \frac{\mathrm{d}\omega'}{2\pi} \int \mathrm{d}\boldsymbol{k} \int \mathrm{d}\boldsymbol{k}' \\
&\qquad \times \left\langle \mathrm{e}^{-\mathrm{i}(\omega+\omega')s} \mathrm{e}^{\mathrm{i}(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{r}_0(s)} \mathrm{e}^{\mathrm{i}(\omega' - \boldsymbol{k}'\cdot\boldsymbol{v}_0)s'}\, \mathrm{i}\boldsymbol{k}\cdot\delta\boldsymbol{E}_{\omega',\boldsymbol{k}'}\delta\boldsymbol{E}_{\omega,\boldsymbol{k}} \right\rangle \\
&= \frac{q}{m}\boldsymbol{E}_{\mathrm{pol}}(\boldsymbol{v}_0) + \lim_{\Delta t\to \text{“}0\text{”}} \frac{1}{\Delta t}\left(\frac{q}{m}\right)^2 \int_0^{\Delta t} \mathrm{d}s \int_0^\infty \mathrm{d}s'\, s' \int \frac{\mathrm{d}\omega}{2\pi} \int \frac{\mathrm{d}\omega'}{2\pi} \int \mathrm{d}\boldsymbol{k} \int \mathrm{d}\boldsymbol{k}' \\
&\qquad \times \mathrm{e}^{-\mathrm{i}(\omega+\omega')s} (2\pi)^3 \delta(\boldsymbol{k}+\boldsymbol{k}') \left\langle \mathrm{e}^{\mathrm{i}(\omega' - \boldsymbol{k}'\cdot\boldsymbol{v}_0)s'}\, \mathrm{i}\boldsymbol{k}\cdot\delta\boldsymbol{E}_{\omega',\boldsymbol{k}'}\delta\boldsymbol{E}_{\omega,\boldsymbol{k}} \right\rangle \\
&= \frac{q}{m}\boldsymbol{E}_{\mathrm{pol}}(\boldsymbol{v}_0) + \lim_{\Delta t\to \text{“}0\text{”}} \frac{1}{\Delta t}\left(\frac{q}{m}\right)^2 \frac{\partial}{\partial \boldsymbol{v}_0} \cdot \int_0^{\Delta t} \mathrm{d}s \int_0^\infty \mathrm{d}s' \int \frac{\mathrm{d}\omega}{2\pi} \int \frac{\mathrm{d}\omega'}{2\pi} \int \mathrm{d}\boldsymbol{k} \\
&\qquad \times \mathrm{e}^{-\mathrm{i}(\omega+\omega')s} (2\pi)^3 \left\langle \mathrm{e}^{\mathrm{i}(\omega' + \boldsymbol{k}\cdot\boldsymbol{v}_0)s'}\, \delta\boldsymbol{E}_{\omega',-\boldsymbol{k}}\delta\boldsymbol{E}_{\omega,\boldsymbol{k}} \right\rangle \\
&= \frac{q}{m}\boldsymbol{E}_{\mathrm{pol}}(\boldsymbol{v}_0) + \lim_{\Delta t\to \text{“}0\text{”}} \frac{1}{\Delta t}\left(\frac{q}{m}\right)^2 \frac{\partial}{\partial \boldsymbol{v}_0} \cdot \int_0^{\Delta t} \mathrm{d}s \int \frac{\mathrm{d}\omega}{2\pi} \int \frac{\mathrm{d}\omega'}{2\pi} \int \mathrm{d}\boldsymbol{k} \\
&\qquad \times \mathrm{e}^{-\mathrm{i}(\omega+\omega')s} (2\pi)^3 \left\langle \pi\delta(\omega' + \boldsymbol{k}\cdot\boldsymbol{v}_0)\, \delta\boldsymbol{E}_{\omega,\boldsymbol{k}}\delta\boldsymbol{E}_{\omega',-\boldsymbol{k}} \right\rangle \\
&= \frac{q}{m}\boldsymbol{E}_{\mathrm{pol}}(\boldsymbol{v}_0) + \lim_{\Delta t\to \text{“}0\text{”}} \frac{1}{\Delta t}\frac{1}{2}\left(\frac{q}{m}\right)^2 \frac{\partial}{\partial \boldsymbol{v}_0} \cdot \int_0^{\Delta t} \mathrm{d}s \int \frac{\mathrm{d}\omega}{2\pi} \int \mathrm{d}\boldsymbol{k} \\
&\qquad \times (2\pi)^3 \left\langle \mathrm{e}^{-\mathrm{i}(\omega - \boldsymbol{k}\cdot\boldsymbol{v}_0)s}\, \delta\boldsymbol{E}_{\omega,\boldsymbol{k}}\delta\boldsymbol{E}_{-\boldsymbol{k}\cdot\boldsymbol{v}_0,-\boldsymbol{k}} \right\rangle \\
&= \frac{q}{m}\boldsymbol{E}_{\mathrm{pol}}(\boldsymbol{v}_0) + \lim_{\Delta t\to \text{“}0\text{”}} \frac{1}{\Delta t}\frac{1}{2}\left(\frac{q}{m}\right)^2 \frac{\partial}{\partial \boldsymbol{v}_0} \cdot \int \frac{\mathrm{d}\omega}{2\pi} \int \mathrm{d}\boldsymbol{k} \\
&\qquad \times (2\pi)^3 \left\langle \frac{\mathrm{e}^{-\mathrm{i}(\omega - \boldsymbol{k}\cdot\boldsymbol{v}_0)\Delta t} - 1}{-\mathrm{i}(\omega - \boldsymbol{k}\cdot\boldsymbol{v}_0)}\, \delta\boldsymbol{E}_{\omega,\boldsymbol{k}}\delta\boldsymbol{E}_{-\boldsymbol{k}\cdot\boldsymbol{v}_0,-\boldsymbol{k}} \right\rangle \\
&= \frac{q}{m}\boldsymbol{E}_{\mathrm{pol}}(\boldsymbol{v}_0) + \frac{1}{2}\left(\frac{q}{m}\right)^2 \frac{\partial}{\partial \boldsymbol{v}_0} \cdot \int \frac{\mathrm{d}\omega}{2\pi} \int \mathrm{d}\boldsymbol{k}\, (2\pi)^3 \left\langle \delta\boldsymbol{E}_{\omega,\boldsymbol{k}}\delta\boldsymbol{E}_{-\boldsymbol{k}\cdot\boldsymbol{v}_0,-\boldsymbol{k}} \right\rangle \\
&= \frac{q}{m}\boldsymbol{E}_{\mathrm{pol}}(\boldsymbol{v}_0) + \frac{1}{2}\left(\frac{q}{m}\right)^2 \frac{\partial}{\partial \boldsymbol{v}_0} \cdot \int \mathrm{d}\boldsymbol{k}\, \left\langle \delta\boldsymbol{E}_{\boldsymbol{k}}\delta\boldsymbol{E}_{-\boldsymbol{k}} \right\rangle_{\omega = \boldsymbol{k}\cdot\boldsymbol{v}_0} \\
&= \frac{q}{m}\boldsymbol{E}_{\mathrm{pol}}(\boldsymbol{v}_0) + \frac{1}{2}\frac{\partial}{\partial \boldsymbol{v}_0} \cdot \boldsymbol{\mathsf{B}}(\boldsymbol{v}_0),
\end{aligned} \tag{VI.3.14}$$

which is precisely (V.4.8)! So, we have learnt that $\boldsymbol{A}$ in the Balescu–Lenard operator corresponds to a particle, on the average, changing its velocity – this is a *drag* term. In the case of $\boldsymbol{E}_{\rm pol}$, it's just polarization drag. For the $\boldsymbol{B}$ piece, it's dynamical friction (i.e., trying to wade through a bunch of waves).

Sometimes, instead of

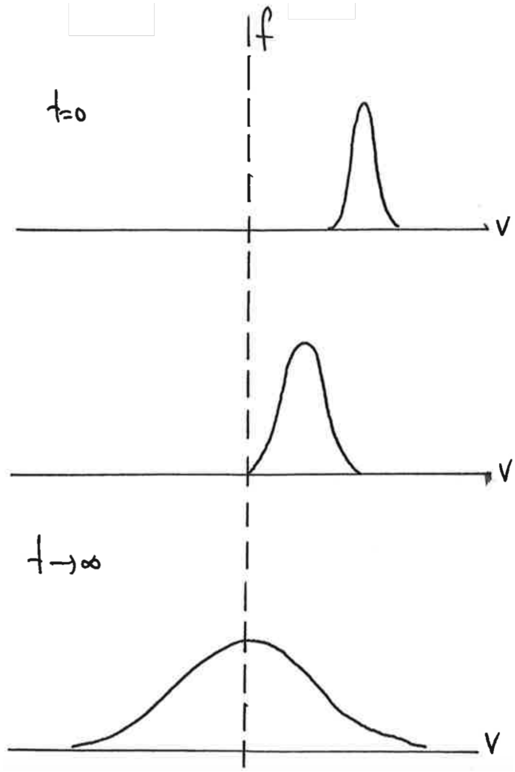
$$-\frac{\partial}{\partial \boldsymbol{v}} \cdot (\boldsymbol{A} f) + \frac{1}{2}\frac{\partial}{\partial \boldsymbol{v}}\frac{\partial}{\partial \boldsymbol{v}} : (\boldsymbol{B} f)$$

for the Fokker-Planck collision operator, you'll see

$$-\frac{\partial}{\partial \boldsymbol{v}} \cdot (\boldsymbol{A}' f) + \frac{1}{2}\frac{\partial}{\partial \boldsymbol{v}} \cdot \left(\boldsymbol{B} \cdot \frac{\partial f}{\partial \boldsymbol{v}}\right),$$

with $\boldsymbol{A}' = (q/m)\boldsymbol{E}_{\rm pol}(\boldsymbol{v})$ representing only the polarization drag. No physical difference, just notation.

Thus, we have drag ($\boldsymbol{A}$) and diffusion ($\boldsymbol{B}$). Pictorially, the relaxation to equilibrium might look like this:



A Maxwellian distribution is thus a balance between drag, which slows down particles, and diffusion, which broadens the distribution.

Had we retained the first term in (VI.3.7) proportional to $\boldsymbol{E}_{\rm pol}(\boldsymbol{v}_0)\boldsymbol{E}_{\rm pol}(\boldsymbol{v}_0)$, we would have obtained a correction to $\boldsymbol{B}_\alpha$ given by

$$\lim_{\Delta t \to \text{“}0\text{”}} \Delta t \left(\frac{q_\alpha}{m_\alpha}\right)^2 \boldsymbol{E}_{\rm pol}(\boldsymbol{v}_0)\boldsymbol{E}_{\rm pol}(\boldsymbol{v}_0).$$

With $\boldsymbol{E}_{\rm pol}(\boldsymbol{v}_0)$ given by (V.4.3), we may estimate the size of this neglected term to be

$$\sim \Delta t\left(\frac{q_\alpha^2 q_\beta^2}{m_\alpha m_\beta}\frac{n_\beta v_0}{v_{{\rm th}\beta}^3}\ln\lambda_{\alpha\beta}\right)^2$$

Let us compare this to its competing term, $(q_\alpha/m_\alpha)^2\int{\rm d}\boldsymbol{k}\,\langle\delta\boldsymbol{E}_{\boldsymbol{k}}\boldsymbol{E}_{-\boldsymbol{k}}\rangle_{\omega=\boldsymbol{k}\cdot\boldsymbol{v}_0}$, which we retained. Using (V.4.6), its size is

$$\sim\frac{q_\alpha^2 q_\beta^2}{m_\alpha^2}\frac{n_\beta}{v_{{\rm th}\beta}}\ln\lambda_{\alpha\beta}.$$

With $v_0\sim v_{{\rm th}\alpha}$, the ratio of these two terms is

$$\sim \Delta t\frac{q_\alpha^2 q_\beta^2 n_\beta\ln\lambda_{\alpha\beta}}{m_\alpha^2 v_{{\rm th}\beta}^3}\frac{m_\alpha T_\alpha}{m_\beta T_\beta}\sim\Delta t\,\nu^{\alpha\beta}\frac{m_\alpha T_\alpha}{m_\beta T_\beta}\ll 1,$$

where the collision frequency $\nu^{\alpha\beta}$ is given later in these notes by (VIII.6.28). Thus, $({\rm drag})^2$ is indeed $\ll$ diffusion, and we rightly dropped the $\boldsymbol{E}_{\rm pol}\boldsymbol{E}_{\rm pol}$ term when calculating $\boldsymbol{A}$ and $\boldsymbol{\mathsf{B}}$.

VI.4. Summary

A summary is in order, before we proceed any further.

We began by obtaining the Balescu–Lenard (and then Landau) collision operator from Klimontovich and the BBGKY hierarchy. This came from a direct solution of the BBGKY hierarchy for the two-particle correlation function, which was possible only after using Bogoliubov's hypothesis: f_α is roughly constant over the timescale on which $g_{\alpha\beta}$ relaxes, which we now see is equivalent to the Markov assumption (that is, the time axis is coarse-grained in units of Δt larger than $\omega_{\rm p}^{-1}$, which measures the interaction of one particle with the Debye cloud of another). The B-L operator captures polarization drag, radiation of plasma waves and the dynamical friction and diffusion that result from many small-angle scatterings off of these fluctuations, and the effects of particle motion on the efficacy of Debye shielding. These effects are neatly categorized into Fokker-Planck coefficients, $\boldsymbol{A}_\alpha$ and $\boldsymbol{\mathsf{B}}_\alpha$, which represent drag and diffusion, respectively. That these coefficients can likewise be obtained using the test-particle superposition principle indicates that we can rigorously view the plasma as being comprised of statistically independent Debye clouds (at least to the order of Λ in which we are working). This is related to the fact that we could split the solution for $g_{\alpha\beta}$ into the product of two one-particle response functions, which are coupled at the next order by a source term describing the Coulomb interaction. Also note that $\boldsymbol{A}_\alpha$ and $\boldsymbol{\mathsf{B}}_\alpha$ were evaluated along unperturbed orbits (see (VI.3.9) and (VI.3.13))!

As part of this summary, below I've matched up the assumptions made and procedures taken when going from BBGKY to the Balescu–Lenard operator and from Fokker-Planck to the Balescu–Lenard operator, color-coded:

(1) Assume that the two-particle correlation $g_{\alpha\beta}$ relaxes on a timescale $\sim\omega_{\rm p}^{-1}$ much less than the timescale on which f_α evolves.

(1) Calculate the plasma response δf_α and the electrostatic response φ due to a test charge in the long-time limit $\Delta t\gg\omega_{\rm p}^{-1}$.

(2) Neglect three-particle correlations $h_{\alpha\beta\gamma}$.

(2) Focus on a single test charge and its dressing.

(3) Take f_α to be homogeneous on $\sim\lambda_{\rm D}$ scales.

(3) Consider jump moments only in velocity space and not in configuration space.

(4) Drop $g_{\alpha\beta}$ from the source term $S_{\alpha\beta}$, so that particles enter into correlations being initially uncorrelated.

(4) Assume small-angle scatterings and adopt the Markov assumption.

(5) Solve the reduced BBGKY hierarchy using the Vlasov's Green function, which assumes straight-line trajectories.

(5) Solve $\mathrm{d}\boldsymbol{v}/\mathrm{d}t = (q/m)\boldsymbol{E}$ and $\mathrm{d}\boldsymbol{r}/\mathrm{d}t = \boldsymbol{v}$ for a single dressed particle in the short-time limit, compute the jump moments $\langle \Delta\boldsymbol{v}\rangle$ and $\langle \Delta\boldsymbol{v}\Delta\boldsymbol{v}\rangle$, and substitute these into the Fokker-Planck operator assuming the test-particle superposition principle.

Both of these approaches ultimately lead to the same Balescu–Lenard operator. Make sure you understand why.

PART VII
The Langevin approach

How dare we speak of the laws of chance? Is not chance the antithesis of all law?

Joseph Bertrand
Calcul des probabilités (1889)

Looking back at a lot of what we've done – obtaining the Balescu–Lenard operator, discussing free-energy conservation, deriving the Fokker–Planck equation and its coefficients – there was always a coarse-graining of the time axis. We were concerned with times longer than ω_p^{-1}, the auto-correlation time on which Debye shielding is set up (and on which reversible physics lives). For example, in the Fokker–Planck approach of the last chapter, we computed the jump moments $\langle \Delta\boldsymbol{v}\rangle$ and $\langle \Delta\boldsymbol{v}\Delta\boldsymbol{v}\rangle$ by expanding in small Δt, "small" meaning $\Delta t \to 0$ while having $\omega_\mathrm{p}\Delta t \gg 1$. These moments were then fed into the Fokker–Planck equation to obtain the long-time physics. They were intrinsically probabilistic, being moments of the transition probability rate $W(t+\Delta t, \boldsymbol{x}; t, \boldsymbol{x}-\Delta\boldsymbol{x})$.

There is an alternative approach, due to Langevin, which computes the moments for all times conditioned on initial conditions. It's easy to imagine that this is much harder than obtaining the Fokker–Planck coefficients, but one can nevertheless do it for some simple problems. And the Langevin approach has use in statistical descriptions of turbulence and a rich history in the context of Brownian motion. For these reasons, and for providing a solid foundation for Fokker–Planck theory, Langevin's method is worth discussing.

We'll start simple.

VII.1. The Langevin equation

Consider a single particle of unit mass with phase-space coordinates (z, v) at time t, subject to an external force $a(t)$ and to a drag force with a rate γ. The equations of motion are

$$\frac{\mathrm{d}z}{\mathrm{d}t} = v, \tag{VII.1.1a}$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\gamma v + a(t). \tag{VII.1.1b}$$

Let $a(t)$ be a non-random oscillatory force, $a(t) = a_0 \exp(-\mathrm{i}\omega_0 t)$. Assuming $z(0) = z_0$ and $v(0) = v_0$, the solution can be obtained by direct inspection, Laplace transforms, or a Green's function approach.

The Laplace way:

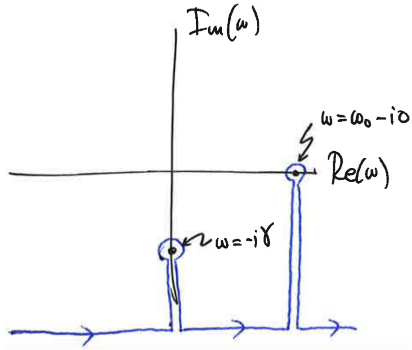
$$\int_0^\infty \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t} \left[\frac{\mathrm{d}v}{\mathrm{d}t} + \gamma v = a(t) \right]$$

$$\Longrightarrow -\mathrm{i}\omega v(\omega) - v_0 + \gamma v(\omega) = a_0 \int_0^\infty \mathrm{d}t\, \mathrm{e}^{\mathrm{i}(\omega-\omega_0)t} = \frac{\mathrm{i}a_0}{\omega - \omega_0}$$

$$\Longrightarrow v(\omega) = \frac{\mathrm{i}v_0}{\omega + \mathrm{i}\gamma} - \frac{a_0}{(\omega-\omega_0)(\omega+\mathrm{i}\gamma)}.$$

$$\Longrightarrow v(t) = \int \frac{\mathrm{d}\omega}{2\pi}\, \mathrm{e}^{-\mathrm{i}\omega t} \left[\frac{\mathrm{i}v_0}{\omega + \mathrm{i}\gamma} - \frac{a_0}{(\omega-\omega_0)(\omega+\mathrm{i}\gamma)} \right]$$
$$= -\frac{2\pi \mathrm{i}}{2\pi} \left(\mathrm{i}v_0 \mathrm{e}^{-\gamma t} - \frac{a_0}{\omega_0 + \mathrm{i}\gamma}\, \mathrm{e}^{-\mathrm{i}\omega_0 t} + \frac{a_0}{\omega_0 + \mathrm{i}\gamma}\, \mathrm{e}^{-\gamma t} \right)$$
$$= v_0 \mathrm{e}^{-\gamma t} + \frac{\mathrm{i}a_0}{\omega_0 + \mathrm{i}\gamma} \left(\mathrm{e}^{-\mathrm{i}\omega_0 t} - \mathrm{e}^{-\gamma t} \right),$$

where the following contour was used to perform the inverse-Laplace transform:



The Green's function way:

$$v(t) = v_0 G(t) + \int_0^t \mathrm{d}t'\, a(t') G(t-t') \quad \text{with} \quad G(s) = \mathrm{e}^{-\gamma s}$$
$$= v_0 \mathrm{e}^{-\gamma t} + \int_0^t \mathrm{d}t'\, a_0 \mathrm{e}^{-\mathrm{i}\omega_0 t'} \mathrm{e}^{-\gamma(t-t')}$$
$$= v_0 \mathrm{e}^{-\gamma t} + \frac{\mathrm{i}a_0}{\omega_0 + \mathrm{i}\gamma} \left(\mathrm{e}^{-\mathrm{i}\omega_0 t} - \mathrm{e}^{-\gamma t} \right).$$

Either way, taking the real part of the solution gives

$$v(t) = \left(v_0 - \frac{a_0 \gamma}{\omega_0^2 + \gamma^2} \right) \mathrm{e}^{-\gamma t} + \frac{a_0}{\omega_0^2 + \gamma^2} \left(\omega_0 \sin \omega_0 t + \gamma \cos \omega_0 t \right). \qquad \text{(VII.1.2)}$$

Note that, as $t \to \infty$, the particle eventually oscillates at the driving frequency.

Now, let us suppose that the external force $a(t)$ is random and independent of v. Then, while $\mathrm{d}v/\mathrm{d}t = -\gamma v + a(t)$ may *look* like a simple differential equation, it is *not*. Rather, it is a *stochastic differential equation*. It's different, because we don't actually know what $a(t)$ is – we just know that it's random.

There are two ways to think about how a solution might be obtained. Suppose that you *did* know $a(t)$ experienced by some particle. In principle, then you could go ahead and solve (VII.1.1). But the next particle under consideration would experience a different $a(t)$. (Remember "deterministic within any particular realization, stochastic between different realizations"?) So you'd have to solve *that* one. Now draw yet another force $a(t)$ from the random-force generator and solve (VII.1.1) for *that* realization. Do this a bunch of times, and this will quickly get exhausting. It's probably best then to speak of some typical solution; i.e., we consider the *ensemble average*. For example, the ensemble-average value of all these randomly generated, but individually deterministic, forces might be zero, i.e., $\langle a(t)\rangle = 0$. The standard deviation, probably not. You could use the statistics of the forces to say something about the statistics of the particles' motion.

The other way is to simply admit that it's unrealistic to actually know $a(t)$ in any specific case. Instead, we only know certain gross features of it, e.g., its average value, or perhaps its standard deviation. (For Gaussian statistics, that's all you need.) Then we could use this crude information to say something about, e.g., the average value of v or maybe of v^2. We adorn these gross features of $a(t)$ with brackets $\langle \dots \rangle$.

This is the goal of solving the *Langevin equation*, which is simply (VII.1.1b) with a random $a(t)$. Clearly, some statistics of $a(t)$ must be provided. Let's see this in action.

Solving (VII.1.1b) with a random $a(t)$, we have

$$v(t) = v_0 \mathrm{e}^{-\gamma t} + \mathrm{e}^{-\gamma t} \int_0^t \mathrm{d}t'\, a(t')\mathrm{e}^{\gamma t'}. \tag{VII.1.3}$$

At this point, we can't go any further. But, let us assume that, on the average, the force vanishes at any given time: $\langle a(t)\rangle = 0$. Then, taking the average of (VII.1.3), we have

$$\langle v(t)\rangle = v_0 \mathrm{e}^{-\gamma t}, \tag{VII.1.4}$$

and so

$$\langle z(t)\rangle = z_0 + \frac{v_0}{\gamma}\bigl(1 - \mathrm{e}^{-\gamma t}\bigr). \tag{VII.1.5}$$

This is not surprising, but there is an interpretive subtlety. These equations are not saying that the velocity and position have been determined, but rather that the ensemble average (or expectation value) is known. (Individual particles in a many-particle system might do different things, but their average behavior is known.)

We can obtain more information by computing some quadratic quantities, such as the variance in position $\langle z^2(t)\rangle - \langle z(t)\rangle^2$ or in velocity $\langle v^2(t)\rangle - \langle v(t)\rangle^2$, or perhaps the two-time correlations $\langle z(t_1)z(t_2)\rangle$ and $\langle v(t_1)v(t_2)\rangle$. The latter tell us information about the correlation between the position and velocity of the particle at time t_2 and at time t_1, given that we know where it was at $t = 0$. For example,

$$\begin{aligned}
\langle v(t_1)v(t_2)\rangle &= \Biggl\langle v_0^2 \mathrm{e}^{-\gamma(t_1+t_2)} + v_0 \mathrm{e}^{-\gamma(t_1+t_2)} \left[\int_0^{t_2} \mathrm{d}t_2'\, a(t_2')\mathrm{e}^{\gamma t_2'} + \int_0^{t_1} \mathrm{d}t_1' a(t_1')\mathrm{e}^{\gamma t_1'}\right] \\
&\qquad + \mathrm{e}^{-\gamma(t_1+t_2)} \int_0^{t_1} \mathrm{d}t_1' \int^{t_2} \mathrm{d}t_2'\, a(t_1')a(t_2')\mathrm{e}^{\gamma(t_1'+t_2')} \Biggr\rangle \\
&= v_0^2 \mathrm{e}^{-\gamma(t_1+t_2)} + \mathrm{e}^{-\gamma(t_1+t_2)} \int^{t_1} \mathrm{d}t_1' \int_0^{t_2} \mathrm{d}t_2'\, \langle a(t_1')a(t_2')\rangle \mathrm{e}^{\gamma(t_1'+t_2')} \\
&= \langle v(t_1)\rangle\langle v(t_2)\rangle + \int_0^{t_1} \mathrm{d}t_1' \int_0^{t_2} \mathrm{d}t_2'\, \langle a(t_1')a(t_2')\rangle \mathrm{e}^{\gamma(t_1'+t_2'-t_1-t_2)}.
\end{aligned} \tag{VII.1.6}$$

Given the two-time correlation for the forcing, we can determine the two-time correlation for the velocity.

In many cases of interest (e.g., Brownian motion), the kicks imparted by the force are both fast and uncorrelated. To get a feeling for what this means, imagine the force is due to collisions between our particle and some other particles. If each collision takes a time $t_{\rm coll}$, then it's obvious that, on timescales less than $t_{\rm coll}$, there will be a correlation between the forces imparted on our particle because these forces are due to the same collisional process that is taking place. But if we consider times $t \gg t_{\rm coll}$, the force will be due to a different collision with a different particle. The statement that the force is uncorrelated means that the force imparted by later collisions knows nothing about earlier collisions. Mathematically, this means

$$\langle a(t_1)a(t_2)\rangle = 0 \quad \text{when} \quad |t_1 - t_2| \gg t_{\rm coll}. \tag{VII.1.7}$$

"Fast" means that the timescales of interest all satisfy $|t_1 - t_2| \gg t_{\rm coll}$, and so we can effectively take $t_{\rm coll} \to 0$. That *doesn't* mean that the correlations vanish; rather,

$$\boxed{\langle a(t_1)a(t_2)\rangle = \varepsilon\,\delta(t_2 - t_1)} \tag{VII.1.8}$$

where ε governs the strength of the correlations. Equation (VII.1.8) is called *white noise*. As written by David Tong in his excellent lecture notes on kinetic theory,[14]

> It is valid whenever the environment relaxes back down to equilibrium much faster than does the system of interest. This guarantees that, although the system is still reeling from the previous kick, the environment remembers nothing of what went before and kicks again, as fresh and random as the first time.

Let's see where this leads.

Using (VII.1.8) in (VII.1.6) and assuming $t_2 \geqslant t_1 > 0$, we have

$$\begin{aligned}\langle v(t_1)v(t_2)\rangle &= \langle v(t_1)\rangle\langle v(t_2)\rangle + \int_0^{t_1} {\rm d}t_1' \int_0^{t_2} {\rm d}t_2'\, \varepsilon\,\delta(t_2' - t_1')\, {\rm e}^{\gamma(t_1'+t_2'-t_1-t_2)} \\ &= \langle v(t_1)\rangle\langle v(t_2)\rangle + \varepsilon \int_0^{t_1} {\rm d}t_1'\, {\rm e}^{\gamma(2t_1'-t_1-t_2)} \\ &= \langle v(t_1)\rangle\langle v(t_2)\rangle + \frac{\varepsilon}{2\gamma}\Big[{\rm e}^{-\gamma(t_2-t_1)} - {\rm e}^{-\gamma(t_2+t_1)}\Big].\end{aligned} \tag{VII.1.9}$$

At late times $(t_1, t_2 \to \infty)$,

$$\langle v(t_1)v(t_2)\rangle \to \frac{\varepsilon}{2\gamma}\,{\rm e}^{-\gamma(t_2-t_1)}; \tag{VII.1.10}$$

i.e., the correlation between velocities decays. So, if you know $v(t_1)$, then $v(t_2)$ will be similar if $t_2 < t_1 + 1/\gamma$; but if you wait any longer, all bets are off on the velocity at t_2 based on $v(t_1)$. (This is called an *Ornstein–Uhlenbeck process*; this process converges to the same PDF for any initial condition. Every Markovian, stationary, Gaussian process in an Ornstein–Uhlenbeck process.)

Note further that

$$\langle v^2(t)\rangle \to \frac{\varepsilon}{2\gamma} \quad \text{as} \quad t \to \infty. \tag{VII.1.11}$$

In this long-time limit, one would expect energy equipartition between the test particle and the noise bath:

$$\langle v^2(t)\rangle = \frac{v_{\rm th}^2}{2} \quad \Longrightarrow \quad \varepsilon = \gamma v_{\rm th}^2, \tag{VII.1.12}$$

which is a *fluctuation-dissipation theorem* (Einstein 1905, 1956). This is remarkable –

[14] www.damtp.cam.ac.uk/user/tong/kinetic.html

there is a relationship between the diffusion of the particle in velocity space (statistically) and the slowing-down of the particle due to drag (statistically).

Before further discussion, let us finish calculating the linear and quadratic expectation values for our 1D-1V problem. We already have

$$\begin{aligned}\langle v(t)\rangle = v_0\mathrm{e}^{-\gamma t}, \quad \langle z(t)\rangle = z_0 + \frac{v_0}{\gamma}\big(1-\mathrm{e}^{-\gamma t}\big),\\ \langle v^2(t)\rangle - \langle v(t)\rangle^2 \doteq \langle \delta v^2(t)\rangle = \frac{\varepsilon}{2\gamma}\big(1-\mathrm{e}^{-2\gamma t}\big);\end{aligned}$$

now compute

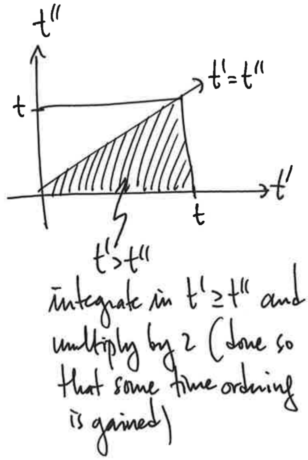
$$\begin{aligned}
\langle z(t)v(t)\rangle &= \left\langle \left[z_0 + \int_0^t \mathrm{d}t'\, v(t')\right] v(t)\right\rangle \\
&= z_0\langle v(t)\rangle + \int_0^t \mathrm{d}t'\, \langle v(t)v(t')\rangle \\
&= z_0\langle v(t)\rangle + \int_0^t \mathrm{d}t'\, \langle v(t)\rangle\langle v(t')\rangle + \frac{\varepsilon}{2\gamma}\int_0^t \mathrm{d}t' \left[\mathrm{e}^{-\gamma(t-t')} - \mathrm{e}^{-\gamma(t+t')}\right] \\
&= \langle v(t)\rangle\left[z_0 + \int_0^t \mathrm{d}t'\, v_0\mathrm{e}^{-\gamma t'}\right] + \frac{\varepsilon}{2\gamma}\int_0^t \mathrm{d}t' \left[\mathrm{e}^{-\gamma(t-t')} - \mathrm{e}^{-\gamma(t+t')}\right] \\
&= \langle v(t)\rangle\langle z(t)\rangle + \frac{\varepsilon}{2\gamma}\left[\frac{1}{\gamma}\big(1-\mathrm{e}^{-\gamma t}\big) + \frac{1}{\gamma}\big(\mathrm{e}^{-2\gamma t} - \mathrm{e}^{-\gamma t}\big)\right]
\end{aligned}$$

$$\Longrightarrow \langle \delta z(t)\delta v(t)\rangle \doteq \langle z(t)v(t)\rangle - \langle z(t)\rangle\langle v(t)\rangle = \frac{\varepsilon}{2\gamma^2}\big(1-\mathrm{e}^{-\gamma t}\big)^2 \tag{VII.1.13}$$

and

$$\begin{aligned}
\langle z^2(t)\rangle &= \left\langle \left[z_0 + \int_0^t \mathrm{d}t'\, v(t')\right]\left[z_0 + \int_0^t \mathrm{d}t''\, v(t'')\right]\right\rangle \\
&= z_0^2 + 2z_0\int_0^t \mathrm{d}t'\, \langle v(t')\rangle + \int_0^t \mathrm{d}t' \int_0^t \mathrm{d}t''\, \langle v(t')v(t'')\rangle \\
&= \langle z(t)\rangle^2 \cancel{- \frac{v_0^2}{\gamma^2}\big(1-\mathrm{e}^{-\gamma t}\big)^2} + \cancel{\left[\int_0^t \mathrm{d}t'\, \langle v(t')\rangle\right]^2} \\
&\quad + \frac{\varepsilon}{2\gamma}\int_0^t \mathrm{d}t' \int_0^t \mathrm{d}t'' \left[\mathrm{e}^{-\gamma(t'-t'')} - \mathrm{e}^{-\gamma(t'+t'')}\right] \\
&= \langle z(t)\rangle^2 + \frac{\varepsilon}{2\gamma}\int_0^t \mathrm{d}t' \int_0^t \mathrm{d}t''\, \underbrace{2H(t'-t'')}_{\substack{\text{gives needed}\\ \text{time ordering}\\ \text{(see figure)}}}\left[\mathrm{e}^{-\gamma(t'-t'')} - \mathrm{e}^{-\gamma(t'+t'')}\right] \\
&= \langle z(t)\rangle^2 + \frac{\varepsilon}{\gamma}\int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' \left[\mathrm{e}^{-\gamma(t'-t'')} - \mathrm{e}^{-\gamma(t'+t'')}\right] \\
&= \langle z(t)\rangle^2 + \frac{\varepsilon}{\gamma}\int_0^t \mathrm{d}t' \left[\frac{1}{\gamma}\Big(1-\mathrm{e}^{-\gamma t'}\Big) + \frac{1}{\gamma}\Big(\mathrm{e}^{-2\gamma t'} - \mathrm{e}^{-\gamma t'}\Big)\right] \\
&= \langle z(t)\rangle^2 + \frac{\varepsilon}{\gamma^2}\left[t + \frac{2}{\gamma}\big(\mathrm{e}^{-\gamma t} - 1\big) + \frac{1}{2\gamma}\big(1-\mathrm{e}^{-2\gamma t}\big)\right]
\end{aligned}$$

$$\Longrightarrow \langle \delta z^2(t)\rangle \doteq \langle z^2(t)\rangle - \langle z(t)\rangle^2 = \frac{\varepsilon}{2\gamma^3}\big(2\gamma t - 3 + 4\mathrm{e}^{-\gamma t} - \mathrm{e}^{-2\gamma t}\big). \tag{VII.1.14}$$



These results are summarized in the table below (which follows Table 6.1 of Krommes (2018)):[15]

	$\gamma t \ll 1$ (short times)	$\gamma t \gg 1$ (long times)
$\langle v \rangle$	$v_0(1-\gamma t)$ (collisional slowing down)	0 (randomization of particle velocity)
$\langle z \rangle$	$z_0 + v_0 t$ (free streaming)	$z_0 + v_0/\gamma \doteq z_0 + \lambda_{\mathrm{mfp}}$ (randomized in a mean free path)
$\langle \delta v^2 \rangle$	εt (velocity-space diffusion)	$\varepsilon/2\gamma \doteq T$ (thermalization)
$\langle \delta z \delta v \rangle$	$\varepsilon t^2/2$	$\varepsilon/2\gamma^2$
$\langle \delta z^2 \rangle$	$\varepsilon t^3/3$	$\varepsilon t/\gamma^2$ (real-space diffusion)

The result $\langle \delta z^2 \rangle \simeq \varepsilon t^3/3$ for $\gamma t \ll 1$ is particularly important; it implies that there is rapidly increasing uncertainty in the particle position at short times. Sometimes this is referred to as "orbit diffusion" or "resonance broadening" (Dupree 1966).

The next thing to do is to connect this exercise, devoid of plasma physics, to everything we've done up until now. Before doing so, a few things are worth noting:

(1) The Einstein relation $\varepsilon = \gamma v_{\mathrm{th}}^2$ gives an excellent way to determine Boltzmann's constant experimentally. We found $\langle \delta z^2(t) \rangle = \varepsilon t/\gamma^2 = v_{\mathrm{th}}^2 t/\gamma$, and so, if we know temperature and the drag rate, we can measure $\langle \delta z^2 \rangle$ versus time and determine Boltzmann's constant. For example, with $\gamma = 6\pi\eta a/m$ (Stokes' law, where η is dynamic viscosity and a is the radius of the particle), we have $k_{\mathrm{Boltz}} = \langle \delta z^2(t) \rangle (3\pi\eta a/Tt)$. In 3D, $\langle |\delta \boldsymbol{r}(t)|^2 \rangle = 3\varepsilon t/\gamma^2$, in which case $k_{\mathrm{Boltz}} = \langle |\delta \boldsymbol{r}(t)|^2 \rangle (\pi\eta a/Tt)$. This formula was used by Jean Baptiste Perrin in 1909 to measure k_{Boltz} (published in 1912); the value was consistent with other values previously determined from tests of the ideal gas law and from measuring blackbody radiation. This experiment constituted the first experimental

[15] $\gamma = 0$, or the short-time limit shown in the table, corresponds to a "Wiener process": $\mathrm{d}v/\mathrm{d}t = a(t)$ for $\langle a(t)a(t') \rangle = \varepsilon\, \delta(t-t')$. This gives Brownian motion.

demonstration of the physical reality of atoms and molecules, and it earned him the Nobel Prize in 1926. (Wikipedia has a very nice page on Brownian motion; the history of the discovery and theoretical development is quite interesting.)

(2) Because $\Delta t \gg t_{\text{coll}}$ and the kicks are statistically independent, $a(t)$ is the sum of many independent random variables. By the central limit theorem, $a(\Delta t)$ is essentially Gaussian – "Gaussian white noise". Moreover, since the sum of two jointly Gaussian variables is also a Gaussian, then z and v are Gaussian. This means that the probability distribution functions of z and v are completely specified by the first and second cumulants $\langle z(t)\rangle$, $\langle v(t)\rangle$, $\langle z(t)v(t')\rangle$, $\langle v(t)v(t')\rangle$, and $\langle z(t)z(t')\rangle$.

(3) Note that $\langle v\rangle = 0$ for $\gamma t \to \infty$ but that $\langle \delta z\rangle = v_0/\gamma \doteq \lambda_{\text{mfp}}$. In other words, the velocity is completely randomized by the kicks, but the particle position has advanced by a mean free path. Thus, spatial diffusion! (Mathematically, $\langle \delta z\rangle \neq \langle \delta v\rangle t$ as $\gamma t \to \infty$.)

(4) Equation (VII.1.8) can be inverted:

$$\int_t^\infty \mathrm{d}t' \left[\langle a(t)a(t')\rangle = \varepsilon\,\delta(t'-t)\right].$$

Writing $t' = t + \tau$, we have

$$\int_0^\infty \mathrm{d}\tau \left[\langle a(t)a(t+\tau)\rangle = \varepsilon\,\delta(\tau)\right].$$

This should hold for any time:

$$\Longrightarrow \quad \boxed{\int_0^\infty \mathrm{d}\tau\,\langle a(0)a(\tau)\rangle = \frac{\varepsilon}{2}} \tag{VII.1.15}$$

(The factor of 1/2 comes from integrating half of a δ-function.) This is *Taylor's formula* for diffusion. Correlation functions are intimately related to transport coefficients.

(5) What if $a(t)$ had a finite correlation time? Recall

$$\langle \delta v^2(t)\rangle = \int_0^t \mathrm{d}t' \int_0^t \mathrm{d}t''\,\langle a(t')a(t'')\rangle\,\mathrm{e}^{\gamma(t'+t''-2t)}. \tag{VII.1.16}$$

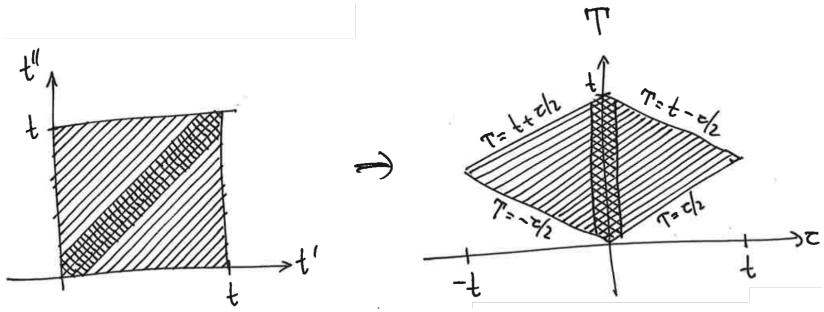
Set $\gamma = 0$, since it will only affect the long-time limit when the precise value of t_{coll} shouldn't matter anyhow. Furthermore, let us assume $a(t)$ to be stationary, i.e.,

$$\langle a(t')a(t'')\rangle = \langle a(t'-t')a(t''-t')\rangle = \langle a(0)a(\tau)\rangle.$$

Then (VII.1.16) becomes

$$\begin{aligned}\langle \delta v^2(t)\rangle &= \int_0^t \mathrm{d}t' \int_0^t \mathrm{d}t''\,\langle a(t')a(t'')\rangle \\ &= \int_{-t}^0 \mathrm{d}\tau\,\langle a(0)a(\tau)\rangle \int_{-\tau/2}^{t+\tau/2} \mathrm{d}T + \int_0^t \mathrm{d}\tau\,\langle a(0)a(\tau)\rangle \int_{\tau/2}^{t-\tau/2} \mathrm{d}T,\end{aligned}$$

where $\tau \doteq t' - t''$ and $T \doteq (1/2)(t' + t'')$. This change of variables looks like this:



where the hatched regions denote the region where substantial auto-correlations in $a(t)$ occur (i.e., $|\tau| < t_{\rm coll}$). Then,

$$\begin{aligned}
\langle \delta v^2(t)\rangle &= \int_{-t}^{0} \mathrm{d}\tau \, \langle a(0)a(\tau)\rangle \Big(t + \frac{\tau}{2} + \frac{\tau}{2}\Big) + \int_0^t \mathrm{d}\tau \, \langle a(0)a(\tau)\rangle \Big(t - \frac{\tau}{2} - \frac{\tau}{2}\Big) \\
&= -\int_t^0 \mathrm{d}\tau \, \langle a(0)a(-\tau)\rangle (t-\tau) + \int_0^t \mathrm{d}\tau \, \langle a(0)a(\tau)\rangle (t-\tau) \\
&= \int_0^t \mathrm{d}\tau \, (t-\tau) \langle a(0)a(-\tau) + a(0)a(\tau)\rangle .
\end{aligned}$$

But $\langle a(0)a(\tau)\rangle = \langle a(-\tau)a(0)\rangle = \langle a(0)a(-\tau)\rangle$, so

$$\langle \delta v^2(t)\rangle = 2t \int_0^t \mathrm{d}\tau \, (1 - \tau/t) \langle a(0)a(\tau)\rangle . \tag{VII.1.17}$$

Note that, if $\langle a(0)a(\tau)\rangle = \varepsilon\,\delta(\tau)$, then $\langle \delta v^2(t)\rangle = \varepsilon t$, as needed. In general, $\langle a(0)a(\tau)\rangle$ has some finite area defined by

$$t_{\rm coll} \doteq \frac{1}{\langle a(0)a(0)\rangle} \int_0^\infty \mathrm{d}\tau \, \langle a(0)a(\tau)\rangle , \tag{VII.1.18}$$

where (to remind you) $t_{\rm coll}$ is the time a collision takes (equivalently, the "auto-correlation time" of the forcing). Then the correction term in (VII.1.17) (i.e., $-\tau/t$) is of order $\sim(t_{\rm coll}/t)$. When $t_{\rm coll} \ll t$, this term is negligible and we recover $\langle \delta v^2(t)\rangle = 2t \int_0^t \mathrm{d}\tau \, \langle a(0)a(\tau)\rangle$, which is Taylor's formula with $\varepsilon \doteq \langle \delta v^2(t)\rangle / t$ as $t \to \infty$. Then, $\varepsilon/2 = \langle a^2(0)\rangle t_{\rm coll}$. (Almost anything looks like a delta function from far enough away!)

(6) We've done all this for scalar fields, but things easily generalize to vectors: the standard Langevin problem can be written as

$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \boldsymbol{v} \quad \text{and} \quad \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\gamma \boldsymbol{v} + \boldsymbol{a}(t),$$

with

$$\langle \boldsymbol{a}(t)\boldsymbol{a}(t')\rangle = \varepsilon\,\delta(t-t')\boldsymbol{I}.$$

Of course, one could imagine variations to this... maybe the drag force is more general, say, $\boldsymbol{A}(\boldsymbol{v})$. Maybe the correlation tensor $\varepsilon\boldsymbol{I}$ is more general, say, $\boldsymbol{B}(\boldsymbol{v})$. (These names are highly suggestive, aren't they?) Perhaps $a_i(t)$ has a correlation with $a_j(t')$ (e.g., if there is a magnetic field), or perhaps only one component is being forced. One useful forcing scheme in kinematic dynamo theory is due to Kazantsev (1968) (and independently introduced in the context of passive scalar advection by Kraichnan (1968)):

$$\langle v_i(t,\boldsymbol{r})v_j(t',\boldsymbol{r}')\rangle = \delta(t-t')\kappa_{ij}(\boldsymbol{r}-\boldsymbol{r}')$$

with

$$\kappa_{ij}(\boldsymbol{y}) = \kappa_0\delta_{ij} - \frac{1}{2}\kappa_2\left(y^2\delta_{ij} - \frac{1}{2}y_iy_j\right) + \dots$$

being a Taylor expansion of the velocity correlation tensor. This drives magnetic-field growth in a way whose statistics are exactly solvable.

(7) Because we've taken $a(t)$ to be independent of velocity and position, things have been nice and simple. We had a *linear* Langevin equation. More generally, $a(t)$ will depend on position and/or velocity: $a = a(t, x(t))$, where $x = (z, v)$ denotes the full phase-space coordinate. If so, then the Langevin equation becomes *non-linear*. We must solve

$$\frac{\mathrm{d}z}{\mathrm{d}t} = v \quad \text{and} \quad \frac{\mathrm{d}v}{\mathrm{d}t} = -\gamma v + a(t, x(t)).$$

The random forcing must be computed along the phase-space trajectory of the particle (cf. (VII.1.16)):

$$\langle \delta v^2(t)\rangle = \int_0^t \mathrm{d}t' \int_0^t \mathrm{d}t''\, \langle a(t', x(t'))a(t'', x(t''))\rangle \mathrm{e}^{\gamma(t'+t''-2t)}.$$

This is a *Lagrangian correlation.* If $a(t)$ is stationary, then (setting $\gamma = 0$ for simplicity) we have

$$\langle \delta v^2(t)\rangle = 2t\int_0^t \mathrm{d}\tau\, \langle a(\tau, x(\tau))a(0, x(0))\rangle,$$

in which case

$$\boxed{\varepsilon = 2\int_0^\infty \mathrm{d}\tau\, \langle a(\tau, x(\tau))a(0, x(0))\rangle} \qquad \text{(VII.1.19)}$$

is the diffusion coefficient. The Lagrangian correlation is the important one, since it takes into consideration the phase-space history of the particle as it moves in a phase-space-dependent environment. Of course, experimentally, Eulerian correlations are measured, i.e., correlations at a fixed location. One must then translate between the two, which is often difficult (if not impossible). But, did you notice that we've already done such a calculation? Remember when we were computing the jump moments to obtain the Fokker–Planck coefficients? We had (cf. (VI.3.7))

$$\langle \delta\boldsymbol{v}(t)\delta\boldsymbol{v}(t)\rangle = \int_0^t \mathrm{d}s \int_0^t \mathrm{d}s'\, \langle \boldsymbol{F}(s, \boldsymbol{x}(s))\boldsymbol{F}(s', \boldsymbol{x}(s'))\rangle,$$

and we solved perturbatively in small times $\omega_\mathrm{p}^{-1} \ll t \ll \nu^{-1}$ so that $\boldsymbol{x}(s) \simeq \boldsymbol{x}_0 + (\mathrm{d}\boldsymbol{x}/\mathrm{d}s)|_{\boldsymbol{x}=\boldsymbol{x}_0}\, s + \dots$. The manifestation of this Lagrangian focus was the $\delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v})$ factor in (VI.3.10) and (VI.3.14), which is related to quasi-linear diffusion (see §5.3.3 of Krommes (2018)).

(8) There is a subtlety when the random force depends upon the phase-space coordinate of the particle. If the force depends on $\boldsymbol{x}(t)$, but the system is getting a delta-function impulse at time t, then $\boldsymbol{x}(t)$ is not well defined during the kick. There are two interpretations that work around this:

- Ito: The forcing is dependent upon the position *just before* the kick occurs (i.e., forward Euler method)
- Stratonovich: The kick is really not a δ-function at all, but rather a process the occurs over a small period of time. Then, the forcing should be determined by the average of the position $\boldsymbol{x}(t)$ over this small (but finite) time interval (i.e., Crank-Nicholson method)

One would hope, of course, that the two give no macroscopic difference, but, alas, that isn't true. For us, what we've done is to consider a forcing with a small but finite auto-correlation time, and then we took $t_{\text{coll}} \to 0$. This gave things like $\int_0^\infty \mathrm{d}x\, \delta(x) = 1/2$. This is in line with the Stratonovich interpretation. (Ito would have $\int_0^\infty \mathrm{d}x\, \delta(x) = 1$.) If there is ever any confusion, it's best to work with an $a(t)$ that has a finite auto-correlation time, and then take that time to zero at the end of the calculation. (In a numerical simulation, the algorithm would determine the interpretation, so be careful.)

VII.2. Relationship between Langevin and Fokker–Planck

There is a close relationship between the Langevin equation and the Fokker–Planck equation. To see this, imagine a particle at some velocity at time t_0, say, v_0. If the subsequent evolution is noisy because of some stochastic forcing, then the Langevin equation can be used. But we will have no idea exactly where that particle will be in velocity space. The best we can do is speak of probabilities: $P(t, v; t_0, v_0)$ is the probability that the particle has velocity v at time t, given that it had velocity v_0 at time t_0. The Langevin approach is to express this uncertainty in terms of correlation functions. Here we ask: what $P(t, v; t_0, v_0)$ would lead to the same correlation functions that arose from the Langevin equation?

Denote the solution to the Langevin equation for a given noise function $f(t)$ as $v_f(t)$. If we knew what this noise is, then there is no uncertainty in $P(t, v)$; it is simply $\delta(v - v_f(t))$. Averaging over all possible noise, however, gives

$$P(t, v) = \langle \delta(v - v_f(t)) \rangle. \tag{VII.2.1}$$

(This should look very familiar from the beginning of the course and the discussion of Liouville-averaging the Klimontovich distribution. Not a coincidence!)

Using this definition, we ask: What is the probability $P(t + \Delta t, v; t, v - \Delta v)$ that the particle has velocity v at time $t + \Delta t$ given that it had velocity $v - \Delta v$ a moment earlier at time t? This is just

$$P(t + \Delta t, v; t, v - \Delta v) = \langle \delta(\Delta v - \widetilde{\Delta v}) \rangle, \tag{VII.2.2}$$

where $\widetilde{\Delta v}$ is a random variable denoting the change in velocity in a time Δt. Next, we Taylor expand the delta function:

$$P(t + \Delta t, v; t, v - \Delta v) = \left(1 + \langle \widetilde{\Delta v} \rangle \frac{\partial}{\partial v} + \frac{1}{2} \langle \widetilde{\Delta v}^2 \rangle \frac{\partial^2}{\partial v^2} + \dots \right) \delta(\Delta v). \tag{VII.2.3}$$

(If that made you queasy, just know that all this will be inside an integral. If that doesn't quell your stomach, then write the delta function as your favorite distribution that reduces to $\delta(x)$ in the limit $\epsilon \to 0$.) This goes into the Chapman–Kolmogorov equation (VI.1.1),

$$P(t + \Delta t, v; t_0, v_0) = \int \mathrm{d}(\Delta v)\, P(t + \Delta t, v; t, v - \Delta v) P(t, v - \Delta v; t_0, v_0).$$

Remember: this states that, given some initial arbitrary velocity v_0 at t_0, the particle must be somewhere in velocity space between then and $t + \Delta t$. Making the substitution, we obtain

$$\begin{aligned}
&P(t+\Delta t, v; t_0, v_0)\\
&\quad= \int \mathrm{d}(\Delta v) \left[\left(1 + \langle\widetilde{\Delta v}\rangle \frac{\partial}{\partial v} + \frac{1}{2}\langle\widetilde{\Delta v}^2\rangle \frac{\partial^2}{\partial v^2} + \dots\right)\delta(\Delta v)\right] P(t, v-\Delta v; t_0, v_0)\\
&\quad\overset{\text{bp}}{=} P(t, v; t_0, v_0) - \frac{\partial}{\partial v}\left[\langle\widetilde{\Delta v}\rangle P(t, v; t_0, v_0)\right] + \frac{1}{2}\frac{\partial^2}{\partial v^2}\left[\langle\widetilde{\Delta v}^2\rangle P(t, v; t_0, v_0)\right] + \dots
\end{aligned} \tag{VII.2.4}$$

For the Langevin problem, $\langle\widetilde{\Delta v}\rangle = -\gamma v \Delta t$ and (remember?) $\langle\widetilde{\Delta v}^2\rangle = \varepsilon\Delta t$ for small Δt (i.e., $\gamma\Delta t \ll 1$ but $\Delta t \gg t_{\mathrm{coll}}$). Expanding (VII.2.4) in small Δt then leads to

$$\begin{aligned}
P(t, v; t_0, v_0) + \Delta t \frac{\partial P(t, v; t_0, v_0)}{\partial t} &= P(t, v; t_0, v_0) - \frac{\partial}{\partial v}\left[-\gamma v \Delta t P(t, v; t_0, v_0)\right]\\
&\quad + \frac{1}{2}\frac{\partial^2}{\partial v^2}\left[\varepsilon \Delta P(t, v; t_0, v_0)\right] + \mathcal{O}(\Delta t^2)
\end{aligned}$$

$$\Longrightarrow \boxed{\frac{\partial P}{\partial t} = \gamma \frac{\partial}{\partial v}(vP) + \frac{\varepsilon}{2}\frac{\partial^2 P}{\partial v^2}} \tag{VII.2.5}$$

which is just the Fokker–Planck equation (VI.2.3) with drag and diffusion!

The equilibrium $\partial P/\partial t = 0$ is

$$\frac{\partial}{\partial v}\left(\gamma v P + \frac{\varepsilon}{2}\frac{\partial P}{\partial v}\right) = 0$$

$$\Longrightarrow P = \left(\frac{\gamma}{\pi\varepsilon}\right)^{1/2} \exp\left(-\frac{\gamma v^2}{\varepsilon}\right) = \frac{1}{\sqrt{\pi} v_{\mathrm{th}}} \exp\left(-\frac{v^2}{v_{\mathrm{th}}^2}\right), \tag{VII.2.6}$$

where, in the final equality, I have used the fluctuation-dissipation relation $\varepsilon = \gamma v_{\mathrm{th}}^2$. A Maxwellian, as expected.

For $\gamma = \text{const}$ and $\varepsilon = \text{const}$, the Fokker–Planck equation can, in fact, be solved exactly. I will use a Green's function approach:

$$P(t, v) = \int \mathrm{d}v_0\, G(t, v; 0, v_0) P(0, v_0), \tag{VII.2.7a}$$

with

$$\begin{aligned}
G(0, v; 0, v_0) &= \delta(v - v_0),\\
\frac{\partial G}{\partial t} &= \gamma \frac{\partial}{\partial v}(vG) + \frac{\varepsilon}{2}\frac{\partial^2 F}{\partial v^2}.
\end{aligned} \tag{VII.2.7b}$$

Start by writing

$$G(t, v; 0, v_0) = \int \frac{\mathrm{d}k}{2\pi} \mathrm{e}^{\mathrm{i}kv} G_k(t).$$

Then, away from $v = v_0$,

$$\frac{\mathrm{d}G_k}{\mathrm{d}t} + \gamma k \frac{\partial G_k}{\partial k} + \frac{\varepsilon k^2}{2} G_k = 0. \tag{VII.2.8}$$

Ansatz:

$$\ln G_k = -\mathrm{i}k v_0 \alpha(t) - \frac{\varepsilon k^2}{2}\beta(t).$$

(It must be Gaussian!) Differentiating in time and wavenumber gives

$$\frac{\mathrm{d}\ln G_k}{\mathrm{d}t} = -\mathrm{i}kv_0\frac{\mathrm{d}\alpha}{\mathrm{d}t} - \frac{\varepsilon k^2}{2}\frac{\mathrm{d}\beta}{\mathrm{d}t} \quad \text{and} \quad \gamma k\frac{\partial \ln G_k}{\partial k} = -\mathrm{i}kv_0\gamma\alpha - \gamma\varepsilon k^2\beta.$$

Plugging these expressions back into (VII.2.8) puts the following constraints on α and β:

$$\begin{aligned} \frac{\mathrm{d}\alpha}{\mathrm{d}t} + \gamma\alpha = 0 \quad &\text{and} \quad \frac{\mathrm{d}\beta}{\mathrm{d}t} + 2\gamma\beta - 1 = 0 \\ \Longrightarrow \alpha = \alpha_0\mathrm{e}^{-\gamma t} \quad &\text{and} \quad \beta = \frac{1}{2\gamma}\big(1 - \mathrm{e}^{-2\gamma t}\big), \end{aligned}$$

and so

$$\ln G_k = -\mathrm{i}kv_0\alpha_0\mathrm{e}^{-\gamma t} - \frac{\varepsilon k^2}{4\gamma}\big(1 - \mathrm{e}^{-2\gamma t}\big).$$

The initial condition $G_k(0) = \exp(-\mathrm{i}kv_0)$ implies $\alpha_0 = 1$, so that

$$G_k(t) = \exp\left[-\mathrm{i}kv_0\mathrm{e}^{-\gamma t} - \frac{\varepsilon k^2}{4\gamma}\big(1 - \mathrm{e}^{-2\gamma t}\big)\right]. \tag{VII.2.9}$$

Completing the square and defining $\sigma^2 \doteq (\varepsilon/2\gamma)(1 - \mathrm{e}^{-2\gamma t})$, equation (VII.2.9) becomes

$$G_k(t) = \exp\left[-\frac{\sigma^2}{2}\left(k + \frac{\mathrm{i}v_0\mathrm{e}^{-\gamma t}}{\sigma^2}\right)^2 - \frac{v_0^2\mathrm{e}^{-2\gamma t}}{2\sigma^2}\right].$$

This form is useful, since the Fourier transform of a Gaussian is a Gaussian:

$$\int\frac{\mathrm{d}k}{2\pi}\,\mathrm{e}^{\mathrm{i}kv}\mathrm{e}^{-k^2\sigma^2/2} = \frac{1}{\sqrt{2\pi\sigma^2}}\mathrm{e}^{-v^2/2\sigma^2}.$$

Defining $K \doteq k + \mathrm{i}v_0\mathrm{e}^{-\gamma t}/\sigma^2$ and proceeding with the inverse Fourier transform...

$$\begin{aligned} G(t,v) &= \int\frac{\mathrm{d}k}{2\pi}\,\mathrm{e}^{\mathrm{i}kv}G_k(t) \\ &= \int\frac{\mathrm{d}k}{2\pi}\,\mathrm{e}^{\mathrm{i}kv}\mathrm{e}^{-K^2\sigma^2/2}\mathrm{e}^{-v_0^2\mathrm{e}^{-2\gamma t}/2\sigma^2} \\ &= \mathrm{e}^{-(v-v_0\mathrm{e}^{-\gamma t})^2/2\sigma^2}\mathrm{e}^{v^2/2\sigma^2}\underbrace{\left(\int\frac{\mathrm{d}K}{2\pi}\,\mathrm{e}^{-K^2\sigma^2/2}\mathrm{e}^{\mathrm{i}Kv}\right)}_{=\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(\frac{-v^2}{2\sigma^2}\right)} \\ &= \frac{1}{\sqrt{2\pi\sigma^2}}\mathrm{e}^{-(v-v_0\mathrm{e}^{-\gamma t})^2/2\sigma^2}. \end{aligned}$$

Simple! With $2\sigma^2 = (\varepsilon/\gamma)(1 - \mathrm{e}^{-2\gamma t}) = v_{\mathrm{th}}^2(1 - \mathrm{e}^{-2\gamma t})$,

$$\boxed{G(t,v;0,v_0) = \frac{1}{\sqrt{\pi}v_{\mathrm{th}}}\frac{\mathrm{e}^{-(v-v_0\mathrm{e}^{-\gamma t})^2/v_{\mathrm{th}}^2(1-\mathrm{e}^{-2\gamma t})}}{\sqrt{1-\mathrm{e}^{-2\gamma t}}}} \tag{VII.2.10}$$

Note that, for $\gamma t \to \infty$, this becomes $\mathrm{e}^{-v^2/v_{\mathrm{th}}^2}/\sqrt{\pi}v_{\mathrm{th}}$! This relaxes any $P(0,v_0)$ towards a Maxwellian, by broadening the standard deviation and pushing the mean towards 0.

The lesson here is that the Fokker–Planck and Langevin approaches are in many ways the same. Both involve a coarse-graining of the time axis over the auto-correlation time of the forcing; both reveal a relaxation towards a Maxwellian equilibrium; and both, accordingly, describe the effects of drag and diffusion. Where they differ is that, in the Fokker–Planck approach, we only required the short-time jump moments. Once those are obtained, they can be fed into the Fokker–Planck equation, and the PDF can be evolved. By contrast, the Langevin approach describes the statistics captured by the PDF in the form of correlations ("cumulants"), which are explicitly calculated *for all time* depending upon the initial conditions. From this, it is clear that the Fokker–Planck approach is logistically superior, since short-time physics is easier to calculate than all-time physics. In fact, there are problems where one cannot explicitly solve the Langevin equation but can at least formulate the equivalent Fokker–Planck equation. See §§7.2,7.3 of Krommes (2018) for more.

PART VIII
Approximate collision operators

The transport will flow along the party line.

Alex Schekochihin, on the "KGB" operator

In practice, the Landau collision operator (IV.5.9) is rarely used, let along the Balescu–Lenard operator (IV.3.22). Simplifications to the Landau operator are instead employed, some more rigorously obtained than others. In this part, we investigate several of these approximate collision operators.

VIII.1. Krook (or BGK) operators

One of the crudest collision operators is the *Krook (or BGK) operator*:

$$C[f_\alpha] = -\nu\,(f_\alpha - f_{\mathrm{M},\alpha}), \quad \text{where} \quad f_{\mathrm{M},\alpha} = \frac{1}{\pi^{3/2} v_{\mathrm{th}\alpha}^3} \mathrm{e}^{-v^2/v_{\mathrm{th}\alpha}^2} \int \mathrm{d}\boldsymbol{v}\, f_\alpha. \tag{VIII.1.1}$$

This operator simply pushes f_α towards a Maxwellian with a specified temperature $T_\alpha = (1/2) m_\alpha v_{\mathrm{th}\alpha}^2$ while conserving particle number at some specified rate ν. (The papers are Bhatnagar *et al.* 1954 and Gross & Krook 1956.) The assumption here is that there are no sharp discontinuities in f_α and so it stays close to Maxwellian. This operator does *not* conserve momentum or energy. One could repair this flaw by constructing an operator that pushes f_α towards a Maxwellian in the correct frame:

$$C[f_\alpha] = -\nu\,(f_\alpha - f_{\mathrm{M},\alpha}), \quad \text{where} \quad f_{\mathrm{M},\alpha} = \frac{1}{\pi^{3/2} v_{\mathrm{th}\alpha}^3} \mathrm{e}^{-|\boldsymbol{v}-\boldsymbol{u}|^2/v_{\mathrm{th}\alpha}^2} \int \mathrm{d}\boldsymbol{v}\, f_\alpha \tag{VIII.1.2}$$

and $\boldsymbol{u} = (\int \boldsymbol{dv}\,\boldsymbol{v} f_\alpha)/(\int \mathrm{d}\boldsymbol{v}\, f_\alpha)$. A similar fix would patch up the temperature for energy conservation, using $(\int \mathrm{d}\boldsymbol{v}\, m_\alpha v^2 f_\alpha)/(\int \mathrm{d}\boldsymbol{v}\, f_\alpha)$ to determine the thermal speed to be used in $f_{\mathrm{M},\alpha}$. Other versions have been constructed for electron-ion collisions (Greene 1973) and for including trapped-particle effects (e.g. Kadomtsev & Pogutse 1970; Tang *et al.* 1976; Tang 1978).

VIII.2. Lenard–Bernstein operator

We have already encountered another approximate collision operator that annihilates a Maxwellian in HW02: the *Lenard–Bernstein operator*

$$C[f_\alpha] = \nu \frac{\partial}{\partial \boldsymbol{v}} \cdot \left(\boldsymbol{v} f_\alpha + \frac{v_{\text{th}\alpha}^2}{2} \frac{\partial f_\alpha}{\partial \boldsymbol{v}} \right), \tag{VIII.2.1}$$

which is described in Lenard & Bernstein (1958) and Dougherty (1964) (see Francisquez *et al.* 2022 for a recent update and additional references). This should also look familiar from our discussion of the Langevin equation (cf. (VII.2.5)), and thus it includes both drag and diffusion *and* pushes f_α towards a stationary Maxwellian. This operator conserves the number density of particles and represents some diffusion in velocity space, but uses the same collision frequency for both drag and diffusion. Thus, there is no distinction between pitch-angle scattering and energy diffusion. Momentum conservation may be gained by modifying (VIII.2.1) to push f_α towards a moving Maxwellian as follows (Kirkwood 1946):

$$C[f_\alpha] = \nu \frac{\partial}{\partial \boldsymbol{v}} \cdot \left[(\boldsymbol{v} - \boldsymbol{u}) f_\alpha + \frac{v_{\text{th}\alpha}^2}{2} \frac{\partial f_\alpha}{\partial \boldsymbol{v}} \right], \quad \boldsymbol{u} \equiv \left(\int \mathrm{d}\boldsymbol{v}\, \boldsymbol{v} f_\alpha \right) \Big/ \left(\int \mathrm{d}\boldsymbol{v}\, f_\alpha \right). \tag{VIII.2.2}$$

If you recall HW02, the Lenard–Bernstein collision operator takes on a particularly elegant form in Hermite space, $-m\nu$, where m is the Hermite number (the velocity-space analogue of the Fourier wavenumber k in real space). One can conserve momentum and energy by demanding that m start at $m = 3$. Nice.

VIII.3. Rosenbluth potentials

Now let us return to the Landau collision operator (IV.5.9),

$$\frac{\partial f_\alpha(t, \boldsymbol{v})}{\partial t} = \sum_\beta \frac{2\pi q_\alpha^2 q_\beta^2 \ln \lambda_{\alpha\beta}}{m_\alpha} \frac{\partial}{\partial \boldsymbol{v}} \cdot \int \mathrm{d}\boldsymbol{v}'\, \boldsymbol{\mathsf{U}}(\boldsymbol{v} - \boldsymbol{v}') \cdot \left(\frac{1}{m_\alpha} \frac{\partial}{\partial \boldsymbol{v}} - \frac{1}{m_\beta} \frac{\partial}{\partial \boldsymbol{v}'} \right) f_\alpha(\boldsymbol{v}) f_\beta(\boldsymbol{v}'),$$

where $\boldsymbol{\mathsf{U}}(\boldsymbol{u}) \doteq (u^2 \boldsymbol{\mathsf{I}} - \boldsymbol{u}\boldsymbol{u})/u^3$. A natural expansion parameter here is the mass ratio m_α/m_β. For example, with $\alpha = e$ and $\beta = i$, $m_\alpha/m_\beta \simeq 1/1836$ for a hydrogenic plasma, and so it's clear that the $(m_\alpha^{-1} \partial/\partial \boldsymbol{v} - m_\beta^{-1} \partial/\partial \boldsymbol{v}')$ part of the Landau operator can be simplified. The easiest way to achieve this is actually not with the Landau form given above, but rather using something called the Fokker–Planck–Rosenbluth form.

The Rosenbluth form (for short) constitutes a different way of interpreting the Fokker–Planck coefficients $\boldsymbol{A}_\alpha$ and $\boldsymbol{\mathsf{B}}_\alpha$, one which is particularly convenient for numerical work. Recall from (IV.4.4) that

$$\boldsymbol{A}_\alpha \doteq \frac{1}{m_\alpha} \sum_\beta \left(\frac{1}{m_\alpha} + \frac{1}{m_\beta} \right) \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\, \boldsymbol{k}\boldsymbol{k} \left| \frac{4\pi q_\alpha q_\beta}{k^2 \mathcal{D}(\boldsymbol{k} \cdot \boldsymbol{v}, \boldsymbol{k})} \right|^2 \int \mathrm{d}\boldsymbol{v}'\, \pi\delta(\boldsymbol{k} \cdot \boldsymbol{v} - \boldsymbol{k} \cdot \boldsymbol{v}') \cdot \frac{\partial f_\beta}{\partial \boldsymbol{v}'},$$

$$\boldsymbol{\mathsf{B}}_\alpha \doteq \frac{1}{m_\alpha} \sum_\beta \left(\frac{1}{m_\alpha} + \frac{1}{m_\alpha} \right) \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\, \boldsymbol{k}\boldsymbol{k} \left| \frac{4\pi q_\alpha q_\beta}{k^2 \mathcal{D}(\boldsymbol{k} \cdot \boldsymbol{v}, \boldsymbol{k})} \right|^2 \int \mathrm{d}\boldsymbol{v}'\, \pi\delta(\boldsymbol{k} \cdot \boldsymbol{v} - \boldsymbol{k} \cdot \boldsymbol{v}') f_\beta(\boldsymbol{v}').$$

As in §IV.5 (see (IV.5.2)), write

$$|\mathcal{D}(\boldsymbol{k} \cdot \boldsymbol{v}, \boldsymbol{k})|^2 = \left[1 + \frac{k_\mathrm{D}^2}{k^2} \operatorname{Re}(\alpha) \right]^2 + \left[\frac{k_\mathrm{D}^2}{k^2} \operatorname{Im}(\alpha) \right]^2,$$

where $\alpha = \alpha(\hat{\boldsymbol{k}} \cdot \boldsymbol{v})$ is only a function of the angle between $\boldsymbol{k}$ and $\boldsymbol{v}$ and not of k itself.

Then,

$$\boldsymbol{B}_\alpha = \sum_\beta \frac{4q_\alpha^2 q_\beta^2}{m_\alpha^2} \int \mathrm{d}\boldsymbol{v}'\, f_\beta(\boldsymbol{v}') \int \mathrm{d}\Omega_{\boldsymbol{k}}\, \hat{\boldsymbol{k}}\hat{\boldsymbol{k}}\, \delta(\hat{\boldsymbol{k}}\cdot\boldsymbol{v} - \hat{\boldsymbol{k}}\cdot\boldsymbol{v}') \int_0^\infty \frac{\mathrm{d}k}{k} \frac{1}{|1 + (k_\mathrm{D}/k)^2 \alpha(\hat{\boldsymbol{k}}\cdot\boldsymbol{v})|^2}. \tag{VIII.3.1}$$

The final integral in (VIII.3.1) – the one over k – may be simplified as follows:

$$\begin{aligned}
\int_0^\infty \frac{\mathrm{d}k}{k} \frac{1}{|1 + (k_\mathrm{D}/k)^2 \alpha(\hat{\boldsymbol{k}}\cdot\boldsymbol{v})|^2} &= \int_0^\infty \frac{\mathrm{d}k^2}{2} \frac{k^2}{\left[k^2 + k_\mathrm{D}^2 \operatorname{Re}(\alpha)\right]^2 + \left[k_\mathrm{D}^2 \operatorname{Im}(\alpha)\right]^2} \\
&= \underbrace{\int_{y_\mathrm{min}}^{y_\mathrm{max}}}_{\substack{\text{Replace } \infty \text{ by} \\ y_\mathrm{max} = k_\mathrm{max}^2 \\ + k_\mathrm{D}^2 \operatorname{Re}(\alpha) \text{ so} \\ \text{that integral} \\ \text{converges}}} \frac{\mathrm{d}y}{2} \frac{y - k_\mathrm{D}^2 \operatorname{Re}(\alpha)}{y^2 + \left[k_\mathrm{D}^2 \operatorname{Im}(\alpha)\right]^2} \qquad (\text{set } y = k^2 + k_\mathrm{D}^2 \operatorname{Re}(\alpha)) \\
&= \frac{1}{4} \ln\left\{ y^2 + \left[k_\mathrm{D}^2 \operatorname{Im}(\alpha)\right]\right\}\Bigg|_{y_\mathrm{min}}^{y_\mathrm{max}} - \frac{\operatorname{Re}(\alpha)}{2|\operatorname{Im}(\alpha)|} \tan^{-1}\left[\frac{y}{k_\mathrm{D}^2 |\operatorname{Im}(\alpha)|}\right]\Bigg|_{y_\mathrm{min}}^{y_\mathrm{max}} \\
&= \ln\left(\frac{k_\mathrm{max}}{k_\mathrm{D}}\right) + \frac{1}{4} \ln\left\{ \frac{[1 + (k_\mathrm{D}^2/k_\mathrm{max}^2)\operatorname{Re}(\alpha)]^2 + [(k_\mathrm{D}^2/k_\mathrm{max}^2)\operatorname{Im}(\alpha)]^2}{[\operatorname{Re}(\alpha)]^2 + [\operatorname{Im}(\alpha)]^2} \right\} \\
&\quad - \frac{\operatorname{Re}(\alpha)}{2|\operatorname{Im}(\alpha)|} \left\{ \frac{\pi}{2} - \tan^{-1}\left[\frac{\operatorname{Re}(\alpha)}{|\operatorname{Im}(\alpha)|}\right]\right\} \\
&= \ln\left(\frac{k_\mathrm{max}}{k_\mathrm{D}}\right) + \mathcal{O}(1) \text{ terms.}
\end{aligned}$$

Chandrasekhar (1943) calls the first term here the "dominant" term, since it diverges logarithmically as $k_\mathrm{max} \to \infty$ while the other terms are finite. Thus, with $\ln(k_\mathrm{max}/k_\mathrm{D}) \to \ln \lambda_{\alpha\beta}$ (i.e., the Coulomb logarithm), equation (VIII.3.1) becomes

$$\boldsymbol{B}_\alpha = \sum_\beta \frac{4q_\alpha^2 q_\beta^2 \ln \lambda_{\alpha\beta}}{m_\alpha^2} \int \mathrm{d}\boldsymbol{v}'\, f_\beta(\boldsymbol{v}') \int \mathrm{d}\Omega_{\boldsymbol{k}}\, \hat{\boldsymbol{k}}\hat{\boldsymbol{k}}\, \delta(\hat{\boldsymbol{k}}\cdot\boldsymbol{v} - \hat{\boldsymbol{k}}\cdot\boldsymbol{v}'). \tag{VIII.3.2}$$

The integral over $\Omega_{\boldsymbol{k}}$ is just what we found in the Landau operator, and so we're effectively capturing Debye shielding via the k_min cutoff. Recall from §IV.5 that

$$\frac{1}{\pi} \int \mathrm{d}\Omega_{\boldsymbol{k}}\, \hat{\boldsymbol{k}}\hat{\boldsymbol{k}}\, \delta(\hat{\boldsymbol{k}}\cdot\boldsymbol{v} - \hat{\boldsymbol{k}}\cdot\boldsymbol{v}') = \frac{u^2 \boldsymbol{I} - \boldsymbol{u}\boldsymbol{u}}{u^3} \doteq \boldsymbol{U}(u) = \frac{\partial^2 u}{\partial\boldsymbol{v}\partial\boldsymbol{v}},$$

where $\boldsymbol{u} \doteq \boldsymbol{v} - \boldsymbol{v}'$. Then (VIII.3.2) may be written as

$$\begin{aligned}
\boldsymbol{B}_\alpha &= \sum_\beta \frac{4\pi q_\alpha^2 q_\beta^2 \ln \lambda_{\alpha\beta}}{m_\alpha^2} \int \mathrm{d}\boldsymbol{v}'\, f_\beta(\boldsymbol{v}') \frac{\partial^2 u}{\partial\boldsymbol{v}\partial\boldsymbol{v}} \\
&= \sum_\beta \frac{4\pi q_\alpha^2 q_\beta^2 \ln \lambda_{\alpha\beta}}{m_\alpha^2} \frac{\partial^2}{\partial\boldsymbol{v}\partial\boldsymbol{v}} \underbrace{\int \mathrm{d}\boldsymbol{v}'\, f_\beta(\boldsymbol{v}') u}_{\doteq\, \psi_\beta(\boldsymbol{v})}.
\end{aligned}$$

Likewise,

$$\begin{aligned}
\boldsymbol{A}_\alpha &= \sum_\beta \frac{2q_\alpha^2 q_\beta^2 \ln\lambda_{\alpha\beta}}{m_\alpha}\left(\frac{1}{m_\alpha}+\frac{1}{m_\beta}\right)\int \mathrm{d}\boldsymbol{v}' \,\frac{\partial f_\beta}{\partial \boldsymbol{v}'}\boldsymbol{\cdot}\int \mathrm{d}\Omega_{\boldsymbol{k}}\,\hat{\boldsymbol{k}}\hat{\boldsymbol{k}}\,\delta(\hat{\boldsymbol{k}}\boldsymbol{\cdot}\boldsymbol{v}-\hat{\boldsymbol{k}}\boldsymbol{\cdot}\boldsymbol{v}') \\
&= \sum_\beta \frac{2\pi q_\alpha^2 q_\beta^2 \ln\lambda_{\alpha\beta}}{m_\alpha}\left(\frac{1}{m_\alpha}+\frac{1}{m_\beta}\right)\underbrace{\int \mathrm{d}\boldsymbol{v}' \,\frac{\partial f_\beta}{\partial \boldsymbol{v}'}\boldsymbol{\cdot}\frac{\partial^2 u}{\partial\boldsymbol{v}\partial\boldsymbol{v}}}_{\substack{\overset{\mathrm{bp}}{=} -\int \mathrm{d}\boldsymbol{v}'\, f_\beta(\boldsymbol{v}')\frac{\partial}{\partial\boldsymbol{v}'}\boldsymbol{\cdot}\frac{\partial^2 u}{\partial\boldsymbol{v}\partial\boldsymbol{v}} \\ = \int \mathrm{d}\boldsymbol{v}'\, f_\beta(\boldsymbol{v}')\frac{\partial}{\partial\boldsymbol{v}}\boldsymbol{\cdot}\frac{\partial^2 u}{\partial\boldsymbol{v}\partial\boldsymbol{v}} \\ = \int \mathrm{d}\boldsymbol{v}'\, f_\beta(\boldsymbol{v}')\frac{\partial}{\partial\boldsymbol{v}}\left(\nabla_v^2 u\right) \\ = \int \mathrm{d}\boldsymbol{v}'\, f_\beta(\boldsymbol{v}')\frac{\partial}{\partial\boldsymbol{v}}\left(\frac{2}{u}\right) \\ = \frac{\partial}{\partial\boldsymbol{v}}\int \mathrm{d}\boldsymbol{v}'\, f_\beta(\boldsymbol{v}')\frac{2}{u}}} \\
&= \sum_\beta \frac{4\pi q_\alpha^2 q_\beta^2 \ln\lambda_{\alpha\beta}}{m_\alpha}\left(\frac{1}{m_\alpha}+\frac{1}{m_\beta}\right)\frac{\partial}{\partial\boldsymbol{v}}\underbrace{\int \mathrm{d}\boldsymbol{v}'\, f_\beta(\boldsymbol{v}')\frac{1}{u}}_{\doteq\,\varphi_\beta(\boldsymbol{v})}.
\end{aligned}$$

Thus,

$$\boxed{\boldsymbol{A}_\alpha = \sum_\beta \frac{4\pi q_\alpha^2 q_\beta^2 \ln\lambda_{\alpha\beta}}{m_\alpha}\left(\frac{1}{m_\alpha}+\frac{1}{m_\beta}\right)\frac{\partial\varphi_\beta}{\partial\boldsymbol{v}}} \tag{VIII.3.3}$$

$$\boxed{\boldsymbol{\mathsf{B}}_\alpha = \sum_\beta \frac{4\pi q_\alpha^2 q_\beta^2 \ln\lambda_{\alpha\beta}}{m_\alpha^2}\frac{\partial^2\psi_\beta}{\partial\boldsymbol{v}\partial\boldsymbol{v}}} \tag{VIII.3.4}$$

with

$$\boxed{\varphi_\beta(\boldsymbol{v}) \doteq \int \mathrm{d}\boldsymbol{v}'\, f_\beta(\boldsymbol{v}')\frac{1}{|\boldsymbol{v}-\boldsymbol{v}'|}} \tag{VIII.3.5}$$

$$\boxed{\psi_\beta(\boldsymbol{v}) \doteq \int \mathrm{d}\boldsymbol{v}'\, f_\beta(\boldsymbol{v}')|\boldsymbol{v}-\boldsymbol{v}'|} \tag{VIII.3.6}$$

The functions φ_β and ψ_β are called *Rosenbluth potentials*, originally introduced by Rosenbluth *et al.* (1957). What's interesting (and why they are called "potentials") is that they satisfy the following Poisson-like equations:[16]

$$\boxed{\nabla_v^2\varphi_\beta = -4\pi f_\beta} \tag{VIII.3.7}$$

$$\boxed{\nabla_v^2\psi_\beta = 2\varphi_\beta} \tag{VIII.3.8}$$

This indicates that there is a close relationship between drag and diffusion, which is not particularly surprising given all the ways we've obtained $\boldsymbol{A}_\alpha$ and $\boldsymbol{\mathsf{B}}_\alpha$. Note the (arbitrary) choice of gauge: we can send $\varphi \to \varphi + a$ and $\psi \to \psi + b + \boldsymbol{c}\boldsymbol{\cdot}\boldsymbol{v}$ with a, b, and $\boldsymbol{c}$ constants and leave both $\boldsymbol{A}_\alpha$ and $\boldsymbol{\mathsf{B}}_\alpha$ unchanged.

There are a few insights gained by writing $\boldsymbol{A}_\alpha$ and $\boldsymbol{\mathsf{B}}_\alpha$ is this form. As Hinton (1983) writes (with his notation changed to conform to mine):

[16]Note: φ_β is usually written as h_β and ψ_β as g_β, but I'm using g and h for other things.

> φ_β is analogous to the potential in real space due to a charge density f_β, ψ_β is the potential which results from considering $-\varphi_\beta/2\pi$ as the charge density. It follows from this analogy that if f_β is spherically symmetric then ψ_β and φ_β are also, and that $\varphi_\beta(v)$ and $\psi_\beta(v)$ are affected only by $f_\beta(v')$ with $v' \leqslant v$. Thus, the dynamical friction vector $\boldsymbol{A}_\alpha(v)$ and the velocity diffusion tensor $\boldsymbol{\mathsf{B}}_\alpha(v)$ acting on a particle with a given speed v, are determined only by interactions with slower particles... It also follows that the dynamical friction force on a fast electron decreases as v^{-2}, by analogy with the electric field outside a spherically symmetric charge distribution.

One can also borrow all the knowledge we have about solving Poisson equations, such as expanding in spherical harmonics and/or using numerical recipes like relaxation or multi-grid methods. Given f_β, you solve for φ_β using (VIII.3.7); using φ_β, you solve for ψ_β using (VIII.3.8). Then you get $\boldsymbol{A}_\alpha$ and $\boldsymbol{\mathsf{B}}_\alpha$ and, thus, the collision operator. Or, perhaps ψ_β is easiest to obtain by computing $\int \mathrm{d}\boldsymbol{v}'\, f_\beta(\boldsymbol{v}')u$. If so, then φ_β is obtained using (VIII.3.8). Either way, the Fokker–Planck collision operator

$$
\begin{aligned}
\left(\frac{\partial f_\alpha}{\partial t}\right)_{\mathrm{c}} &= -\frac{\partial}{\partial \boldsymbol{v}} \cdot \left[\boldsymbol{A}_\alpha f_\alpha(\boldsymbol{v})\right] + \frac{1}{2}\frac{\partial}{\partial \boldsymbol{v}}\frac{\partial}{\partial \boldsymbol{v}} : \left[\boldsymbol{\mathsf{B}}_\alpha f_\alpha(\boldsymbol{v})\right] \\
&= \sum_\beta \frac{4\pi q_\alpha^2 q_\beta^2 \ln\lambda_{\alpha\beta}}{m_\alpha^2}\left\{-\frac{\partial}{\partial \boldsymbol{v}}\cdot\left[\left(1+\frac{m_\alpha}{m_\beta}\right)\frac{\partial \varphi_\beta}{\partial \boldsymbol{v}} f_\alpha(\boldsymbol{v})\right] + \frac{1}{2}\frac{\partial}{\partial \boldsymbol{v}}\frac{\partial}{\partial \boldsymbol{v}} : \left[\frac{\partial^2 \psi_\beta}{\partial \boldsymbol{v}\partial \boldsymbol{v}} f_\alpha(\boldsymbol{v})\right]\right\} \\
&= \sum_\beta \frac{4\pi q_\alpha^2 q_\beta^2 \ln\lambda_{\alpha\beta}}{m_\alpha^2}\frac{\partial}{\partial \boldsymbol{v}}\cdot\left[-\left(\cancel{1}+\frac{m_\alpha}{m_\beta}\right)\frac{\partial \varphi_\beta}{\partial \boldsymbol{v}} f_\alpha(\boldsymbol{v}) + \cancel{\frac{1}{2}\left(\frac{\partial}{\partial \boldsymbol{v}}\nabla_v^2\psi_\beta\right)} f_\alpha(\boldsymbol{v})\right. \\
&\qquad\qquad \left. + \frac{1}{2}\frac{\partial^2 \psi_\beta}{\partial \boldsymbol{v}\partial \boldsymbol{v}}\cdot\frac{\partial f_\alpha}{\partial \boldsymbol{v}}\right],
\end{aligned} \tag{VIII.3.9}
$$

where (VIII.3.8) has been used to cancel the two slashed terms. Thus,

$$
\boxed{\left(\frac{\partial f_\alpha}{\partial t}\right)_{\mathrm{c}} = \sum_\beta \frac{4\pi q_\alpha^2 q_\beta^2 \ln\lambda_{\alpha\beta}}{m_\alpha^2}\frac{\partial}{\partial \boldsymbol{v}}\cdot\left[-\frac{m_\alpha}{m_\beta}\frac{\partial \varphi_\beta}{\partial \boldsymbol{v}} f_\alpha(\boldsymbol{v}) + \frac{1}{2}\frac{\partial^2 \psi_\beta}{\partial \boldsymbol{v}\partial \boldsymbol{v}}\cdot\frac{\partial f_\alpha}{\partial \boldsymbol{v}}\right]} \tag{VIII.3.10}
$$

This is the *Rosenbluth form* of the Landau collision operator. This form will be used in the next three sections (§§VIII.4–VIII.6) to investigate electron–ion collisions, ion–electron collisions, and collisions with a Maxwellian background.

VIII.4. Electron–ion and ion–impurity collisions

First, consider $\alpha = e$ and $\beta = i$. The first term in (VIII.3.10) is then $\sim\mathcal{O}(m_e/m_i)$ and can be dropped. As for the second term, proportional to $\partial^2\psi_i/\partial\boldsymbol{v}\partial\boldsymbol{v}$, we can treat the ion distribution function $f_i(\boldsymbol{v})$ as if it were a delta function:

$$
f_i(\boldsymbol{v}) \simeq n_i \delta(\boldsymbol{v} - \boldsymbol{u}_i),
$$

where n_i is the number density and $\boldsymbol{u}_i$ is the mean "fluid" velocity of the ions. This is because, for $T_i \sim T_e$, the thermal speeds of the two species are widely different:

$$
v_{\mathrm{th}i} = \sqrt{\frac{2T_i}{m_i}} = \left(\frac{T_i}{T_e}\right)^{1/2}\left(\frac{m_e}{m_i}\right)^{1/2} v_{\mathrm{th}e} \ll v_{\mathrm{th}e}.
$$

In other words, from the standpoint of a thermal electron, the ion distribution is extremely narrow. Thus,

$$
\psi_i(\boldsymbol{v}) \doteq \int \mathrm{d}\boldsymbol{v}'\, f_i(\boldsymbol{v}')|\boldsymbol{v} - \boldsymbol{v}'| \simeq n_i|\boldsymbol{v} - \boldsymbol{u}_i| \tag{VIII.4.1}
$$

and (VIII.3.10) becomes

$$\left(\frac{\partial f_e}{\partial t}\right)_{\mathrm{c},i} \simeq \frac{2\pi Z^2 e^4 n_i \ln\lambda_{ei}}{m_e^2}\frac{\partial}{\partial \boldsymbol{v}}\cdot\left(\frac{\partial^2 |\boldsymbol{v}-\boldsymbol{u}_i|}{\partial \boldsymbol{v}\partial \boldsymbol{v}}\cdot\frac{\partial f_e}{\partial \boldsymbol{v}}\right). \tag{VIII.4.2}$$

This operator will drive the electrons towards an isotropic distribution in the frame of the ions, so that $f_e \to f_e(|\boldsymbol{v}-\boldsymbol{u}_i|)$. Let us show this.

With $\boldsymbol{u}_i/v_{\mathrm{the}} \sim v_{\mathrm{thi}}/v_{\mathrm{the}} \sim \mathcal{O}(\sqrt{m_e/m_i})$, note that

$$\psi_i(\boldsymbol{v}) \simeq n_i v\left(1-\frac{\boldsymbol{v}\cdot\boldsymbol{u}_i}{v^2}\right)+\mathcal{O}\left(\frac{m_e}{m_i}\right).$$

We then expand $f_e(\boldsymbol{v}) = f_e^{(0)}(\boldsymbol{v}) + f_e^{(1)}(\boldsymbol{v}) + \dots$ in the small parameter $\sqrt{m_e/m_i}$, so that $f_e^{(0)} \sim \mathcal{O}(1)$, $f_e^{(1)} \sim \mathcal{O}(\sqrt{m_e/m_i})$, and so on. Equation (VIII.4.2) becomes

$$\left(\frac{\partial f_e}{\partial t}\right)_{\mathrm{c},i} = \frac{2\pi Z^2 e^4 n_i \ln\lambda_{ei}}{m_e^2}\frac{\partial}{\partial \boldsymbol{v}}\cdot\left[\frac{\partial^2 v}{\partial \boldsymbol{v}\partial \boldsymbol{v}}\cdot\frac{\partial f_e^{(0)}}{\partial \boldsymbol{v}} + \frac{\partial^2 v}{\partial \boldsymbol{v}\partial \boldsymbol{v}}\cdot\frac{\partial f_e^{(1)}}{\partial \boldsymbol{v}} - \frac{\partial^2}{\partial \boldsymbol{v}\partial \boldsymbol{v}}\left(\frac{\boldsymbol{v}\cdot\boldsymbol{u}_i}{v}\right)\cdot\frac{\partial f_e^{(0)}}{\partial \boldsymbol{v}}\right] + \mathcal{O}\left(\frac{m_e}{m_i}\right). \tag{VIII.4.3}$$

Recall $\partial^2 v/\partial\boldsymbol{v}\partial\boldsymbol{v} = (v^2\mathsf{I} - \boldsymbol{v}\boldsymbol{v})/v^3$, which is orthogonal to $\boldsymbol{v}$. If $f_e^{(0)}$ is isotropic, then $\partial f_e^{(0)}/\partial\boldsymbol{v} \propto \boldsymbol{v}$ and so the first term in (VIII.4.3) vanishes. (This is why we're keeping terms of $\mathcal{O}(\sqrt{m_e/m_i})$!) If $f_e^{(0)}$ is not isotropic, then that first term is

$$\frac{\partial}{\partial \boldsymbol{v}}\cdot\left(\frac{\partial^2 v}{\partial \boldsymbol{v}\partial \boldsymbol{v}}\cdot\frac{\partial f_e^{(0)}}{\partial \boldsymbol{v}}\right) = \frac{\partial}{\partial \boldsymbol{v}}\cdot\left[\frac{1}{v}\frac{\partial f_e^{(0)}}{\partial \boldsymbol{v}} - \frac{\boldsymbol{v}}{v^3}\left(\boldsymbol{v}\cdot\frac{\partial f_e^{(0)}}{\partial \boldsymbol{v}}\right)\right].$$

Adopting spherical coordinates in velocity space, (v,θ,ϕ), with the z axis oriented along $\boldsymbol{u}_i$, this expression becomes

$$\frac{1}{v^3}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f_e^{(0)}}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 f_e^{(0)}}{\partial\phi^2}\right].$$

No v derivatives! Physically, this is because electron–ion collisions do not change the magnitude of the electron velocity but rather only its direction (at least to the order in mass ratio at which we're working). The kinetic energy of an ion barely changes during an encounter with an electron and so, because energy should be conserved in such an interaction, the kinetic energy of the electron barely changes as well; only angular changes in the electron distribution occur, as the electrons are deflected in pitch angle. This is so important that it has its own symbol:

$$\boxed{\mathcal{L}[f] \doteq \frac{1}{2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}\right]} \tag{VIII.4.4}$$

Thus, the first term in brackets on the right-hand side of (VIII.4.3) is

$$\frac{4\pi Z^2 e^4 n_i \ln\lambda_{ei}}{m_e^2}\frac{1}{v^3}\mathcal{L}\left[f_e^{(0)}\right] = \frac{3\sqrt{\pi}}{4}\underbrace{\left(\frac{4\sqrt{2\pi}Z^2 e^4 n_i\ln\lambda_{ei}}{3\sqrt{m_e}T_e^{3/2}}\right)}_{\doteq\, \tau_{ei}^{-1}\ \text{(recall (V.3.11))}}\left(\frac{v_{\mathrm{the}}}{v}\right)^3\mathcal{L}\left[f_e^{(0)}\right]. \tag{VIII.4.5}$$

This is the *Lorentz* or *pitch-angle-scattering* operator. The reason for the latter name is that, defining the pitch angle as $\xi \doteq \cos\theta \in [-1, 1]$,

$$\mathcal{L} = \frac{1}{2}\left[\frac{\partial}{\partial\xi}\left(1-\xi^2\right)\frac{\partial}{\partial\xi} + \frac{1}{1-\xi^2}\frac{\partial^2}{\partial\phi^2}\right]; \tag{VIII.4.6}$$

thus, this part of the electron–ion collision operator tries to make the electron distribution isotropic via diffusion in pitch angle. Returning to equation (VIII.4.3), the electron–ion collision operator may then be written as

$$\begin{aligned}\left(\frac{\partial f_e}{\partial t}\right)_{\mathrm{c},i} &\simeq \frac{3\sqrt{\pi}}{4\tau_{ei}}\left(\frac{v_{\mathrm{the}}}{v}\right)^3 \mathcal{L}[f_e^{(0)}] \\ &\quad + \frac{2\pi Z^2 e^4 n_i \ln\lambda_{ei}}{m_e^2}\frac{\partial}{\partial \boldsymbol{v}}\boldsymbol{\cdot}\left[\frac{\partial^2 v}{\partial\boldsymbol{v}\partial\boldsymbol{v}}\boldsymbol{\cdot}\frac{\partial f_e^{(1)}}{\partial\boldsymbol{v}} - \frac{\partial^2}{\partial\boldsymbol{v}\partial\boldsymbol{v}}\left(\frac{\boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{u}_i}{v}\right)\boldsymbol{\cdot}\frac{\partial f_e^{(0)}}{\partial\boldsymbol{v}}\right],\end{aligned} \tag{VIII.4.7}$$

dropping $\mathcal{O}(m_e/m_i)$ terms.

Next, the $\mathcal{O}(\sqrt{m_e/m_i})$ terms (i.e., the second line of (VIII.4.7)). The first term there, proportional to $\partial f_e^{(1)}/\partial\boldsymbol{v}$, is simple – this is, again, the Lorentz operator. The second term can be simplified considerably if we anticipate $f_e^{(0)}$ being isotropic, as would result from the Lorentz operator. In this case,

$$\begin{aligned}-\frac{\partial}{\partial\boldsymbol{v}}\boldsymbol{\cdot}\left[\frac{\partial^2}{\partial\boldsymbol{v}\partial\boldsymbol{v}}\left(\frac{\boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{u}_i}{v}\right)\boldsymbol{\cdot}\frac{\partial f_e^{(0)}}{\partial\boldsymbol{v}}\right] &= -\frac{\partial}{\partial\boldsymbol{v}}\boldsymbol{\cdot}\left[\frac{\partial^2}{\partial\boldsymbol{v}\partial\boldsymbol{v}}\left(\frac{\boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{u}_i}{v}\right)\boldsymbol{\cdot}\frac{\boldsymbol{v}}{v}\frac{\partial f_e^{(0)}}{\partial v}\right] \\ &= -\frac{\partial}{\partial\boldsymbol{v}}\boldsymbol{\cdot}\left[\frac{\partial^3 v}{\partial\boldsymbol{v}\partial\boldsymbol{v}\partial\boldsymbol{v}}\boldsymbol{:}\boldsymbol{u}_i\frac{\boldsymbol{v}}{v}\frac{\partial f_e^{(0)}}{\partial v}\right] \quad \left(\text{since } \frac{\partial}{\partial\boldsymbol{v}}\frac{v_k}{v} = \frac{\partial^2 v}{\partial\boldsymbol{v}\partial v_k}\right) \\ &= -\frac{\partial}{\partial\boldsymbol{v}}\boldsymbol{\cdot}\left[\boldsymbol{v}\boldsymbol{\cdot}\frac{\partial}{\partial\boldsymbol{v}}\left(\frac{\partial^2 v}{\partial\boldsymbol{v}\partial\boldsymbol{v}}\right)\boldsymbol{\cdot}\frac{\boldsymbol{u}_i}{v}\frac{\partial f_e^{(0)}}{\partial v}\right] \quad \left(\text{since } \frac{\partial^2 v}{\partial v_j\partial v_k} = \frac{\partial^2 v}{\partial v_k\partial v_j}\right) \\ &= -\frac{\partial}{\partial\boldsymbol{v}}\boldsymbol{\cdot}\left(-\frac{\partial^2 v}{\partial\boldsymbol{v}\partial\boldsymbol{v}}\boldsymbol{\cdot}\frac{\boldsymbol{u}_i}{v}\frac{\partial f_e^{(0)}}{\partial v}\right),\end{aligned} \tag{VIII.4.8}$$

since

$$\begin{aligned}v_k\frac{\partial}{\partial v_k}\left(\frac{\partial^2 v}{\partial v_\ell\partial v_m}\right) &= v_k\frac{\partial}{\partial v_k}\left(\frac{v^2\delta_{\ell m} - v_\ell v_m}{v^3}\right) \\ &= -\delta_{\ell m}\frac{v_k}{v^2}\frac{\partial v}{\partial v_k} - \frac{v_k}{v^3}\left(\delta_{k\ell}v_m + \delta_{km}v_\ell\right) + \frac{3v_k v_\ell v_m}{v^4}\frac{\partial v}{\partial v_k} \\ &= -\delta_{\ell m}\frac{1}{v} + \frac{v_\ell v_m}{v^3} \\ &= -\frac{\partial^2 v}{\partial v_\ell\partial v_m}.\end{aligned}$$

Picking the calculation back up...

$$\begin{aligned}\text{(VIII.4.8)} &= \frac{\partial}{\partial\boldsymbol{v}}\boldsymbol{\cdot}\left(\frac{\partial^2 v}{\partial\boldsymbol{v}\partial\boldsymbol{v}}\boldsymbol{\cdot}\frac{\boldsymbol{u}_i}{v}\frac{\partial f_e^{(0)}}{\partial v}\right) \\ &= \frac{\partial}{\partial\boldsymbol{v}}\boldsymbol{\cdot}\left[\frac{\partial^2 v}{\partial\boldsymbol{v}\partial\boldsymbol{v}}\boldsymbol{\cdot}\frac{\partial}{\partial\boldsymbol{v}}\left(\frac{\boldsymbol{v}}{v}\frac{\partial f_e^{(0)}}{\partial v}\right)\boldsymbol{\cdot}\boldsymbol{u}_i\right],\end{aligned} \tag{VIII.4.9}$$

since

$$u_k\left(\frac{\partial^2 v}{\partial v_\ell \partial v_m}\right)\frac{\partial}{\partial v_m}\left(\frac{v_k}{v}\frac{\partial f}{\partial v}\right) = u_k\left(\frac{\partial^2 v}{\partial v_\ell \partial v_m}\right)\left[v_k\frac{\partial}{\partial v_m}\left(\frac{1}{v}\frac{\partial f}{\partial v}\right) + \delta_{km}\frac{1}{v}\frac{\partial f}{\partial v}\right]$$
$$= u_k v_k\left(\frac{\partial^2 v}{\partial v_\ell \partial v_m}\right)\frac{v_m}{v}\frac{\partial}{\partial v}\left(\frac{1}{v}\frac{\partial f}{\partial v}\right) + u_k\left(\frac{\partial^2 v}{\partial v_\ell \partial v_k}\right)\frac{1}{v}\frac{\partial f}{\partial v}$$
$$= (\boldsymbol{u}\cdot\boldsymbol{v})\underbrace{(\boldsymbol{v}\cdot\boldsymbol{U})}_{\substack{=0\\ \text{since}\\ \boldsymbol{v}\perp\boldsymbol{U}}}\frac{1}{v}\frac{\partial}{\partial v}\left(\frac{1}{v}\frac{\partial f}{\partial v}\right) + \frac{\partial^2 v}{\partial\boldsymbol{v}\partial\boldsymbol{v}}\cdot\frac{\boldsymbol{u}}{v}\frac{\partial f}{\partial v},$$

and so

$$(\text{VIII.4.9}) = \frac{\partial}{\partial\boldsymbol{v}}\cdot\left[\frac{\partial^2 v}{\partial\boldsymbol{v}\partial\boldsymbol{v}}\cdot\frac{\partial}{\partial\boldsymbol{v}}\left(\frac{\boldsymbol{v}\cdot\boldsymbol{u}_i}{v}\frac{\partial f_e^{(0)}}{\partial v}\right)\right] = \frac{2}{v^3}\mathcal{L}\left[\frac{\boldsymbol{v}\cdot\boldsymbol{u}_i}{v}\frac{\partial f_e^{(0)}}{\partial v}\right]. \tag{VIII.4.10}$$

All together, then, the *electron–ion collision operator* is

$$\boxed{\left(\frac{\partial f_e}{\partial t}\right)_{\mathrm{c},i} \simeq \frac{3\sqrt{\pi}}{4\tau_{ei}}\left(\frac{v_{\mathrm{the}}}{v}\right)^3\mathcal{L}\left[f_e^{(0)} + f_e^{(1)} + \frac{\boldsymbol{v}\cdot\boldsymbol{u}_i}{v}\frac{\partial f_e^{(0)}}{\partial v}\right]} \tag{VIII.4.11}$$

This operator pushes

$$f_e^{(1)} \to -\frac{\boldsymbol{v}\cdot\boldsymbol{u}_i}{v}\frac{\partial f_e^{(0)}}{\partial\boldsymbol{v}} + g_e^{(1)}(v),$$

where $g_e^{(1)}(v)$ is some $\mathcal{O}(\sqrt{m_e/m_i})$ isotropic distribution. Thus, under the action of (VIII.4.11), the total distribution tends towards

$$f_e(\boldsymbol{v}) = f_e^{(0)}(v) + f_e^{(1)} = f_e^{(0)}(v) + g_e^{(1)}(v) - \frac{\boldsymbol{v}\cdot\boldsymbol{u}_i}{v}\frac{\partial f_e^{(0)}}{\partial v}$$
$$= f_e^{(0)}(v) - \frac{\boldsymbol{v}\cdot\boldsymbol{u}_i}{v}\frac{\partial f_e^{(0)}}{\partial v} \quad (\text{just renaming } f_e^{(0)})$$
$$= f_e^{(0)}(|\boldsymbol{v}-\boldsymbol{u}_i|) + \mathcal{O}\left(\frac{m_e}{m_i}\right). \tag{VIII.4.12}$$

So, indeed, the electron–ion collision operator pushes $f_e(\boldsymbol{v})$ towards an isotropic distribution in the frame of the ions.

Note that the electron–ion collision operator (VIII.4.11) depends in no way on the ion mass (only the ion charge). Thus, we can just sum over all ion species and declare some Z_{eff}. The only assumption here was $m_e/m_i \ll 1$, and so we could just as well apply this operator to ion–impurity collisions if $m_Z \gg m_i$, where m_Z is the mass of the impurity.

While we have the electron–ion collision operator fresh in our minds, let's (i) make sure it conserves energy and (ii) calculate the friction force between electrons and ion and compare it with the polarization drag we calculated in (V.3.11). First, the collisional energy gained or lost by the electrons is

$$\begin{aligned}
&\int \mathrm{d}\boldsymbol{v}\, \frac{1}{2} m_e v^2 \left(\frac{\partial f_e}{\partial t}\right)_{\mathrm{c},i} \\
&\quad = \frac{3\sqrt{\pi}}{8\tau_{ei}} T_e \int \mathrm{d}\boldsymbol{v}\, \frac{v_{\mathrm{th}e}}{v} \left[\frac{\partial}{\partial \xi}(1-\xi^2)\frac{\partial}{\partial \xi} + \frac{1}{1-\xi^2}\frac{\partial^2}{\partial \phi^2}\right] \left(f_e + \frac{\boldsymbol{v}\cdot\boldsymbol{u}_i}{v}\frac{\partial f_e^{(0)}}{\partial v}\right) \\
&\quad \overset{\mathrm{bp}}{=} 0. \qquad \text{(VIII.4.13)}
\end{aligned}$$

Good. Energy is conserved.[17] Next, the friction force is

$$\begin{aligned}
\boldsymbol{R}_{ei} &= \int \mathrm{d}\boldsymbol{v}\, m_e \boldsymbol{v} \left(\frac{\partial f_e}{\partial t}\right)_{\mathrm{c},i} \\
&= \frac{3\sqrt{\pi}}{4\tau_{ei}} \int \mathrm{d}\boldsymbol{v}\, m_e \boldsymbol{v} \left(\frac{v_{\mathrm{th}e}}{v}\right)^3 \mathcal{L}\Bigg[\underbrace{f_e}_{①} + \underbrace{\frac{\boldsymbol{v}\cdot\boldsymbol{u}_i}{v}\frac{\partial f_e^{(0)}}{\partial v}}_{②}\Bigg] \qquad \text{(VIII.4.14)}
\end{aligned}$$

Let's handle each term separately:

$$\begin{aligned}
② &= \frac{3\sqrt{\pi}}{4\tau_{ei}} m_e \int_0^{2\pi} \mathrm{d}\phi \int_{-1}^{+1} \mathrm{d}(\cos\theta) \int_0^\infty \mathrm{d}v\, v^2 \big[v\cos\theta\, \hat{\boldsymbol{z}} + v\sin\theta(\cos\phi\, \hat{\boldsymbol{x}} + \sin\phi\, \hat{\boldsymbol{y}})\big] \\
&\qquad \times \left(\frac{v_{\mathrm{th}e}}{v}\right)^3 \mathcal{L}\left[\frac{\boldsymbol{v}\cdot\boldsymbol{u}_i}{v}\frac{\partial f_e^{(0)}}{\partial v}\right] \\
&= \frac{3\sqrt{\pi}}{4\tau_{ei}} m_e \times 2\pi v_{\mathrm{th}e}^3 \hat{\boldsymbol{z}} \int_{-1}^{+1} \mathrm{d}(\cos\theta) \int_0^\infty \mathrm{d}v\, \cos\theta\, \mathcal{L}\left[\frac{\boldsymbol{v}\cdot\boldsymbol{u}_i}{v}\frac{\partial f_e^{(0)}}{\partial v}\right] \\
&\qquad \text{(since neither } \mathcal{L}\text{, nor } f_e^{(0)}\text{, nor } \boldsymbol{v}\cdot\boldsymbol{u}_i \text{ depend upon } \phi) \\
&= \frac{3\pi^{3/2}}{4\tau_{ei}} m_e v_{\mathrm{th}e}^3 u_i \hat{\boldsymbol{z}} \underbrace{\int_{-1}^{+1} \mathrm{d}\xi\, \xi \frac{\partial}{\partial \xi}(1-\xi^2)\frac{\partial}{\partial \xi}\xi}_{=-4/3} \underbrace{\int_0^\infty \mathrm{d}v\, \frac{\partial f_e^{(0)}}{\partial v}}_{=-f_e^{(0)}(0)} \\
&= \frac{\pi^{3/2} m_e v_{\mathrm{th}e}^3}{\tau_{ei}} \boldsymbol{u}_i f_e^{(0)}(0). \qquad \text{(VIII.4.15)}
\end{aligned}$$

So that's one piece. Apparently, only $v = 0$ electrons are important here, which makes sense: slow electrons are more likely to have strong interactions with ions. The other

[17] Energy transfer between the electrons and ions is best calculated using the ion–electron collision operator. Again, it's not that there is zero energy exchange; it's just that it occurs at higher order in the mass-ratio expansion. See §VIII.5.

piece is

$$
\begin{aligned}
① &= \frac{3\sqrt{\pi}}{8\tau_{ei}} m_e \int \mathrm{d}\boldsymbol{v}\, \boldsymbol{v} \left(\frac{v_{\rm the}}{v}\right)^3 \left[\frac{\partial}{\partial\xi}(1-\xi^2)\frac{\partial}{\partial\xi} + \frac{1}{1-\xi^2}\frac{\partial^2}{\partial\phi^2}\right] f_e^{(1)} \\
&\overset{\rm bp}{=} -\frac{3\sqrt{\pi}}{8\tau_{ei}} m_e \int \mathrm{d}\boldsymbol{v} \left(\frac{v_{\rm the}}{v}\right)^3 \left[\frac{\partial\boldsymbol{v}}{\partial\xi}(1-\xi^2)\frac{\partial}{\partial\xi} + \frac{1}{1-\xi^2}\frac{\partial\boldsymbol{v}}{\partial\phi}\frac{\partial}{\partial\phi}\right] f_e^{(1)} \\
&\overset{\rm bp}{=} \frac{3\sqrt{\pi}}{8\tau_{ei}} m_e \int \mathrm{d}\boldsymbol{v} \left(\frac{v_{\rm the}}{v}\right)^3 \left\{\frac{\partial}{\partial\xi}\left[\frac{\partial\boldsymbol{v}}{\partial\xi}(1-\xi^2)\right] + \frac{1}{1-\xi^2}\frac{\partial^2\boldsymbol{v}}{\partial\phi^2}\right\} f_e^{(1)} \\
&= \frac{3\sqrt{\pi}}{8\tau_{ei}} m_e \int \mathrm{d}\boldsymbol{v} \left(\frac{v_{\rm the}}{v}\right)^3 \left(-2\boldsymbol{v} + \frac{\boldsymbol{v}_\perp}{1-\xi^2} - \frac{\boldsymbol{v}_\perp}{1-\xi^2}\right) f_e^{(1)} \\
&= -\frac{3\sqrt{\pi}}{4\tau_{ei}} m_e v_{\rm the}^3 \int \mathrm{d}\boldsymbol{v}\, \frac{\boldsymbol{v}}{v^3} f_e^{(1)}. \qquad \text{(VIII.4.16)}
\end{aligned}
$$

Again, the slowest electrons suffer the most friction. Inserting (VIII.4.15) and (VIII.4.16) back into (VIII.4.14) leads to

$$
\boxed{\boldsymbol{R}_{ei} = \frac{m_e \pi^{3/2} v_{\rm the}^3}{\tau_{ei}} \left[\boldsymbol{u}_i f_e^{(0)}(0) - \frac{3}{4\pi}\int \mathrm{d}\boldsymbol{v}\, \frac{\boldsymbol{v}}{v^3} f_e^{(1)}\right]} \qquad \text{(VIII.4.17)}
$$

If we assume that $f_e(\boldsymbol{v})$ is a Maxwellian with mean velocity $\boldsymbol{u}_e$, so that $f_e^{(1)} = (2\boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{u}_e/v_{\rm the}^2) f_{\rm Me}(v)$, then the second term in (VIII.4.17) becomes (after some straightforward algebra and integration) $-n_e\boldsymbol{u}_e/\pi^{3/2}v_{\rm the}^3$. With $f_{\rm Me}^{(0)}(0) = n_e/\pi^{3/2}v_{\rm the}^3$,

$$
\boxed{\boldsymbol{R}_{ei}\ \text{(Maxwellian electrons)} = \frac{m_e n_e}{\tau_{ei}}(\boldsymbol{u}_i - \boldsymbol{u}_e)} \qquad \text{(VIII.4.18)}
$$

That makes sense.

VIII.5. Ion–electron collisions

As was stated in the previous section, electron–ion collision operator conserves energy because the ion barely moves when an electron "hits" it. To calculate the energy transfer, it is better to use the ion–electron collision operator, which describes the effect of collisions with electrons on the ion distribution function. The situation is similar to Brownian motion, since each ion is subjected to a bombardment of light, fast particles.

Again, we employ a mass-ratio expansion, this time on

$$
\begin{aligned}
\left(\frac{\partial f_i}{\partial t}\right)_{\rm c,e} &= \frac{4\pi Z^2 e^4 \ln\lambda_{ie}}{m_i^2}\frac{\partial}{\partial\boldsymbol{v}}\boldsymbol{\cdot}\left[-\frac{m_i}{m_e}\frac{\partial\varphi_e}{\partial\boldsymbol{v}} f_i(\boldsymbol{v}) + \frac{1}{2}\frac{\partial^2\psi_e}{\partial\boldsymbol{v}\partial\boldsymbol{v}}\boldsymbol{\cdot}\frac{\partial f_i}{\partial\boldsymbol{v}}\right] \\
&= \frac{m_e n_e}{m_i n_i}\frac{1}{\tau_{ei}}\frac{\partial}{\partial\boldsymbol{v}}\boldsymbol{\cdot}\left[-\frac{\partial\varphi_e}{\partial\boldsymbol{v}} f_i(\boldsymbol{v}) + \frac{m_e}{2m_i}\frac{\partial^2\psi_e}{\partial\boldsymbol{v}\partial\boldsymbol{v}}\boldsymbol{\cdot}\frac{\partial f_i}{\partial\boldsymbol{v}}\right] \times \frac{3}{4\pi}\frac{\pi^{3/2}v_{\rm the}^3}{n_e} \qquad \text{(VIII.5.1)}
\end{aligned}
$$

Following the method employed in the previous section, we expand $f_e(\boldsymbol{v}') = f_e^{(0)}(v') + f_e^{(1)}(\boldsymbol{v}') + \dots$ in powers of $\sqrt{m_e/m_i}$, and calculate the Rosenbluth potentials φ_e and ψ_e to leading order. First,

$$
\varphi_e(\boldsymbol{v}) = \int \mathrm{d}\boldsymbol{v}'\, f_e(\boldsymbol{v}')\frac{1}{|\boldsymbol{v}-\boldsymbol{v}'|}. \qquad \text{(VIII.5.2)}
$$

We could proceed as before by expanding $|\boldsymbol{v}-\boldsymbol{v}'|$ in powers of $\sqrt{m_e/m_i}$, with $v \sim v_{\rm thi}$ and $v' \sim v_{\rm the}$, but then we would miss contributions to the potential near $v' = 0$ (since

0 is smaller than an infinitesimal number). We must tread carefully. Instead, start by expanding $|\boldsymbol{v}-\boldsymbol{v}'|^{-1}$ in Legendre polynomials, so that equation (VIII.5.2) becomes

$$\varphi_e(\boldsymbol{v}) = \int\limits_{v'<v} \mathrm{d}\boldsymbol{v}'\, f_e(\boldsymbol{v}')\frac{1}{v}\sum_{\ell=0}^{\infty}\left(\frac{v'}{v}\right)^{\ell} P_\ell(\xi') + \int\limits_{v<v'} \mathrm{d}\boldsymbol{v}'\, f_e(\boldsymbol{v}')\frac{1}{v'}\sum_{\ell=0}^{\infty}\left(\frac{v}{v'}\right)^{\ell} P_\ell(\xi'), \tag{VIII.5.3}$$

where P_ℓ are the Legendre polynomials and their argument $\xi \doteq \boldsymbol{v}\cdot\boldsymbol{v}'/vv'$. The first integral in (VIII.5.3) captures contributions from those (few) electrons that are slower than an ion having speed $v \sim v_{\mathrm{th}i}$. Because the electron distribution is much broader than the ion distribution – a consequence of the mass-ratio expansion – the electron distribution in this integral may be approximated accurately by its value at $v'=0$. Because there aren't many electrons that satisfy $v'<v$, it is enough to retain only the leading-order contribution to the electron distribution, $f_e^{(0)}(0)$. Then

$$\int\limits_{v'<v} \mathrm{d}\boldsymbol{v}'\, f_e(\boldsymbol{v}')\frac{1}{v}\sum_{\ell=0}^{\infty}\left(\frac{v'}{v}\right)^{\ell} P_\ell(\xi') \simeq 4\pi\int_0^v \mathrm{d}v'\, v'^2 f_e^{(0)}(0)\frac{1}{v} = \frac{4\pi}{3}v^2 f_e^{(0)}(0). \tag{VIII.5.4}$$

The second integral in (VIII.5.3) is slightly more complicated, because it contains contributions both from thermal elections that are statistically much faster than an ion with speed v, and from sub-thermal electrons whose speeds are comparable to, but slightly larger than, an ion with speed $v \sim v_{\mathrm{th}i}$. These two contributions may be separated after expanding $f_e(\boldsymbol{v}')$ in mass ratio:

$$\begin{aligned}
&\int\limits_{v<v'} \mathrm{d}\boldsymbol{v}'\left[f_e^{(0)}(v') + f_e^{(1)}(\boldsymbol{v}') + \dots\right]\frac{1}{v'}\sum_{\ell=0}^{\infty}\left(\frac{v}{v'}\right)^{\ell} P_\ell(\xi') \\
&\qquad \simeq 4\pi\int_v^{v_{\mathrm{th}i}} \mathrm{d}v'\, v'^2 f_e^{(0)}(0)\frac{1}{v'} + \int\limits_{v<v'} \mathrm{d}\boldsymbol{v}'\, f_e^{(1)}(\boldsymbol{v}')\frac{v}{v'^2}P_1(\xi') + \dots \\
&\qquad = -2\pi(v^2 - v_{\mathrm{th}i}^2) f_e^{(0)}(0) + \int\limits_{v<v'} \mathrm{d}\boldsymbol{v}'\, f_e^{(1)}(\boldsymbol{v}')\frac{\boldsymbol{v}\cdot\boldsymbol{v}'}{v'^3} + \dots
\end{aligned} \tag{VIII.5.5}$$

Combining (VIII.5.4)–(VIII.5.5) and taking the required velocity-space gradient yields

$$\frac{\partial \varphi_e^{(1)}}{\partial \boldsymbol{v}} \simeq -\frac{4\pi}{3}\boldsymbol{v} f_e^{(0)}(0) + \int \mathrm{d}\boldsymbol{v}'\, f_e^{(1)}(\boldsymbol{v}')\frac{\boldsymbol{v}'}{v'^3}. \tag{VIII.5.6}$$

Then, using (VIII.4.17) to eliminate the above velocity-space integral of $f_e^{(1)}(\boldsymbol{v}')$ in favor of the electron–ion friction force $\boldsymbol{R}_{ei}$, equation (VIII.5.6) may be recast as

$$\frac{\partial \varphi_e^{(1)}}{\partial \boldsymbol{v}} \simeq -\frac{4\pi}{3}\boldsymbol{v} f_e^{(0)}(0) - \frac{4\pi}{3}\left[\frac{\tau_{ei}\boldsymbol{R}_{ei}}{m_e \pi^{3/2} v_{\mathrm{th}e}^3} - \boldsymbol{u}_i f_e^{(0)}(0)\right]. \tag{VIII.5.7}$$

This will go into (VIII.5.1). Next we calculate ψ_e, which may be obtained from (VIII.3.6):

$$\psi_e(\boldsymbol{v}) = \int \mathrm{d}\boldsymbol{v}'\, f_e(\boldsymbol{v}')|\boldsymbol{v}-\boldsymbol{v}'| \simeq \int \mathrm{d}\boldsymbol{v}'\, f_e^{(0)}(v')|\boldsymbol{v}-\boldsymbol{v}'|. \tag{VIII.5.8}$$

Here we need only retain the leading-order contribution from f_e, because the competing term in (VIII.5.1) is multiplied by $m_i/m_e \gg 1$. Taking two velocity-space gradients of

(VIII.5.8) and using (IV.5.8),

$$\frac{\partial^2 \psi_e^{(0)}}{\partial \boldsymbol{v} \partial \boldsymbol{v}} \simeq \mathsf{I}\, \frac{8\pi}{3} \int \mathrm{d}v'\, v' f_e^{(0)}(v'). \tag{VIII.5.9}$$

Finally, inserting (VIII.5.7) and (VIII.5.9) back into (VIII.5.1) and rearranging terms gives the *ion–electron collision operator*,

$$\boxed{\begin{aligned}\left(\frac{\partial f_i}{\partial t}\right)_{\mathrm{c},e} = \frac{\boldsymbol{R}_{ei}}{m_i n_i} \cdot \frac{\partial f_i}{\partial \boldsymbol{v}} + \frac{m_e n_e}{m_i n_i} \frac{1}{\tau_{ei}} \frac{\partial}{\partial \boldsymbol{v}} \cdot \Biggl[(\boldsymbol{v} - \boldsymbol{u}_i) f_i(\boldsymbol{v}) \, \frac{f_e^{(0)}(0)\pi^{3/2} v_{\mathrm{the}}^3}{n_e} \\ + \frac{m_e}{m_i} \frac{\partial f_i}{\partial \boldsymbol{v}} \int_0^\infty \mathrm{d}v'\, v' \, \frac{f_e^{(0)}(v')\pi^{3/2} v_{\mathrm{the}}^3}{n_e} \Biggr]\end{aligned}} \tag{VIII.5.10}$$

As usual, let's evaluate (VIII.5.10) for Maxwellian electrons:

$$\begin{aligned}\left(\frac{\partial f_i}{\partial t}\right)_{\mathrm{c},e} (\text{Maxwellian electrons}) &= \frac{\boldsymbol{R}_{ei}}{m_i n_i} \cdot \frac{\partial f_i}{\partial \boldsymbol{v}} \\ &+ \frac{m_e n_e}{m_i n_i} \frac{1}{\tau_{ei}} \frac{\partial}{\partial \boldsymbol{v}} \cdot \left[(\boldsymbol{v} - \boldsymbol{u}_i) f_i(\boldsymbol{v}) + \frac{T_e}{m_i} \frac{\partial f_i}{\partial \boldsymbol{v}} \right]\end{aligned} \tag{VIII.5.11}$$

with $\boldsymbol{R}_{ei}$ given by (VIII.4.18). (Does the second line in (VIII.5.11) look familiar? If not, re-read §VIII.2.) For a Maxwellian plasma, equation (VIII.5.11) becomes

$$\begin{aligned}\left(\frac{\partial f_i}{\partial t}\right)_{\mathrm{c},e} (\text{Maxwellian plasma}) &= -\frac{\boldsymbol{R}_{ei}}{n_i T_i} \cdot (\boldsymbol{v} - \boldsymbol{u}_i) f_{\mathrm{M}i} \\ &- \frac{2 m_e n_e}{m_i n_i} \frac{1}{\tau_{ei}} \left(1 - \frac{T_e}{T_i}\right) \left(\frac{|\boldsymbol{v} - \boldsymbol{u}_i|^2}{v_{\mathrm{th}i}^2} - \frac{3}{2} \right) f_{\mathrm{M}i}.\end{aligned} \tag{VIII.5.12}$$

From this form of the ion–electron collision operator, it is clear that it acts to equilibrate the ion and electron temperatures – see (VIII.5.16) below for more.

Next, we compute the ion–electron friction force and energy exchange. For general $\widehat{f}_e^{(0)}(v) \doteq f_e^{(0)}(v)\pi^{3/2} v_{\mathrm{the}}^3 / n_e$, the former is

$$\begin{aligned}\boldsymbol{R}_{ie} &= \int \mathrm{d}\boldsymbol{v}\, m_i \boldsymbol{v} \left(\frac{\partial f_i}{\partial t}\right)_{\mathrm{c},e} \\ &= \frac{\boldsymbol{R}_{ei}}{m_i n_i} \cdot \underbrace{\int \mathrm{d}\boldsymbol{v}\, m_i \boldsymbol{v} \frac{\partial f_i}{\partial \boldsymbol{v}}}_{\stackrel{\mathrm{bp}}{=} -m_i n_i \mathsf{I}} + \frac{m_e n_e}{m_i n_i} \frac{\widehat{f}_e^{(0)}(0)}{\tau_{ei}} \underbrace{\int \mathrm{d}\boldsymbol{v}\, m_i \boldsymbol{v} \frac{\partial}{\partial \boldsymbol{v}} \cdot [(\boldsymbol{v} - \boldsymbol{u}_i) f_i(\boldsymbol{v})]}_{= 0 \text{ by parts and def'n of } \boldsymbol{u}_i} \\ &+ \frac{m_e^2 n_e}{m_i^2 n_i} \frac{1}{\tau_{ei}} \underbrace{\int \mathrm{d}\boldsymbol{v}\, m_i \boldsymbol{v} \nabla_v^2 f_i}_{\stackrel{\mathrm{bp}}{=} 0} \int_0^\infty \mathrm{d}v'\, v' \widehat{f}_e^{(0)}(v') \\ &= -\boldsymbol{R}_{ei}.\end{aligned} \tag{VIII.5.13}$$

Good! Newton would be proud. Now energy conservation (for general $\widehat{f}_e^{(0)}$):

$$
\begin{aligned}
Q_{ie} &= \int \mathrm{d}\boldsymbol{v}\, \frac{1}{2} m_i v^2 \left(\frac{\partial f_i}{\partial t}\right)_{\mathrm{c},e} \\
&= \frac{\boldsymbol{R}_{ei}}{m_i n_i} \cdot \underbrace{\int \mathrm{d}\boldsymbol{v}\, \frac{1}{2} m_i v^2 \frac{\partial f_i}{\partial \boldsymbol{v}}}_{\substack{\overset{\text{bp}}{=} -\int \mathrm{d}\boldsymbol{v}\, m_i \boldsymbol{v} f_i \\ = -m_i n_i \boldsymbol{u}_i \ (\text{def'n})}} + \frac{m_e n_e}{m_i n_i} \frac{\widehat{f}_e^{(0)}(0)}{\tau_{ei}} \underbrace{\int \mathrm{d}\boldsymbol{v}\, \frac{1}{2} m_i v^2 \frac{\partial}{\partial \boldsymbol{v}} \cdot \left[(\boldsymbol{v} - \boldsymbol{u}_i) f_i(\boldsymbol{v})\right]}_{\substack{\overset{\text{bp}}{=} -\int \mathrm{d}\boldsymbol{v}\, m_i \boldsymbol{v} \cdot (\boldsymbol{v} - \boldsymbol{u}_i) f_i(\boldsymbol{v}) \\ = -\int \mathrm{d}\boldsymbol{v}\, m_i |\boldsymbol{v} - \boldsymbol{u}_i|^2 f_i(\boldsymbol{v})}} \\
&\quad + \frac{m_e^2 n_e}{m_i^2 n_i} \frac{1}{\tau_{ei}} \underbrace{\int \mathrm{d}\boldsymbol{v}\, \frac{1}{2} m_i v^2 \nabla_v^2 f_i}_{\substack{\overset{\text{bp}}{=} 3m_i \int \mathrm{d}\boldsymbol{v}\, f_i \\ = 3 m_i n_i \ (\text{def'n})}} \int_0^\infty \mathrm{d}v'\, v' \widehat{f}_e^{(0)}(v') \\
&= -\boldsymbol{u}_i \cdot \boldsymbol{R}_{ei} - \frac{m_e n_e}{m_o n_i} \frac{\widehat{f}_e^{(0)}(0)}{\tau_{ei}} \int \mathrm{d}\boldsymbol{v}\, m_i |\boldsymbol{v} - \boldsymbol{u}_i|^2 f_i(\boldsymbol{v}) + \frac{3 m_e^2 n_e}{m_i \tau_{ei}} \int_0^\infty \mathrm{d}v'\, v' \widehat{f}_e^{(0)}(v').
\end{aligned}
\tag{VIII.5.14}
$$

The first term here is clearly the work done by the friction force. What of the other two terms? Let's again take $f_e^{(0)}$ Maxwellian:

$$
Q_{ie} \text{ (Maxwellian electrons)} = -\boldsymbol{u}_i \cdot \boldsymbol{R}_{ei} - \frac{m_e n_e}{m_i n_i} \int \mathrm{d}\boldsymbol{v}\, m_i |\boldsymbol{v} - \boldsymbol{u}_i|^2 f_i(\boldsymbol{v}) + \frac{3 m_e n_e T_e}{m_i \tau_{ei}}.
\tag{VIII.5.15}
$$

The final term here looks like the energy exchange related to temperature equilibration. Now take $f_i(\boldsymbol{v} - \boldsymbol{u}_i)$ to be Maxwellian; then

$$
Q_{ie} \text{ (Maxwellian plasma)} = -\boldsymbol{u}_i \cdot \boldsymbol{R}_{ei} - \frac{3 m_e n_e}{m_i \tau_{ei}} (T_i - T_e).
\tag{VIII.5.16}
$$

Temperature equilibration, indeed. But it occurs on a long timescale, $\sim(m_i/m_e)\tau_{ei}$, which means that ions and electrons can have many collisions and their distribution functions can become Maxwellian long before their temperatures become equal. Finally, because of energy conservation,

$$
\begin{aligned}
Q_{ei} = -Q_{ie} &= \boldsymbol{u}_i \cdot \boldsymbol{R}_{ei} - \frac{3 m_e n_e}{m_i \tau_{ei}} (T_e - T_i) \\
&= \underbrace{\boldsymbol{u}_e \cdot \boldsymbol{R}_{ei}}_{\substack{\text{work done} \\ \text{by friction}}} + \underbrace{(\boldsymbol{u}_i - \boldsymbol{u}_e) \cdot \boldsymbol{R}_{ei}}_{\substack{\text{Joule heating} \\ \text{(current disruption} \\ \text{by collisions} \\ \text{gives heating)}}} - \underbrace{\frac{3 m_e n_e}{m_i \tau_{ei}} (T_e - T_i)}_{\substack{\text{temperature} \\ \text{equilibration}}}.
\end{aligned}
\tag{VIII.5.17}
$$

VIII.6. Collisions with a Maxwellian background

We next consider collisions between an arbitrary species α and a Maxwellian species β, which we take to be at rest:

$$
f_\beta(\boldsymbol{v}) = f_\beta(v) = \frac{n_\beta}{\pi^{3/2} v_{\mathrm{th}\beta}^3} \exp\left(-\frac{v^2}{v_{\mathrm{th}\beta}^2}\right).
\tag{VIII.6.1}
$$

We must compute the Rosenbluth potentials for a Maxwellian species.

Let's start with φ_β. In spherical velocity-space coordinates, equation (VIII.3.7) is

$$\nabla_v^2 \varphi_\beta = \frac{1}{v^2}\frac{\mathrm{d}}{\mathrm{d}v}\left(v^2 \frac{\mathrm{d}\varphi_\beta}{\mathrm{d}v}\right) = -4\pi f_\beta = -\frac{4n_\beta}{\sqrt{\pi}}\frac{\mathrm{e}^{-v^2/v_{\mathrm{th}\beta}^2}}{v_{\mathrm{th}\beta}^3}. \tag{VIII.6.2}$$

Integrating (VIII.6.2) once over velocity gives

$$v^2 \frac{\mathrm{d}\varphi_\beta}{\mathrm{d}v} = -\frac{4n_\beta}{\sqrt{\pi}} \int_0^{v/v_{\mathrm{th}\beta}} \mathrm{d}x\, x^2 \mathrm{e}^{-x^2}. \tag{VIII.6.3}$$

The integral on the right-hand side of this equation is special. Let's chew on it:

$$\begin{aligned}
\int_0^a \mathrm{d}x\, x^2 \mathrm{e}^{-x^2} &\overset{\mathrm{bp}}{=} -\frac{1}{2} x \mathrm{e}^{-x^2}\bigg|_0^a + \frac{1}{2}\int_0^a \mathrm{d}x\, \mathrm{e}^{-x^2} \\
&= -\frac{a}{2}\mathrm{e}^{-a^2} + \frac{\sqrt{\pi}}{4}\varPhi(a), \\
&= \frac{\sqrt{\pi}}{4}\bigl[\varPhi(a) - a\varPhi'(a)\bigr],
\end{aligned} \tag{VIII.6.4}$$

where

$$\varPhi(a) \doteq \frac{2}{\sqrt{\pi}}\int_0^a \mathrm{d}x\, \mathrm{e}^{-x^2} \tag{VIII.6.5}$$

is the error function (see figure 6). Using (VIII.6.4) in (VIII.6.3) and integrating once more over velocity, we find

$$\begin{aligned}
\varphi_\beta &= \frac{n_\beta}{v_{\mathrm{th}\beta}} \int_{v/v_{\mathrm{th}\beta}}^\infty \mathrm{d}x\, \frac{1}{x^2}\bigl[\varPhi(x) - x\varPhi'(x)\bigr] \\
&= -\frac{n_\beta}{v_{\mathrm{th}\beta}} \int_{v/v_{\mathrm{th}\beta}}^\infty \mathrm{d}x\, \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{x}\varPhi(x)\right] \\
\Longrightarrow \quad \varphi_\beta(v) &= \frac{n_\beta}{v}\varPhi\!\left(\frac{v}{v_{\mathrm{th}\beta}}\right).
\end{aligned} \tag{VIII.6.6}$$

Next up, ψ_β. We must solve (see (VIII.3.8))

$$\frac{1}{v^2}\frac{\mathrm{d}}{\mathrm{d}v}\left(v^2\frac{\mathrm{d}\psi_\beta}{\mathrm{d}v}\right) = 2\varphi_\beta = \frac{2n_\beta}{v}\varPhi\!\left(\frac{v}{v_{\mathrm{th}\beta}}\right). \tag{VIII.6.7}$$

Integrating this equation once over velocity,

$$v^2 \frac{\mathrm{d}\psi_\beta}{\mathrm{d}v} = 2n_\beta v_{\mathrm{th}\beta}^2 \int_0^{v/v_{\mathrm{th}\beta}} \mathrm{d}x\, x\varPhi(x) \tag{VIII.6.8}$$

The integral on the right-hand side of this equation is

$$\int_0^a \mathrm{d}x\, x\varPhi(x) \overset{\mathrm{bp}}{=} x\left[x\varPhi(x) + \frac{1}{2}\varPhi'(x)\right]\bigg|_0^a - \int_0^a \mathrm{d}x\left[x\varPhi(x) + \frac{1}{2}\varPhi'(x)\right]$$

$$\left(\text{since } \int \mathrm{d}x\, \varPhi(x) = x\varPhi(x) + \frac{1}{2}\varPhi'(x)\right)$$

$$\Longrightarrow \quad \int_0^a \mathrm{d}x\, x\varPhi(x) = \frac{1}{2}\left[a^2\varPhi(a) + \frac{a}{2}\varPhi'(a)\right] - \frac{1}{4}\varPhi(a). \tag{VIII.6.9}$$

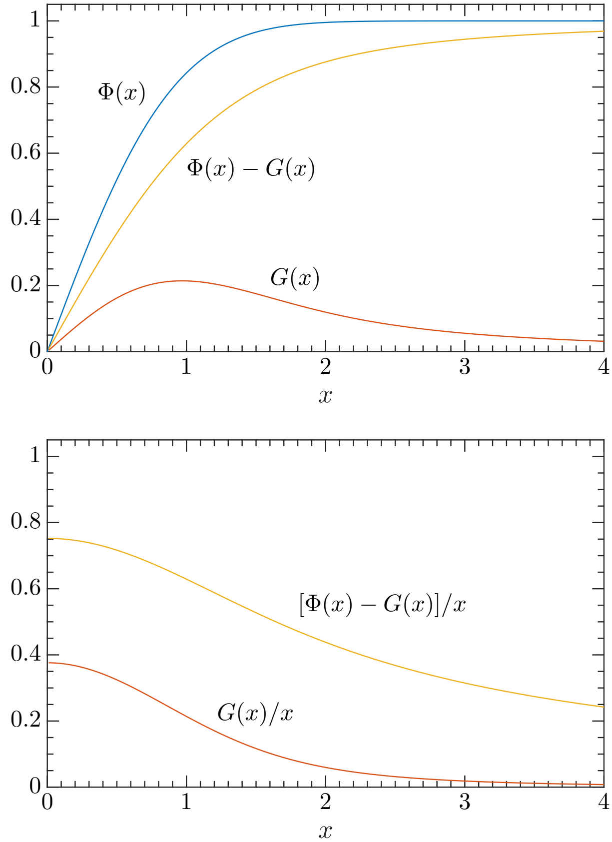


FIGURE 6. Error function $\Phi(x)$, Chandrasekhar function $G(x)$ (see (VIII.6.11)), and some important combinations.

Using (VIII.6.9) in (VIII.6.8) and integrating once more over velocity, we find

$$\begin{aligned}\psi_\beta &= \frac{n_\beta v_{\mathrm{th}\beta}}{2} \int_0^{v/v_{\mathrm{th}\beta}} \mathrm{d}x\, \frac{1}{x^2} \left[\left(2x^2 - 1\right) \Phi(x) + x\Phi'(x) \right] \\ &= \frac{n_\beta v_{\mathrm{th}\beta}}{2} \int_0^{v/v_{\mathrm{th}\beta}} \mathrm{d}x\, \frac{\mathrm{d}}{\mathrm{d}x} \left[\Phi'(x) + \left(1 + 2x^2\right) \frac{\Phi(x)}{x} \right] \quad \left(\text{since } \Phi''(x) = -2x\Phi'(x)\right)\end{aligned}$$

$$\implies \quad \psi_\beta(v) = \frac{n_\beta v_{\mathrm{th}\beta}}{2} \left[\Phi'\left(\frac{v}{v_{\mathrm{th}\beta}}\right) + \left(1 + \frac{2v^2}{v_{\mathrm{th}\beta}^2}\right) \frac{v_{\mathrm{th}\beta}}{v} \Phi\left(\frac{v}{v_{\mathrm{th}\beta}}\right) \right] + \text{const.} \quad \text{(VIII.6.10)}$$

Now, there is a particular combination of the error function and its derivative that occurs frequently, so much so that it deserves a name:

$$\boxed{G(x) \doteq \frac{\Phi(x) - x\Phi'(x)}{2x^2},} \quad \text{(VIII.6.11)}$$

the *Chandrasekhar function* (see Chandrasekhar 1943); its form is shown in figure 6. Written in terms of $G(x)$, the first Fokker–Planck coefficient (see (VIII.3.3)) is

$$\boxed{\begin{aligned}\boldsymbol{A}_\alpha(\boldsymbol{v}) &= \sum_\beta \frac{4\pi q_\alpha^2 q_\beta^2 \ln\lambda_{\alpha\beta}}{m_\alpha^2}\left(1+\frac{m_\alpha}{m_\beta}\right)\frac{\partial\varphi_\beta(v)}{\partial\boldsymbol{v}} \\ &= -\frac{\boldsymbol{v}}{v}\sum_\beta \frac{4\pi q_\alpha^2 q_\beta^2 n_\beta \ln\lambda_{\alpha\beta}}{m_\alpha T_\beta}\left(1+\frac{m_\beta}{m_\alpha}\right)G\left(\frac{v}{v_{\mathrm{th}\beta}}\right)\end{aligned}} \tag{VIII.6.12}$$

Note that (i) the drag force is greatest when $v \sim v_{\mathrm{th}\beta}$, and (ii) $\boldsymbol{A}_\alpha$ decreases at large v and so the drag force gets smaller and smaller as $v \to \infty$. (Why, physically?) In order to write the second Fokker–Planck coefficient in terms of Φ and G, note that

$$\frac{\mathrm{d}\psi_\beta}{\mathrm{d}v} = n_\beta\left[\Phi\left(\frac{v}{v_{\mathrm{th}\beta}}\right) - G\left(\frac{v}{v_{\mathrm{th}\beta}}\right)\right],$$

$$\Phi'(x) - G'(x) = \frac{2}{x}G(x),$$

and

$$\frac{\partial^2}{\partial\boldsymbol{v}\partial\boldsymbol{v}} = \left(\mathsf{I} - \frac{\boldsymbol{v}\boldsymbol{v}}{v^2}\right)\frac{1}{v}\frac{\partial}{\partial v} + \frac{\boldsymbol{v}\boldsymbol{v}}{v^2}\frac{\partial^2}{\partial v^2} \quad \text{(when operating on an isotropic function).}$$

Then, after some straightforward manipulation, the second Fokker–Planck coefficient (see (VIII.3.4)) becomes

$$\boxed{\begin{aligned}\mathsf{B}_\alpha(\boldsymbol{v}) &= \sum_\beta \frac{4\pi q_\alpha^2 q_\beta^2 \ln\lambda_{\alpha\beta}}{m_\alpha^2}\frac{\partial^2\psi_\beta(v)}{\partial\boldsymbol{v}\partial\boldsymbol{v}} \\ &= \left(\mathsf{I} - \frac{\boldsymbol{v}\boldsymbol{v}}{v^2}\right)\sum_\beta \frac{4\pi q_\alpha^2 q_\beta^2 n_\beta \ln\lambda_{\alpha\beta}}{m_\alpha^2 v}\left[\Phi\left(\frac{v}{v_{\mathrm{th}\beta}}\right) - G\left(\frac{v}{v_{\mathrm{th}\beta}}\right)\right] \\ &\quad + \frac{\boldsymbol{v}\boldsymbol{v}}{v^2}\sum_\beta \frac{8\pi q_\alpha^2 q_\beta^2 n_\beta \ln\lambda_{\alpha\beta}}{m_\alpha^2 v}G\left(\frac{v}{v_{\mathrm{th}\beta}}\right)\end{aligned}} \tag{VIII.6.13}$$

Note that both parallel and perpendicular diffusion decrease with velocity.

These coefficients, (VIII.6.12) and (VIII.6.13), go into the Fokker–Planck collision operator,

$$\left(\frac{\partial f_\alpha}{\partial t}\right)_{\mathrm{c}} = -\frac{\partial}{\partial\boldsymbol{v}}\boldsymbol{\cdot}\left[\boldsymbol{A}_\alpha(\boldsymbol{v})f_\alpha(\boldsymbol{v})\right] + \frac{1}{2}\frac{\partial}{\partial\boldsymbol{v}}\frac{\partial}{\partial\boldsymbol{v}}\boldsymbol{:}\left[\mathsf{B}_\alpha(\boldsymbol{v})f_\alpha(\boldsymbol{v})\right]. \tag{VIII.6.14}$$

Recall that $\boldsymbol{A}_\alpha$ and B_α originally come from the jump moments,

$$\boldsymbol{A}_\alpha \doteq \frac{\langle\Delta\boldsymbol{v}\rangle}{\Delta t} \quad \text{and} \quad \mathsf{B}_\alpha \doteq \frac{\langle\Delta\boldsymbol{v}\Delta\boldsymbol{v}\rangle}{\Delta t},$$

with $\Delta t \to$ "0" (see §VI.2). Since $\boldsymbol{A}_\alpha$ is parallel to $\boldsymbol{v}$ (see (VIII.6.12)), let us define a collision frequency associated with this drag force:

$$\nu_{\mathrm{s}}^{\alpha\beta}(v) \doteq -\frac{\langle\Delta v_\parallel/v\rangle^{\alpha\beta}}{\Delta t}, \tag{VIII.6.15}$$

where "s" stands for *slowing down*. Writing $\partial\varphi_\beta(v)/\partial\boldsymbol{v} = (\boldsymbol{v}/v)(\mathrm{d}\varphi_\beta/\mathrm{d}v)$ and using

(VIII.6.12), we have

$$\boxed{\begin{aligned}\nu_{\mathrm{s}}^{\alpha\beta}(v) &= -\frac{4\pi q_\alpha^2 q_\beta^2 \ln\lambda_{\alpha\beta}}{m_\alpha^2}\left(1+\frac{m_\alpha}{m_\beta}\right)\frac{1}{v}\frac{\mathrm{d}\varphi_\beta}{\mathrm{d}v} \\ &= \frac{4\pi q_\alpha^2 q_\beta^2 n_\beta \ln\lambda_{\alpha\beta}}{m_\alpha T_\beta}\left(1+\frac{m_\beta}{m_\alpha}\right)\frac{1}{v}\,G\!\left(\frac{v}{v_{\mathrm{th}\beta}}\right)\end{aligned}} \tag{VIII.6.16}$$

This describes the rate at which a particle of species α is decelerated by collisions with particles of species β. Likewise, recall that $\boldsymbol{B}_\alpha$ has two pieces, parallel and perpendicular to $\boldsymbol{v}$ (see (VIII.6.13)). Define

$$\nu_\parallel^{\alpha\beta}(v) \doteq \frac{\langle(\Delta v_\parallel/v)^2\rangle^{\alpha\beta}}{\Delta t} \tag{VIII.6.17}$$

as the *parallel diffusion frequency*. Using (VIII.6.13) and

$$\frac{\partial^2\psi_\beta}{\partial\boldsymbol{v}\partial\boldsymbol{v}} = \frac{\partial^2 v}{\partial\boldsymbol{v}\partial\boldsymbol{v}}\frac{\mathrm{d}\psi_\beta}{\mathrm{d}v} + \frac{\boldsymbol{v}\boldsymbol{v}}{v^2}\frac{\mathrm{d}^2\psi_\beta}{\mathrm{d}v^2} = \left(\boldsymbol{I} - \frac{\boldsymbol{v}\boldsymbol{v}}{v^2}\right)\frac{1}{v}\frac{\mathrm{d}\psi_\beta}{\mathrm{d}v} + \frac{\boldsymbol{v}\boldsymbol{v}}{v^2}\frac{\mathrm{d}^2\psi_\beta}{\mathrm{d}v^2},$$

we have

$$\boxed{\begin{aligned}\nu_\parallel^{\alpha\beta}(v) &= \frac{4\pi q_\alpha^2 q_\beta^2 \ln\lambda_{\alpha\beta}}{m_\alpha^2}\frac{1}{v^2}\frac{\mathrm{d}^2\psi_\beta}{\mathrm{d}v^2} \\ &= \frac{8\pi q_\alpha^2 q_\beta^2 n_\beta \ln\lambda_{\alpha\beta}}{m_\alpha^2}\frac{1}{v^3}\,G\!\left(\frac{v}{v_{\mathrm{th}\beta}}\right)\end{aligned}} \tag{VIII.6.18}$$

This represents the rate at which a particle of species α diffuses in $v_\parallel$ by collisions with particles of species β. Finally, define

$$\nu_\perp^{\alpha\beta}(v) \doteq \frac{\langle(\Delta v_\perp/v)^2\rangle^{\alpha\beta}}{\Delta t} \tag{VIII.6.19}$$

as the *perpendicular diffusion frequency*. Using (VIII.6.13) and

$$\left(\frac{\partial^2\psi_\beta}{\partial\boldsymbol{v}\partial\boldsymbol{v}}\right)_\perp = \left(\boldsymbol{I} - \frac{\boldsymbol{v}\boldsymbol{v}}{v^2}\right)\frac{1}{v}\frac{\mathrm{d}\psi_\beta}{\mathrm{d}v},$$

we have

$$\boxed{\begin{aligned}\nu_\perp^{\alpha\beta}(v) &= \frac{8\pi q_\alpha^2 q_\beta^2 \ln\lambda_{\alpha\beta}}{m_\alpha^2}\frac{1}{v^3}\frac{\mathrm{d}\psi_\beta}{\mathrm{d}v} \\ &= \frac{8\pi q_\alpha^2 q_\beta^2 n_\beta \ln\lambda_{\alpha\beta}}{m_\alpha^2}\frac{1}{v^3}\left[\Phi\!\left(\frac{v}{v_{\mathrm{th}\beta}}\right) - G\!\left(\frac{v}{v_{\mathrm{th}\beta}}\right)\right]\end{aligned}} \tag{VIII.6.20}$$

This is the rate at which a particles of species α is diffusing in $v_\perp$ by collisions with particles of species β.

In a frame oriented with $\boldsymbol{v}$ along the z axis,

$$\boldsymbol{A}_\alpha = \frac{1}{\Delta t}\begin{bmatrix}0\\0\\\langle\Delta v_\parallel\rangle\end{bmatrix} = -v\begin{bmatrix}0\\0\\\sum_\beta \nu_{\mathrm{s}}^{\alpha\beta}\end{bmatrix}, \tag{VIII.6.21}$$

$$\boldsymbol{B}_\alpha = \frac{1}{\Delta t}\begin{bmatrix} \langle \Delta v_\perp^2\rangle/2 & 0 & 0 \\ 0 & \langle \Delta v_\perp^2\rangle/2 & 0 \\ 0 & 0 & \langle \Delta v_\parallel^2\rangle \end{bmatrix} = v^2 \begin{bmatrix} \sum_\beta \nu_\perp^{\alpha\beta}/2 & 0 & 0 \\ 0 & \sum_\beta \nu_\perp^{\alpha\beta}/2 & 0 \\ 0 & 0 & \sum_\beta \nu_\parallel^{\alpha\beta} \end{bmatrix}, \tag{VIII.6.22}$$

or, in vector notation,

$$\boldsymbol{A}_\alpha = -\boldsymbol{v}\sum_\beta \nu_{\rm s}^{\alpha\beta}, \tag{VIII.6.23}$$

$$\boldsymbol{B}_\alpha = \boldsymbol{v}\boldsymbol{v}\sum_\beta \nu_\parallel^{\alpha\beta} + \frac{1}{2}\left(v^2\boldsymbol{I} - \boldsymbol{v}\boldsymbol{v}\right)\sum_\beta \nu_\perp^{\alpha\beta}. \tag{VIII.6.24}$$

Plugging these into (VIII.6.14) gives

$$\left(\frac{\partial f_\alpha}{\partial t}\right)_{\rm c} = \frac{\partial}{\partial \boldsymbol{v}}\boldsymbol{\cdot}\left[\boldsymbol{v}\sum_\beta \nu_{\rm s}^{\alpha\beta} f_\alpha(\boldsymbol{v})\right] + \frac{1}{2}\frac{\partial}{\partial \boldsymbol{v}}\frac{\partial}{\partial \boldsymbol{v}}\boldsymbol{:}\left[\boldsymbol{v}\boldsymbol{v}\sum_\beta \nu_\parallel^{\alpha\beta} f_\alpha(\boldsymbol{v}) + \frac{1}{2}\left(v^2\boldsymbol{I} - \boldsymbol{v}\boldsymbol{v}\right)\sum_\beta \nu_\perp^{\alpha\beta} f_\alpha(\boldsymbol{v})\right]. \tag{VIII.6.25}$$

Let's simplify each of the colored terms in (VIII.6.25):

$$\begin{aligned} \text{red} &= \sum_\beta \nu_{\rm s}^{\alpha\beta}\,\boldsymbol{v}\boldsymbol{\cdot}\frac{\partial f_\alpha}{\partial \boldsymbol{v}} + f_\alpha(\boldsymbol{v})\frac{\partial}{\partial \boldsymbol{v}}\boldsymbol{\cdot}\left(\boldsymbol{v}\sum_\beta \nu_{\rm s}^{\alpha\beta}\right) \\ &= \sum_\beta \nu_{\rm s}^{\alpha\beta}\,v\frac{\partial f_\alpha(\boldsymbol{v})}{\partial v} + f_\alpha(\boldsymbol{v})\frac{1}{v^2}\frac{{\rm d}}{{\rm d}v}\left(v^3\sum_\beta \nu_{\rm s}^{\alpha\beta}\right) \\ &= \frac{1}{v^2}\frac{\partial}{\partial v}\left[v^3\sum_\beta \nu_{\rm s}^{\alpha\beta} f_\alpha(\boldsymbol{v})\right]; \end{aligned}$$

$$\begin{aligned} \text{blue} &= \frac{1}{4}\nabla_v^2\left[v^2\sum_\beta \nu_\perp^{\alpha\beta} f_\alpha(\boldsymbol{v})\right] \\ &= \frac{1}{4v^2}\frac{\partial}{\partial v}\left[v^2\frac{\partial}{\partial v}v^2\sum_\beta \nu_\perp^{\alpha\beta} f_\alpha(\boldsymbol{v})\right] + \sum_\beta \nu_\perp^{\alpha\beta}\frac{1}{2}\mathcal{L}[f_\alpha]. \end{aligned}$$

The remaining terms in (VIII.6.25) are equal to

$$\begin{aligned} &\frac{1}{2}\frac{\partial}{\partial \boldsymbol{v}}\frac{\partial}{\partial \boldsymbol{v}}\boldsymbol{:}\left[\boldsymbol{v}\boldsymbol{v}\sum_\beta\left(\nu_\parallel^{\alpha\beta} - \frac{1}{2}\nu_\perp^{\alpha\beta}\right)f_\alpha(\boldsymbol{v})\right] \\ &= \frac{1}{2v^2}\frac{\partial}{\partial v}\left\{v^2\frac{\partial}{\partial v}\left[v^2\sum_\beta\left(\nu_\parallel^{\alpha\beta} - \frac{1}{2}\nu_\perp^{\alpha\beta}\right)f_\alpha(\boldsymbol{v})\right] + 2v^3\sum_\beta\left(\nu_\parallel^{\alpha\beta} - \frac{1}{2}\nu_\perp^{\alpha\beta}\right)f_\alpha(\boldsymbol{v})\right\} \\ &= \frac{1}{2v^2}\frac{\partial}{\partial v}\left[\sum_\beta\left(\nu_\parallel^{\alpha\beta} - \frac{1}{2}\nu_\perp^{\alpha\beta}\right)\left(4v^3 + v^4\frac{\partial}{\partial v}\right)f_\alpha(\boldsymbol{v}) + v^4 f_\alpha(\boldsymbol{v})\frac{{\rm d}}{{\rm d}v}\sum_\beta\left(\nu_\parallel^{\alpha\beta} - \frac{1}{2}\nu_\perp^{\alpha\beta}\right)\right]. \end{aligned}$$

To further simplify this expression, note that

$$\frac{\mathrm{d}\nu_\parallel^{\alpha\beta}}{\mathrm{d}v} = -\frac{2}{v}\left(1+\frac{m_\alpha}{m_\beta}\right)^{-1}\nu_\mathrm{s}^{\alpha\beta} - \frac{4}{v}\nu_\parallel^{\alpha\beta} + \frac{1}{v}\nu_\perp^{\alpha\beta},$$
$$\frac{\mathrm{d}\nu_\perp^{\alpha\beta}}{\mathrm{d}v} = \frac{2}{v}\nu_\parallel^{\alpha\beta} - \frac{3}{v}\nu_\perp^{\alpha\beta}.$$

All together then, equation (VIII.6.25) simplifies to

$$\boxed{\left(\frac{\partial f_\alpha}{\partial t}\right)_\mathrm{c} = \frac{1}{2}\sum_\beta\left\{\nu_\perp^{\alpha\beta}\mathcal{L}[f_\alpha] + \frac{1}{v^2}\frac{\partial}{\partial v}\left[v^3\frac{2m_\alpha}{m_\alpha+m_\beta}\nu_\mathrm{s}^{\alpha\beta}f_\alpha(\boldsymbol{v}) + v^4\nu_\parallel^{\alpha\beta}\frac{\partial f_\alpha(\boldsymbol{v})}{\partial v}\right]\right\}} \tag{VIII.6.26}$$

Each of the three terms in (VIII.6.26) has a clean physical interpretation. The first corresponds to perpendicular diffusion at fixed energy, i.e., pitch-angle scattering. Often, $\nu_\mathrm{D}^{\alpha\beta} \doteq \nu_\perp^{\alpha\beta}/2$ is defined as the *deflection frequency*. The second term corresponds to slowing down, i.e., drag. The third and final term corresponds to parallel diffusion (i.e., energy diffusion). Noting that $\nu_\parallel$ and ν_s are closely related (cf. (VIII.6.15) and (VIII.6.18)), an alternative way to write (VIII.6.26) is

$$\left(\frac{\partial f_\alpha}{\partial t}\right)_\mathrm{c} = \sum_\beta\left\{\nu_\mathrm{D}^{\alpha\beta}\mathcal{L}[f_\alpha] + \frac{1}{v^2}\frac{\partial}{\partial v}\left[v^2\,\nu_\mathrm{s}^{\alpha\beta}\frac{m_\alpha}{m_\alpha+m_\beta}\left(vf_\alpha(\boldsymbol{v}) + \frac{T_\beta}{m_\alpha}\frac{\partial f_\alpha(\boldsymbol{v})}{\partial v}\right)\right]\right\}. \tag{VIII.6.27}$$

Physically, collisions with species β are trying to make species α relax to a Maxwellian at the same temperature as species β.

The operator (VIII.6.26) is good for collisions of a tenuous ($n_\alpha \ll n_\beta$) group of test particles whose presence doesn't change the background (Maxwellian) distribution very much. It is not useful for self-collisions within a single species, our next target (§VIII.7). But, first, two things:

(1) A summary of collision frequencies. Denoting the generic pre-factor

$$\nu_0^{\alpha\beta} \doteq \frac{4\pi q_\alpha^2 q_\beta^2 n_\beta \ln\lambda_{\alpha\beta}}{m_\alpha^2 v_{\mathrm{th}\beta}^3}, \tag{VIII.6.28}$$

we found the following:

$$\boxed{\begin{aligned}
\nu_\mathrm{s}^{\alpha\beta} &= -\nu_0^{\alpha\beta}\frac{v_{\mathrm{th}\beta}^3}{n_\beta}\left(1+\frac{m_\alpha}{m_\beta}\right)\frac{1}{v}\frac{\mathrm{d}\varphi_\beta}{\mathrm{d}v}\\
&= 2\nu_0^{\alpha\beta}\left(1+\frac{m_\alpha}{m_\beta}\right)\frac{v_{\mathrm{th}\beta}}{v}\,G\!\left(\frac{v}{v_{\mathrm{th}\beta}}\right)\\
\nu_\parallel^{\alpha\beta} &= \nu_0^{\alpha\beta}\frac{v_{\mathrm{th}\beta}^3}{n_\beta}\frac{1}{v^2}\frac{\mathrm{d}^2\psi_\beta}{\mathrm{d}v^2}\\
&= 2\nu_0^{\alpha\beta}\frac{v_{\mathrm{th}\beta}^3}{v^3}\,G\!\left(\frac{v}{v_{\mathrm{th}\beta}}\right)\\
\nu_\perp^{\alpha\beta} &= 2\nu_\mathrm{D}^{\alpha\beta} = \nu_0^{\alpha\beta}\frac{v_{\mathrm{th}\beta}^3}{n_\beta}\frac{2}{v^3}\frac{\mathrm{d}\psi_\beta}{\mathrm{d}v}\\
&= 2\nu_0^{\alpha\beta}\frac{v_{\mathrm{th}\beta}^3}{v^3}\left[\varPhi\!\left(\frac{v}{v_{\mathrm{th}\beta}}\right) - G\!\left(\frac{v}{v_{\mathrm{th}\beta}}\right)\right]
\end{aligned}}$$

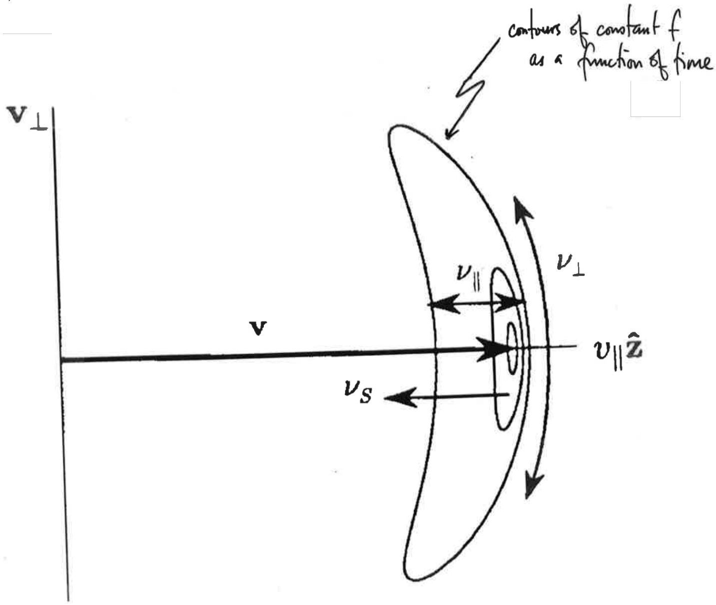
where

$$\Phi(x) \doteq \frac{2}{\sqrt{\pi}} \int_0^x \mathrm{d}v\, \mathrm{e}^{-v^2} \quad \text{and} \quad G(x) \doteq \frac{\Phi(x) - x\Phi'(x)}{2x^2}.$$

One can further define an *energy-loss rate*:

$$\begin{aligned}
\nu_\varepsilon^{\alpha\beta} &\doteq -\frac{\langle (\Delta v/v)^2 \rangle^{\alpha\beta}}{\Delta t} \\
&= 2\nu_{\mathrm{s}}^{\alpha\beta} - \nu_\perp^{\alpha\beta} - \nu_\parallel^{\alpha\beta} \\
&= 2\nu_0^{\alpha\beta} \left[2\left(1 + \frac{m_\alpha}{m_\beta}\right) \frac{v_{\mathrm{th}\beta}}{v}\, G\left(\frac{v}{v_{\mathrm{th}\beta}}\right) - \frac{v_{\mathrm{th}\beta}^3}{v^3}\, \Phi\left(\frac{v}{v_{\mathrm{th}\beta}}\right) \right].
\end{aligned}$$

Pictorially,



It's useful to know further that

$$G(x) \approx \begin{cases} \dfrac{2x}{3\sqrt{\pi}}, & x \to 0 \\[2mm] \dfrac{1}{2x^2}, & x \to \infty \end{cases} \quad \text{and} \quad \Phi(x) \approx \begin{cases} \dfrac{2x}{\sqrt{\pi}}, & x \to 0 \\[2mm] 1, & x \to \infty \end{cases}$$

Why? Because...

(2) Runaway electrons. The fact that $G(x)$ decreases for large x is important. The average friction force on a particle is

$$\frac{m_\alpha \langle \Delta v_\parallel \rangle^{\alpha\beta}}{\Delta t} = -m_\alpha v_\parallel \nu_{\mathrm{s}}^{\alpha\beta} \propto G\left(\frac{v}{v_{\mathrm{th}\beta}}\right),$$

which decreases for $v \gtrsim v_{\mathrm{th}\beta}$. It actually *vanishes* as $v/v_{\mathrm{th}\beta} \to \infty$ (although relativistic effects intervene as $v \to c$). The reason is that the momentum exchanged in a collision decreases with the incident particle's speed if the impact parameter is held constant, and the number of particles with which the test particle is interacting does not increase as $v/v_{\mathrm{th}\beta} \to \infty$ (recall that the drag and diffusion are determined only by interactions with slower particles – a feature made clear by working with Rosenbluth potentials and from the fact that the integral in the error function goes from 0 to a finite value, its argument) since $f_\beta \to 0$ as $v \to \infty$. The implication is that, if some persistent force is applied to the particle (e.g., a constant electric field), this force will always be larger than the frictional drag force for sufficiently fast particles. Some electrons in the tail of the distribution can thus be accelerated to arbitrarily high energy and form a population of *runaway electrons*. If the applied force is strong enough, even ordinary thermal electrons can run away. This occurs when the electric field being applied outstrips the drag force; roughly, when

$$\boxed{E > E_{\mathrm{D}} \doteq \frac{4\pi n_e e^3 \ln \lambda}{T_e}} \tag{VIII.6.29}$$

E_{D} is known as the *Dreicer field* (Dreicer 1959). (Note that $E_{\mathrm{D}} \sim e/\lambda_{\mathrm{D}}^2$.) This has practical importance: if a tokamak disrupts, significant numbers of runaways can be produced that can damage the device. Disruptions are thus a significant problem for large machines like ITER. See §24.6 of Krommes (2018) for more.

VIII.7. Ion–ion and electron–electron collisions

Up to now, we have either exploited the possibility of m_α/m_β being a small number or just assumed that one of the species is (and stays) Maxwellian. These are obviously not good assumptions for like-particle collisions. Both the ion–ion and electron–electron collision operators are very difficult to approximate, as $\boldsymbol{v}$ and $\boldsymbol{v}'$ are comparable and the distribution function itself is unknown. But, if we assume that f_α is *near* Maxwellian, then some progress can be made. This is because the Landau operator is *bi-linear* in its arguments. Ignoring perturbations to the Coulomb logarithm, we have

$$\begin{aligned} C[f_\alpha, f_\beta] + C[g_\alpha, f_\beta] &= C[f_\alpha + g_\alpha, f_\beta], \\ C[f_\alpha, f_\beta] + C[f_\alpha, g_\beta] &= C[f_\alpha, f_\beta + g_\beta], \\ C[a_\alpha f_\alpha, b_\beta f_\beta] &= a_\alpha b_\beta C[f_\alpha, f_\beta] \end{aligned}$$

for any distribution functions f_α, f_β, g_α, g_β and constants a_α and b_β. This means that the collision operator for self-collisions, $C[f_\alpha, f_\alpha]$, is nonlinear:

$$C[2f_\alpha, 2f_\alpha] = 4C[f_\alpha, f_\alpha].$$

But, if f_α is close to Maxwellian, $f_\alpha = f_{\mathrm{M}\alpha} + \delta f_\alpha$ with $\delta f_\alpha \ll f_{\mathrm{M}\alpha}$, then

$$\begin{aligned} C[f_\alpha, f_\alpha] &= \cancelto{0}{C[f_{\mathrm{M}\alpha}, f_{\mathrm{M}\alpha}]} + C[f_{\mathrm{M}\alpha}, \delta f_\alpha] + C[\delta f_\alpha, f_{\mathrm{M}\alpha}] + C[\delta f_\alpha, \delta f_\alpha] \\ &\simeq C[f_{\mathrm{M}\alpha}, \delta f_\alpha] + C[\delta f_\alpha, f_{\mathrm{M}\alpha}] \\ &\doteq C^\ell[f_\alpha], \end{aligned} \tag{VIII.7.1}$$

where the superscript ℓ denotes "linearized". Physically, this states that a collision between a test α particle and the other particles in f_α can be treated as a sum of:

(1) a collision between a Maxwellian test particle and a Maxwellian background;

(2) a collision between a Maxwellian test particle and the perturbed distribution;

(3) a collision between a perturbed test particle and a Maxwellian background;

(4) a nonlinear term that can be neglected if δf_α is small.

We have already computed the third term in §VIII.6. The first term vanishes by the H theorem. The last term is dropped as being small. The second term is the new one, as it involves the Rosenbluth potentials of the non-Maxwellian distribution δf_α. While complicated, this term is required for momentum conservation. Let's examine it.

From (VIII.3.10) and (VIII.6.28), we have

$$C[f_{\mathrm{M}\alpha},\delta f_\alpha] = \nu_0^{\alpha\alpha}\frac{v_{\mathrm{th}\alpha}^3}{n_\alpha}\frac{\partial}{\partial \boldsymbol{v}}\cdot\left[-\frac{\partial\delta\varphi_\alpha}{\partial\boldsymbol{v}}f_{\mathrm{M}\alpha}(v)+\frac{1}{2}\frac{\partial^2\delta\psi_\alpha}{\partial\boldsymbol{v}\partial\boldsymbol{v}}\cdot\frac{\partial f_{\mathrm{M}\alpha}}{\partial\boldsymbol{v}}\right]. \tag{VIII.7.2}$$

Using $\partial f_{\mathrm{M}\alpha}/\partial\boldsymbol{v} = -(2\boldsymbol{v}/v_{\mathrm{th}\alpha}^2)f_{\mathrm{M}\alpha}$ along with

$$\nabla_v^2\delta\psi_\alpha = 2\delta\varphi_\alpha \quad\text{and}\quad \nabla_v^2\delta\varphi_\alpha = -4\pi\delta f_\alpha$$

(see (VIII.3.8) and (VIII.3.7)), equation (VIII.7.2) becomes

$$C[f_{\mathrm{M}\alpha},\delta f_\alpha] = \nu_0^{\alpha\alpha}\frac{v_{\mathrm{th}\alpha}^3 f_{\mathrm{M}\alpha}}{n_\alpha}\left(4\pi\delta f_\alpha - \frac{2}{v_{\mathrm{th}\alpha}^2}\delta\varphi_\alpha + \frac{2v^2}{v_{\mathrm{th}\alpha}^4}\frac{\partial^2\delta\psi_\alpha}{\partial v^2}\right). \tag{VIII.7.3}$$

Recall from §VIII.6 that

$$C[\delta f_\alpha, f_{\mathrm{M}\alpha}] = \nu_0^{\alpha\alpha}\frac{v_{\mathrm{th}\alpha}^3}{n_\alpha}\left[\frac{1}{v^3}\frac{\mathrm{d}\psi_\alpha}{\mathrm{d}v}\mathcal{L}[\delta f_\alpha] + \frac{1}{v^2}\frac{\partial}{\partial v}\left(\frac{v^2}{2}\frac{\mathrm{d}^2\psi_\alpha}{\mathrm{d}v^2}\frac{\partial\delta f_\alpha}{\partial v} - v^2\frac{\mathrm{d}\varphi_\alpha}{\mathrm{d}v}\delta f_\alpha\right)\right], \tag{VIII.7.4}$$

with the background potentials satisfying

$$\begin{aligned}
\frac{\mathrm{d}\psi_\alpha}{\mathrm{d}v} &= n_\alpha\left[\Phi\left(\frac{v}{v_{\mathrm{th}\alpha}}\right) - G\left(\frac{v}{v_{\mathrm{th}\alpha}}\right)\right],\\
\frac{\mathrm{d}^2\psi_\alpha}{\mathrm{d}v^2} &= \frac{2n_\alpha}{v}G\left(\frac{v}{v_{\mathrm{th}\alpha}}\right),\\
\frac{\mathrm{d}\varphi_\alpha}{\mathrm{d}v} &= -\frac{2n_\alpha}{v_{\mathrm{th}\alpha}^2}G\left(\frac{v}{v_{\mathrm{th}\alpha}}\right).
\end{aligned}$$

Provided we can obtain $\delta\varphi_\alpha$ and $\mathrm{d}^2\delta\psi_\alpha/\mathrm{d}v^2$ from δf_α, we can write down $C^\ell[f_\alpha]$.

Now, if this doesn't seem to you like much progress, I can't completely fault you. We still have an integro-differential equation to solve. But there *is* something useful here: the linearized collision operator forms the basis of a formal expansion of δf_α in terms of spherical harmonics $Y_{\ell m}(\theta,\phi)$ – or, if δf_α is gyrotropic in some direction, Legendre polynomials. This is because the $Y_{\ell m}$'s are the angular eigenfunctions of the Laplace operator

$$\nabla_v^2 = \frac{1}{v^2}\frac{\partial}{\partial v}v^2\frac{\partial}{\partial v} + 2\mathcal{L},$$

so that, if δf_α can be written as

$$\delta f_\alpha(\boldsymbol{v}) = \sum_{\ell,m} F_\alpha(v) Y_{\ell m}(\theta,\phi),$$

then the perturbed Rosenbluth potentials have the same angular structure, and thus so does the full linearized collision operators.

The linearized collision operator $C^\ell[f_\alpha]$ has another important property – it is *self-adjoint*, i.e.,

$$\boxed{\int \mathrm{d}\boldsymbol{v}\,\frac{1}{f_{\mathrm{M}\alpha}}\delta g_\alpha C^\ell[f_\alpha] = \int \mathrm{d}\boldsymbol{v}\frac{1}{f_{\mathrm{M}\alpha}}\delta f_\alpha C^\ell[g_\alpha]} \tag{VIII.7.5}$$

where f_α and g_α are reasonably well behaved and do not diverge at ∞. The proof mainly consists of several integrations by parts. First,

$$\int \mathrm{d}\boldsymbol{v}\,\frac{1}{f_{\mathrm{M}\alpha}}\delta g_\alpha C[f_{\mathrm{M}\alpha},\delta f_\alpha] = \int \mathrm{d}\boldsymbol{v}\,\delta g_\alpha\,\nu_0^{\alpha\alpha}\frac{v_{\mathrm{th}\alpha}^3}{n_\alpha}\left(4\pi\delta f_\alpha - \frac{2}{v_{\mathrm{th}\alpha^2}}\delta\varphi_\alpha + \frac{2v^2}{v_{\mathrm{th}\alpha}^4}\frac{\partial^2\delta\psi_\alpha}{\partial v^2}\right).$$

Defining $\delta\Phi_\alpha$ and $\delta\Psi_\alpha$ via $\nabla_v^2\delta\Phi_\alpha = -4\pi\delta g_\alpha$ and $\nabla_v^2\delta\Psi_\alpha = 2\delta\Phi_\alpha$, the above expression

$$\begin{aligned}
&= \nu_0^{\alpha\alpha}\frac{v_{\mathrm{th}\alpha}^3}{n_\alpha}\int \mathrm{d}\boldsymbol{v}\left[4\pi\delta g_\alpha\delta f_\alpha - \underbrace{\frac{2}{v_{\mathrm{th}\alpha}^2}\left(-\frac{\nabla_v^2\delta\Phi_\alpha}{4\pi}\right)\delta\varphi_\alpha}_{\text{integrate by parts}} + \underbrace{\delta g_\alpha\frac{2v^2}{v_{\mathrm{th}\alpha}^4}\frac{\partial^2\delta\psi_\alpha}{\partial v^2}}_{\text{use def'n of } \delta\psi_\alpha}\right]\\
&= \nu_0^{\alpha\alpha}\frac{v_{\mathrm{th}\alpha}^3}{n_\alpha}\int \mathrm{d}\boldsymbol{v}\,4\pi\delta g_\alpha\delta f_\alpha + \frac{4\pi q_\alpha^4\ln\lambda_{\alpha\alpha}}{m_\alpha^2}\int \mathrm{d}\boldsymbol{v}\,\frac{2}{v_{\mathrm{th}\alpha}^2}\delta\Phi_\alpha\underbrace{\left(\frac{\nabla_v^2\delta\varphi_\alpha}{4\pi}\right)}_{=-\delta f_\alpha}\\
&\quad + \nu_0^{\alpha\alpha}\frac{v_{\mathrm{th}\alpha}^3}{n_\alpha}\int \mathrm{d}\boldsymbol{v}\int \mathrm{d}\boldsymbol{v}'\,\underbrace{\delta g_\alpha(\boldsymbol{v})\frac{2v^2}{v_{\mathrm{th}\alpha}^4}\frac{\partial^2}{\partial v^2}\big[\delta f_\alpha(\boldsymbol{v}')|\boldsymbol{v}-\boldsymbol{v}'|\big]}_{\text{use symmetries}}\\
&= \nu_0^{\alpha\alpha}\frac{v_{\mathrm{th}\alpha}^3}{n_\alpha}\int \mathrm{d}\boldsymbol{v}\,4\pi\delta g_\alpha\delta f_\alpha - \frac{4\pi q_\alpha^4\ln\lambda_{\alpha\alpha}}{m_\alpha^2}\int \mathrm{d}\boldsymbol{v}\,\frac{2}{v_{\mathrm{th}\alpha}^2}\delta\Phi_\alpha\delta f_\alpha\\
&\quad + \nu_0^{\alpha\alpha}\frac{v_{\mathrm{th}\alpha}^3}{n_\alpha}\underbrace{\int \mathrm{d}\boldsymbol{v}\int \mathrm{d}\boldsymbol{v}'\,\delta f_\alpha(\boldsymbol{v}')\frac{2v'^2}{v_{\mathrm{th}\alpha}^4}\frac{\partial^2}{\partial v'^2}\big[\delta g_\alpha(\boldsymbol{v})|\boldsymbol{v}-\boldsymbol{v}'|\big]}_{\text{switch dummy integration variables, } \boldsymbol{v}\leftrightarrow\boldsymbol{v}'}\\
&= \nu_0^{\alpha\alpha}\frac{v_{\mathrm{th}\alpha}^3}{n_\alpha}\int \mathrm{d}\boldsymbol{v}\,\delta f_\alpha\left(4\pi\delta g_\alpha - \frac{2}{v_{\mathrm{th}\alpha}^2}\delta\Phi_\alpha\right)\\
&\quad + \nu_0^{\alpha\alpha}\frac{v_{\mathrm{th}\alpha}^3}{n_\alpha}\int \mathrm{d}\boldsymbol{v}\,\delta f_\alpha(\boldsymbol{v})\frac{2v^2}{v_{\mathrm{th}\alpha}^4}\frac{\partial^2}{\partial v^2}\underbrace{\int \mathrm{d}\boldsymbol{v}'\,\delta g_\alpha(\boldsymbol{v}')|\boldsymbol{v}-\boldsymbol{v}'|}_{=\delta\Psi_\alpha(v)}\\
&= \int \mathrm{d}\boldsymbol{v}\,\frac{1}{f_{\mathrm{M}\alpha}}\delta f_\alpha C[f_{\mathrm{M}\alpha},\delta g_\alpha].
\end{aligned}$$

Second,

$$\int \mathrm{d}\boldsymbol{v}\, \frac{1}{f_{\mathrm{M}\alpha}} \delta g_\alpha C[\delta f_\alpha, f_{\mathrm{M}\alpha}]$$
$$= \int \mathrm{d}\boldsymbol{v}\, \frac{1}{f_{\mathrm{M}\alpha}} \delta g_\alpha\, \nu_0^{\alpha\alpha} \frac{v_{\mathrm{th}\alpha}^3}{n_\alpha} \left[\underbrace{\frac{1}{v^3} \frac{\mathrm{d}\psi_\alpha}{\mathrm{d}v} \mathcal{L}[\delta f_\alpha]}_{\substack{\text{int. by parts}\\ \text{to move } \mathcal{L}\\ \text{onto } \delta g_\alpha}} + \frac{1}{v^2}\frac{\partial}{\partial v}\left(\frac{v^2}{2} \frac{\mathrm{d}^2\psi_\alpha}{\mathrm{d}v^2} \frac{\partial \delta f_\alpha}{\partial v} - v^2 \frac{\mathrm{d}\varphi_\alpha}{\mathrm{d}v} \delta f_\alpha \right)\right]$$
$$= \nu_0^{\alpha\alpha} \frac{v_{\mathrm{th}\alpha}^3}{n_\alpha} \int \mathrm{d}\boldsymbol{v}\, \frac{1}{f_{\mathrm{M}\alpha}} \mathcal{L}[\delta g_\alpha]\, \frac{1}{v^3} \frac{\mathrm{d}\psi_\alpha}{\mathrm{d}v} \delta f_\alpha$$
$$+ \nu_0^{\alpha\alpha} \frac{v_{\mathrm{th}\alpha}^3}{n_\alpha} \int \mathrm{d}\Omega_{\boldsymbol{v}} \underbrace{\int_0^\infty \mathrm{d}v\, \frac{1}{f_{\mathrm{M}\alpha}} \delta g_\alpha \frac{\partial}{\partial v}\left(\frac{v^2}{2} \frac{\mathrm{d}^2\psi_\alpha}{\mathrm{d}v^2} \frac{\partial \delta f_\alpha}{\partial v} - v^2 \frac{\mathrm{d}\varphi_\alpha}{\mathrm{d}v} \delta f_\alpha \right)}_{\text{int. by parts to move } \partial/\partial v \text{ onto } \delta g_\alpha / f_{\mathrm{M}\alpha}}$$

$$= \int \mathrm{d}\boldsymbol{v}\, \frac{1}{f_{\mathrm{M}\alpha}} \delta f_\alpha \nu_0^{\alpha\alpha} \frac{v_{\mathrm{th}\alpha}^3}{n_\alpha} \frac{1}{v^3} \frac{\mathrm{d}\psi_\alpha}{\mathrm{d}v} \mathcal{L}[\delta g_\alpha]$$
$$- \nu_0^{\alpha\alpha} \frac{v_{\mathrm{th}\alpha}^3}{n_\alpha} \underbrace{\int \mathrm{d}\Omega_{\boldsymbol{v}} \int_0^\infty \mathrm{d}v\, \frac{\partial}{\partial v}\left(\frac{\delta g_\alpha}{f_{\mathrm{M}\alpha}}\right) \left(\frac{v^2}{2} \frac{\mathrm{d}^2\psi_\alpha}{\mathrm{d}v^2} \frac{\partial \delta f_\alpha}{\partial v} - v^2 \frac{\mathrm{d}\varphi_\alpha}{\mathrm{d}v} \delta f_\alpha \right)}_{\text{do lots of int. by parts and algebra on scrap paper}}$$
$$= \int \mathrm{d}\boldsymbol{v}\, \frac{1}{f_{\mathrm{M}\alpha}} \delta f_\alpha \nu_0^{\alpha\alpha} \frac{v_{\mathrm{th}\alpha}^3}{n_\alpha} \left[\frac{1}{v^3} \frac{\mathrm{d}\psi_\alpha}{\mathrm{d}v} \mathcal{L}[\delta g_\alpha] + \frac{1}{v^2}\frac{\partial}{\partial v}\left(\frac{v^2}{2} \frac{\mathrm{d}^2\psi_\alpha}{\mathrm{d}v^2} \frac{\partial \delta g_\alpha}{\partial v} - v^2 \frac{\mathrm{d}\varphi_\alpha}{\mathrm{d}v} \delta g_\alpha \right)\right]$$
$$= \int \mathrm{d}\boldsymbol{v}\, \frac{1}{f_{\mathrm{M}\alpha}} \delta f_\alpha C[\delta g_\alpha, f_{\mathrm{M}\alpha}].$$

Thus,

$$\int \mathrm{d}\boldsymbol{v}\, \frac{1}{f_{\mathrm{M}\alpha}} \delta g_\alpha \Big(C[f_{\mathrm{M}\alpha}, \delta f_\alpha] + C[\delta f_\alpha, f_{\mathrm{M}\alpha}] \Big) = \int \mathrm{d}\boldsymbol{v}\, \frac{1}{f_{\mathrm{M}\alpha}} \delta f_\alpha \Big(C[f_{\mathrm{M}\alpha}, \delta g_\alpha] + C[\delta g_\alpha, f_{\mathrm{M}\alpha}] \Big).$$

Q.E.D.

Self-adjointness implies positive entropy production (see Problem 3.3 in Helander & Sigmar (2005)), and also allows for an easy proof of conservation laws. For example, with $\delta g_\alpha = f_{\mathrm{M}\alpha}$, equation (VIII.7.5) implies

$$\int \mathrm{d}\boldsymbol{v} \Big(C[f_{\mathrm{M}\alpha}, \delta f_\alpha] + C[\delta f_\alpha, f_{\mathrm{M}\alpha}] \Big) = \int \mathrm{d}\boldsymbol{v}\, \frac{1}{f_{\mathrm{M}\alpha}} \delta f_\alpha \Big(C[f_{\mathrm{M}\alpha}, f_{\mathrm{M}\alpha}] + C[f_{\mathrm{M}\alpha}, f_{\mathrm{M}\alpha}] \Big) = 0.$$

Voila, number conservation. Likewise for momentum (use $\delta g_\alpha = \boldsymbol{v} f_{\mathrm{M}\alpha}$) and energy (use $\delta g_\alpha = v^2 f_{\mathrm{M}\alpha}$). Simple.

For completeness, here is the *linearized operator for collision between arbitrary species*:

$$\boxed{\begin{aligned} C^\ell[f_\alpha] &= C[f_{\mathrm{M}\alpha}, \delta f_\beta] + C[\delta f_\alpha, f_{\mathrm{M}\beta}] \\ &= \sum_\beta \nu_0^{\alpha\beta} \frac{v_{\mathrm{th}\beta}^3 f_{\mathrm{M}\alpha}}{n_\beta} \left[4\pi \frac{m_\alpha}{m_\beta} \delta f_\beta - \frac{2v}{v_{\mathrm{th}\alpha}^2}\left(1 - \frac{m_\alpha}{m_\beta}\right) \frac{\partial \delta\varphi_\beta}{\partial v} - \frac{2}{v_{\mathrm{th}\alpha}^2} \delta\varphi_\beta + \frac{2v^2}{v_{\mathrm{th}\alpha}^4} \frac{\partial^2 \delta\psi_\alpha}{\partial v^2} \right] \\ &\quad + \nu_0^{\alpha\beta} \frac{v_{\mathrm{th}\beta}^3}{n_\beta} \left[\frac{1}{v^3} \frac{\mathrm{d}\psi_\beta}{\mathrm{d}v} \mathcal{L}[\delta f_\alpha] + \frac{1}{v^2}\frac{\partial}{\partial v}\left(\frac{v^2}{2} \frac{\mathrm{d}^2\psi_\beta}{\mathrm{d}v^2} \frac{\partial \delta f_\alpha}{\partial v} - v^2 \frac{m_\alpha}{m_\beta} \frac{\mathrm{d}\varphi_\beta}{\mathrm{d}v} \delta f_\alpha \right)\right] \end{aligned}}$$

where

$$\nu_0^{\alpha\beta} \doteq \frac{4\pi q_\alpha^2 q_\beta^2 n_\beta \ln \lambda_{\alpha\beta}}{m_\alpha^2 v_{\mathrm{th}\beta}^3}.$$

Equation (VIII.7.6) conserves number, total momentum, and total energy. It's also possible to show that it is self-adjoint, and that it obeys an H theorem: entropy stays constant only for

$$\delta f_\alpha(\boldsymbol{v}) = f_{\mathrm{M}\alpha}(v)\left[\frac{\delta n_\alpha}{n_\alpha} + \frac{m_\alpha \delta\boldsymbol{u}\boldsymbol{\cdot}(\boldsymbol{v}-\boldsymbol{u})}{T} + \frac{\delta T}{T}\left(\frac{m_\alpha|\boldsymbol{v}-\boldsymbol{u}|^2}{2T} - \frac{3}{2}\right)\right],$$
$$\delta f_\beta(\boldsymbol{v}) = f_{\mathrm{M}\beta}(v)\left[\frac{\delta n_\beta}{n_\beta} + \frac{m_\beta \delta\boldsymbol{u}\boldsymbol{\cdot}(\boldsymbol{v}-\boldsymbol{u})}{T} + \frac{\delta T}{T}\left(\frac{m_\beta|\boldsymbol{v}-\boldsymbol{u}|^2}{2T} - \frac{3}{2}\right)\right],$$

which are just perturbations to two full Maxwellians with equal temperatures and mean velocities. Otherwise, entropy increases.

In summary, one can identify four collision frequencies:

$\nu_{ee} \sim m_e^{-1/2}$	equilibration of electrons with each other
$\nu_{ei} \sim m_e^{-1/2}$	equilibration of electrons, momentum transfer between electrons and ions
$\nu_{ii} \sim m_i^{-1/2}$	equilibration of ions with each other
$\nu_{ie} \sim \dfrac{m_e}{m_i}\nu_{ei}$	momentum and enegy transfer between ions and electrons

Thus,

$$\boxed{\nu_{ee}, \nu_{ei} \;:\; \nu_{ii} \;:\; \nu_{ie} \;=\; 1 \;:\; \left(\frac{m_e}{m_i}\right)^{1/2} \;:\; \left(\frac{m_e}{m_i}\right)} \qquad \text{(VIII.7.6)}$$

PART IX

Classical transport

We now embark on the final part of the this course, that which corresponds to the final entries in Bogoliubov's hierarchy of timescales (see §I.1):

(3) Macroscopic force balance emerges on a crossing time $\sim L/v_{\mathrm{th}} = \nu^{-1}(L/\lambda_{\mathrm{mfp}}) \gg \nu^{-1}$. (Recall that this timescale was not included in Bogoliubov's original hierarchy, but it appears in the Chapman–Enskog–Braginskii expansion and is thus important in this course.)

(4) Hydrodynamic diffusion occurs on macroscopic spatial and temporal scales, and attempts to relax the system to a global, space- and time-independent Maxwellian. (Boundary conditions that enforce density or temperature gradients prevent this from occurring.) This occurs on a diffusive timescale $\sim L^2/D$; e.g., $\sim\nu^{-1}(L/\lambda_{\mathrm{mfp}})^2$ in an unmagnetized plasma, or $\sim\nu^{-1}(L/\rho)^2$ across the magnetic field in a magnetized plasma.

This part is what ties all that we've done up to this point with the more familiar topics of collisional hydrodynamics and magnetohydrodynamics and the more tangible phenomenon of spatial irreversibility. The focus is on transport processes that are spatially local, that is, the fluxes of particles, momentum, and energy are due to forces at approximately the same location. This requires the plasma to be dominated by collisions, with collisional mean free paths much smaller than the gradient lengthscales (in the direction of the magnetic field if the system is magnetized; in this case, either the mean free path of the mean gyroradius must be much shorter than the gradient lengthscales in the directions perpendicular to the magnetic field). Then, the particles comprising the plasma are affected only by forces within a mean free path (or a gyroradius).

The connection to the prior topics of this course is captured well by this excerpt from the review article by Hinton (1983):

> When the transport processes are local, the plasma may be considered to be made up of many approximately closed subsystems, with slightly different densities, mean velocities and temperatures. Charged-particle collisions tend to force each subsystem to local thermodynamic equilibrium, with the subsystem entropies being maximized, subject to the constraints imposed by particle, momentum and energy conservation. Because of the small differences between subsystems, the velocity distributions for these subsystems depart slightly from Maxwellians. For example, the distribution of the velocity component in the direction of the temperature gradient is skewed somewhat in the direction of motion of those particles coming from the hotter region. As a result, there are small fluxes of particles, momentum and energy between subsystems, which are approximately linear in the thermodynamic forces (e.g. the density and temperature gradients). The resulting entropy fluxes between subsystems then make the plasma as a whole tend towards a state of global thermal equilibrium. Because of the boundary conditions and other external constraints, such as applied electromotive forces, the plasma generally is not able to reach this equilibrium state but remains in a nonequilibrium steady state. The charged particles and energy are lost from the plasma at the same rate that they are produced in the plasma in this steady state. It is the goal of transport theory to calculate these loss rates, assuming they are due to Coulomb collisions.

With this viewpoint borne in mind, we focus on three calculations:

(1) The Spitzer–Härm problem (§IX.1): how collisional conductivity is established in a plasma subject to a macroscopic electric field

(2) The Chapman–Enskog expansion (§IX.2): how Navier–Stokes emerges from the kinetic equation for a collisional plasma

(3) The Braginskii-MHD equations (§IX.3): how transport in a collisional, magnetized plasma is constrained by the magnetic field

IX.1. The Spitzer–Härm problem

For our first example of transport theory, consider the conductivity of a fully ionized, collisional plasma. The problem can be stated and solved with varying degrees of complexity and difficulty. Let's start simple.

A constant electric field is applied to an infinite, homogeneous plasma. We seek a relationship between the steady-state current that results and the applied field:

$$\boldsymbol{j} = \sigma \boldsymbol{E}. \tag{IX.1.1}$$

Here, σ is called the *electrical conductivity*. As a first pass at calculating σ, let us recall (VIII.4.17):

$$\boldsymbol{R}_{ei} = \frac{m_e n_e}{\tau_{ei}} (\boldsymbol{u}_i - \boldsymbol{u}_e),$$

which is the frictional force on electrons due to collisions with ions if (*if!*) the electrons are Maxwellian and the collision operator is approximated by the Lorentz operator (see (VIII.4.11)):

$$C[f_e] = \frac{3\sqrt{\pi}}{4\tau_{ei}}\left(\frac{v_{\rm the}}{v}\right)^3 \mathcal{L}\left[f_e + 2\boldsymbol{v}\cdot\boldsymbol{u}_i\frac{\partial f_e}{\partial v^2}\right].$$

In steady state, force balance on the electrons is then

$$\begin{aligned}0 &= -en_e\boldsymbol{E} + \boldsymbol{R}_{ei}\\ &= -en_e\boldsymbol{E} + \frac{m_e n_e}{\tau_{ei}}(\boldsymbol{u}_i - \boldsymbol{u}_e)\\ &= -en_e\boldsymbol{E} + \frac{m_e n_e}{\tau_{ei}}\frac{\boldsymbol{j}}{en_e}\end{aligned}$$

$$\Longrightarrow \quad \boxed{\boldsymbol{j} = \frac{e^2 n_e \tau_{ei}}{m_e}\boldsymbol{E} \doteq \sigma\boldsymbol{E}} \tag{IX.1.2}$$

Now, what we have done here is treated the small distortion of the electron distribution generated by the electric field as an induced flow, with no distortion in the thermal part of the distribution. Not really kinetics, but we do have an answer! The disappointing, yet somewhat reassuring, thing is that doing a proper kinetic treatment only affords a more accurate numerical prefactor. A lot of work for an $\mathcal{O}(1)$ detail, but it's an important detail, and finding such details is good for one's training and character.

So how do we do better? We really ought to be solving

$$-\frac{e}{m_e}\boldsymbol{E}\cdot\frac{\partial f_e}{\partial \boldsymbol{v}} = C[f_e], \tag{IX.1.3}$$

but the full collision operator is complicated. For our a first pass at this, ignore self-collisions and adopt the Lorentz collision operator for electron-ion collisions:

$$C = \frac{3\sqrt{\pi}}{4\tau_{ei}}\left(\frac{v_{\rm the}}{v}\right)^3 \mathcal{L}.$$

We'll also neglect any ion motion, and assume that $E < E_{\rm D}$ (see (VIII.6.29)) so that the distortion in f_e will be small and there will be few runaways. Then (IX.1.3) can be solved perturbatively by introducing the small parameter

$$\epsilon \doteq \frac{E}{E_{\rm D}} \sim \frac{J}{en_e v_{\rm the}} \sim \frac{u_e}{v_{\rm the}} \ll 1 \tag{IX.1.4}$$

and by expanding the electron distribution function in powers of ϵ:

$$f_e = f_{e0} + \epsilon f_{e1} + \epsilon^2 f_{e2} + \dots. \tag{IX.1.5}$$

Equation (IX.1.3) becomes

$$-\frac{e}{m_e}\boldsymbol{E}\cdot\frac{\partial}{\partial\boldsymbol{v}}\left(f_{e0} + \epsilon f_{e1} + \epsilon^2 f_{e2} + \dots\right) = C\big[f_{e0}\big] + C\big[f_{e1}\big] + C\big[f_{e2}\big] + \dots \tag{IX.1.6}$$

Now examine (IX.1.6) order by order in ϵ. At zeroth order, we have

$$0 = C\big[f_{e0}\big] = \frac{3\sqrt{\pi}}{4\tau_{ei}}\left(\frac{v_{\rm the}}{v}\right)^3 \mathcal{L}\big[f_{e0}\big] \quad\Longrightarrow\quad f_{e0} = f_{e0}(v); \tag{IX.1.7}$$

i.e., the zeroth-order electron distribution function is independent of the (cosine of the) pitch angle $\xi \doteq v_\parallel/v$, where $\parallel$ denotes the direction parallel to the applied electric field.

While not enforced by this collision operator, let us assume that

$$f_{e0}(v) = f_{\mathrm{M},e}(v) \doteq \frac{n_e}{\pi^{3/2} v_{\mathrm{th}e}^3} \exp\left(-\frac{v^2}{v_{\mathrm{th}e}^2}\right).$$

(This would be enforced by a more realistic collision operator.) Proceeding to first order in ϵ, we then have

$$-\frac{e}{m_e}\boldsymbol{E}\cdot\frac{\partial f_{e0}}{\partial \boldsymbol{v}} = \frac{e}{T_e}\boldsymbol{E}\cdot\boldsymbol{v} f_{\mathrm{M},e} = \frac{3\sqrt{\pi}}{4\tau_{ei}}\left(\frac{v_{\mathrm{th}e}}{v}\right)^3 \mathcal{L}[f_{e1}],$$

or, substituting in (VIII.4.6) for $\mathcal{L}$,

$$\frac{eEv}{T_e}\xi f_{\mathrm{M},e}(v) = \frac{3\sqrt{\pi}}{8\tau_{ei}}\left(\frac{v_{\mathrm{th}e}}{v}\right)^3 \frac{\partial}{\partial \xi}(1-\xi^2)\frac{\partial}{\partial \xi} f_{e1}. \tag{IX.1.8}$$

Equation (IX.1.8) can be solved by recognizing that $\xi = P_1(\xi)$ and exploiting the orthogonality of the Legendre polynomials or, simply, by direct integration:

$$\frac{eEv}{T_e}\frac{\xi^2}{2} f_{\mathrm{M},e}(v) = \frac{3\sqrt{\pi}}{8\tau_{ei}}\left(\frac{v_{\mathrm{th}e}}{v}\right)^3 (1-\xi^2)\frac{\partial f_{e1}}{\partial \xi} + \mathrm{const.} \tag{IX.1.9}$$

To keep $\partial f_{e1}/\partial\xi$ finite at $\xi = \pm 1$, the constant of integration must be $(eEv/2T_e) f_{\mathrm{M},e}(v)$. Then (IX.1.9) becomes

$$\frac{eEv}{T_e}\frac{\xi^2-1}{2} f_{\mathrm{M},e}(v) = \frac{3\sqrt{\pi}}{8\tau_{ei}}\left(\frac{v_{\mathrm{th}e}}{v}\right)^3 (1-\xi^2)\frac{\partial f_{e1}}{\partial \xi}.$$

Integrating once more over ξ, with $f_{e1}(\xi = 0) = 0$, and solving for f_{e1} gives

$$\boxed{f_{e1} = -\frac{eEv}{T_e}\xi\frac{4\tau_{ei}}{3\sqrt{\pi}}\left(\frac{v}{v_{\mathrm{th}e}}\right)^3 f_{\mathrm{M},e}} \tag{IX.1.10}$$

Note that this distortion is not just a drifting Maxwellian, for which f_{e1} would be $(m_e v u_e \xi/T_e) f_{\mathrm{M},e}$. The greater distortion at higher speeds in this Lorentz model occurs because the friction force is proportional to v^{-2}, so that the electric field causes a greater distortion at higher speeds. (A complete collision operator with electron–electron collisions would reduce the magnitude of this distortion for $v \gg v_{\mathrm{th}e}$.) Now we can compute the current that is flowing (recall that we've assumed stationary ions):

$$\begin{aligned}
&\boldsymbol{j} = -e\int \mathrm{d}\boldsymbol{v}\,\boldsymbol{v} f_{e1} = \frac{e^2 E}{T_e}\frac{4\tau_{ei}}{3\sqrt{\pi}}\int \mathrm{d}\boldsymbol{v}\,\boldsymbol{v} v\xi\left(\frac{v}{v_{\mathrm{th}e}}\right)^3 f_{\mathrm{M},e}(v) \\
\Longrightarrow\quad & j_\parallel = \frac{e^2 E n_e}{T_e}\frac{8\tau_{ei}}{3\sqrt{\pi}}\underbrace{\int_{-1}^{+1}\mathrm{d}\xi\,\xi^2}_{=2/3}\int_0^\infty \mathrm{d}v\, v^2\left(\frac{v}{v_{\mathrm{th}e}}\right)^5 \frac{\mathrm{e}^{-v^2/v_{\mathrm{th}e}^2}}{\sqrt{\pi} v_{\mathrm{th}e}} \\
&\quad = \frac{e^2 E n_e}{T_e}\frac{8\tau_{ei}}{9\sqrt{\pi}}\frac{v_{\mathrm{th}e}^2}{\sqrt{\pi}}\underbrace{\int_0^\infty \mathrm{d}x\, x^3 \mathrm{e}^{-x}}_{=3!=6} \\
&\quad = \frac{32}{3\pi}\frac{e^2 n_e \tau_{ei}}{m_e}E = \frac{32}{3\pi}\sigma E
\end{aligned}$$

$$\Longrightarrow\quad \boxed{\sigma_{\mathcal{L}} = \frac{32}{3\pi}\sigma} \tag{IX.1.11}$$

Now, $32/3\pi \simeq 3.40$ is greater than 1 (obviously). The increase is because more current is being carried by high-speed electrons, whose frictional drag is smaller. Indeed, $f_{e1} \propto v^4 \exp(-v^2/v_{\mathrm{the}}^2)$, which peaks at $v = \sqrt{2} v_{\mathrm{the}}$.

To ensure consistency with our expansion, let us check when $f_{e1}/f_{\mathrm{M},e} \ll 1$. From (IX.1.10),

$$\frac{f_{e1}}{f_{\mathrm{M},e}} \sim \frac{eE}{T_e} v_{\mathrm{the}} \tau_{ei} = \frac{eE}{T_e} \lambda_{\mathrm{mfp}} \ll 1;$$

i.e., the work done on an electron over a distance comparable to the collisional mean free path must be smaller than the typical electron kinetic energy. In other words, the energy gain between collisions must not be too large, so that the Maxwellian distribution function is not significantly distorted. Another way of stating this inequality is by using $E/E_{\mathrm{D}} \sim \epsilon$ to find $\sqrt{\lambda_{\mathrm{mfp}}(e^2/T_e)} \sim \lambda_{\mathrm{D}}$; i.e., the Debye length should be intermediate between the mean free path and the Landau length.

Doing better than this constitutes the *Spitzer–Härm problem*:

$$-\frac{e}{m_e} \boldsymbol{E} \boldsymbol{\cdot} \frac{\partial f_e}{\partial \boldsymbol{v}} = C[f_e, f_e] + C[f_e, f_i] \tag{IX.1.12}$$

(Spitzer & Härm 1953; Spitzer 1962), where the collision operator on the right-hand side of (IX.1.12) is the full Landau operator. The solution for the Spitzer–Härm conductivity σ_{SH} was obtained numerically (see Spitzer 1962; Braginskii 1965):

$$\sigma_{\mathrm{SH}} = \frac{\sigma}{\alpha_e} \quad \text{with} \quad \begin{array}{r|cccccc} Z & 1 & 2 & 3 & 4 & 16 & \infty \\ \alpha_e\ (\text{Spitzer}) & 0.506 & 0.431 & \text{—} & 0.375 & 0.319 & \text{—} \\ \alpha_e\ (\text{Braginskii}) & 0.51 & 0.44 & 0.40 & 0.38 & \text{—} & 0.29 \end{array} \tag{IX.1.13}$$

Note that the $Z = \infty$ solution, for which $\alpha_e = 0.29 \simeq 3\pi/32$, corresponds to our Lorentz-operator solution, in which electron–electron collisions are negligible. The reduction of the conductivity when $Z = 1$ is because electron self-collisions reduce the high-energy tails (which was caused by the v^4 factor of the Lorentz solution (IX.1.10)) by pushing f_e towards a Maxwellian.

The Spitzer problem can, in fact, be solved analytically to any desired accuracy by expanding the perturbed distribution function f_{e1} in a suitable basis of orthogonal functions, taking moments of the kinetic equation, and then solving the resulting set of matrix equations for the coefficients in the expansion. This approach was developed in Braginskii (1958, 1965), Hirshman (1977, 1978), Hirshman & Sigmar (1981), and Hinton (1983). The appropriate basis of orthogonal functions is that of Laguerre polynomials (which were shown also to be the appropriate basis functions for determining the most compact set of moment equation by Grad; see Grad (1949*a*,*b*)):

$$f_{e1} = \frac{2 v_\parallel}{v_{\mathrm{the}}^2} f_{\mathrm{M},e} \sum_{k=0}^{N} u_{e,k} L_k^{(3/2)}\left(\frac{v^2}{v_{\mathrm{the}}^2}\right), \tag{IX.1.14}$$

where $v_\parallel = v\xi$ and

$$L_0^{(3/2)}(x) = 1, \quad L_1^{(3/2)}(x) = \frac{5}{2} - x, \quad L_2^{(3/2)}(x) = \frac{35}{8} - \frac{7}{2}x + \frac{1}{2}x^2, \ldots.$$

The Laguerre polynomials constitute a good basis because, for the oft-occuring integral weighting function of

$$\int_0^\infty \frac{\mathrm{d}v\, v^2}{v_{\mathrm{th}}^3} \frac{2v^2}{v_{\mathrm{th}}^2} \,\mathrm{e}^{-v^2/v_{\mathrm{th}}^2} = \int_0^\infty \mathrm{d}x\, x^{3/2} \mathrm{e}^{-x},$$

they have the orthogonality relation

$$\int_0^\infty \mathrm{d}x\, x^{3/2}\mathrm{e}^{-x} L_p^{(3/2)}(x) L_q^{(3/2)}(x) = \frac{(p+3/2)!}{p!}\,\delta_{pq} = \frac{\Gamma(p+5/2)}{\Gamma(p+1)}\,\delta_{pq}.$$

Here are the first few integrals, which will ultimately come in handy:

$$\begin{aligned}
\int_0^\infty \mathrm{d}x\, x^{3/2}\mathrm{e}^{-x}\big[L_0^{(3/2)}(x)\big]^2 &= \frac{3\sqrt{\pi}}{4},\\
\int_0^\infty \mathrm{d}x\, x^{3/2}\mathrm{e}^{-x}\big[L_1^{(3/2)}(x)\big]^2 &= \frac{15\sqrt{\pi}}{8},\\
\int_0^\infty \mathrm{d}x\, x^{3/2}\mathrm{e}^{-x}\big[L_2^{(3/2)}(x)\big]^2 &= \frac{105\sqrt{\pi}}{32}.
\end{aligned}$$

While the Laguerre polynomials are not eigenfunctions of the Landau collision operator, they are particularly useful for kinetic transport problems in which the background distribution is Maxwellian. (Because the lowest-order equation is $0 = C[f_{e0}]$, with C being the full Landau collision operator, a Maxwellian background is guaranteed by the H theorem.) This can be seen by examining the moments. For example, the momentum density parallel to the electric field is

$$\begin{aligned}
n_e u_{\|e} &= \int \mathrm{d}\boldsymbol{v}\, v_\| f_e = \int \mathrm{d}\boldsymbol{v}\, v_\| \big(\cancelto{0}{f_{e0}} + f_{e1} + \dots\big)\\
&\simeq \int \mathrm{d}\boldsymbol{v}\, v\xi f_{e1}\\
&= \int \mathrm{d}\boldsymbol{v}\, \frac{2v^2}{v_{\mathrm{th}e}^2}\xi^2 f_{\mathrm{M},e}\left[u_{e,0}L_0^{(3/2)}\!\left(\frac{v^2}{v_{\mathrm{th}e}^2}\right) + u_{e,1}L_1^{(3/2)}\!\left(\frac{v^2}{v_{\mathrm{th}e}^2}\right) + \dots\right]\\
&= \frac{4\pi}{3}\frac{n_e}{\pi\sqrt{\pi}}\int_0^\infty \mathrm{d}x\, x^{3/2}\mathrm{e}^{-x}\, \underbrace{L_0^{(3/2)}(x)}_{\substack{\text{can insert}\\ \text{here, since}\\ \text{it equals 1}}} \Big[u_{e,0}L_0^{(3/2)}(x) + u_{e,1}L_1^{(3/2)}(x) + \dots\Big]\\
&= \frac{4}{3\sqrt{\pi}} n_e u_{e,0} \times \frac{3\sqrt{\pi}}{4} = n_e u_{e,0}.
\end{aligned} \tag{IX.1.15}$$

Evidently, the zeroth Laguerre coefficient is the parallel electron fluid flow. Likewise, the parallel heat flux of electrons is[18]

$$
\begin{aligned}
\widehat{q}_{\|e} &= \int \mathrm{d}\boldsymbol{v}\, v_\| \left(\frac{m_e v^2}{2} - \frac{5T_e}{2}\right) f_e = \int \mathrm{d}\boldsymbol{v}\, v_\| \left(\frac{m_e v^2}{2} - \frac{5T_e}{2}\right) (\cancelto{0}{f_{e0}} + f_{e1} + \dots) \\
&\simeq \int \mathrm{d}\boldsymbol{v}\, v\xi \left(\frac{m_e v^2}{2} - \frac{5T_e}{2}\right) f_{e1} \\
&= T_e \int \mathrm{d}\boldsymbol{v}\, \frac{2v^2}{v_{\mathrm{th}e}^2} \xi^2 f_{\mathrm{M},e} \left(\frac{v^2}{v_{\mathrm{th}e}^2} - \frac{5}{2}\right) \left[u_{e,0} L_0^{(3/2)}\left(\frac{v^2}{v_{\mathrm{th}e}^2}\right) + u_{e,1} L_1^{(3/2)}\left(\frac{v^2}{v_{\mathrm{th}e}^2}\right) + \dots\right] \\
&= \frac{4\pi}{3} \frac{n_e T_e}{\pi\sqrt{\pi}} \int_0^\infty \mathrm{d}x\, x^{3/2} \mathrm{e}^{-x} \underbrace{\left(x - \frac{5}{2}\right)}_{= -L_1^{(3/2)}(x)} \left[u_{e,0} L_0^{(3/2)}(x) + u_{e,1} L_1^{(3/2)}(x) + \dots\right] \\
&= -\frac{4}{3\sqrt{\pi}} n_e T_e u_{e,1} \times \frac{15\sqrt{\pi}}{8} = -\frac{5}{2} p_e u_{e,1}.
\end{aligned}
\tag{IX.1.16}
$$

The first Laguerre coefficient is related to the parallel electron heat flux.

Using (IX.1.15) and (IX.1.16), the first-order electron distribution function (IX.1.14) can thus be written as

$$
\boxed{f_{e1} = \frac{2v_\|}{v_{\mathrm{th}e}^2} f_{\mathrm{M},e} \left[u_{\|e} L_0^{(3/2)}(x) - \frac{2}{5} \frac{\widehat{q}_{\|e}}{p_e} L_1^{(3/2)}(x) + \dots\right]; \quad x \doteq \frac{v^2}{v_{\mathrm{th}e}^2}}
\tag{IX.1.17}
$$

This expression goes into the first-order kinetic equation

$$
-\frac{e}{m_e} \boldsymbol{E} \cdot \frac{\partial f_{e0}}{\partial \boldsymbol{v}} = \frac{eEv_\|}{T_e} f_{\mathrm{M},e} = C[f_{e1}],
\tag{IX.1.18}
$$

where

$$
C[f_{e1}] = C[f_{e1}, f_{\mathrm{M},e}] + C[f_{\mathrm{M},e}, f_{e1}] + C[f_{e1}, f_{\mathrm{M},i}] + C[f_{\mathrm{M},e}, f_{i1}].
$$

is the first-order collision operator. Thus, to solve the Spitzer problem, we substitute (IX.1.17) into (IX.1.18), take $\int \mathrm{d}\boldsymbol{v}\, v_\| L_k^{(3/2)}(x)$ moments of the resulting kinetic equation to determine the $u_{e,k}$ coefficients, and invert the resulting matrix equation to solve for $u_{e,k}$ in terms of E. We can continue to higher and higher k to obtain ever more accurate answers. All this, just to invert a collision operator! Let us proceed with this programme.

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[18] This is, of course, not the *conductive* parallel electron heat flux, which is customarily given by $q_{\|e} \doteq \int \mathrm{d}\boldsymbol{v}\, w_\| (m_e w^2/2) f_e$, where $\boldsymbol{w} \doteq \boldsymbol{v} - \boldsymbol{u}_e$ is the peculiar velocity. Using $\langle w_\| f_e \rangle = 0$, this expression is also equivalent to $q_{\|e} = \int \mathrm{d}\boldsymbol{v}\, w_\| (m_e w^2/2 - 5T_e/2) f_e = -T_e \int \mathrm{d}\boldsymbol{v}\, w_\| L_1^{(3/2)}(w^2/v_{\mathrm{th}e}^2) f_e$. Mathematically, the distinction between this definition and that given in (IX.1.16) doesn't matter; it just matters what we dub the "heat flux" and whether we chose to Laguerre expand about a stationary or drifting Maxwellian. But to avoid confusion, the heat flux defined in (IX.1.16) is adorned by a hat to distinguish it from $q_{\|e}$. To be absolutely precise, the vector $\widehat{\boldsymbol{q}} = \int \mathrm{d}\boldsymbol{v}\, (mv^2/2 - 5T/2) \boldsymbol{v} f = \boldsymbol{q} + \boldsymbol{\Pi} \cdot \boldsymbol{u} + (mnu^2/2) \boldsymbol{u}$, where $\boldsymbol{q} \equiv \int \mathrm{d}\boldsymbol{w}\, \boldsymbol{w} (mw^2/2) f$ is the conductive heat flux, $\boldsymbol{\Pi} \equiv \int \mathrm{d}\boldsymbol{w}\, m(\boldsymbol{w}\boldsymbol{w} - w^2 \mathsf{I}/3) f$ is the viscous stress, and $\boldsymbol{u} \equiv (1/n) \int \mathrm{d}\boldsymbol{v}\, \boldsymbol{v} f$ is the fluid velocity.

The left-hand side of (IX.1.18) is easy: for each k in the sum,

$$\int \mathrm{d}\boldsymbol{v}\, \frac{eE}{m_e}\frac{2v_\parallel^2}{v_{\mathrm{th}e}^2} f_{\mathrm{M},e} L_k^{(3/2)}\left(\frac{v^2}{v_{\mathrm{th}e}^2}\right) = \frac{eE}{m_e}\int \mathrm{d}\boldsymbol{v}\, \frac{2v^2}{v_{\mathrm{th}e}^2}\xi^2 f_{\mathrm{M},e} L_k^{(3/2)}\left(\frac{v^2}{v_{\mathrm{th}e}^2}\right)$$
$$= \frac{4\pi}{3}\frac{eE}{m_e}\frac{n_e}{\pi\sqrt{\pi}}\int_0^\infty \mathrm{d}x\, x^{3/2}\mathrm{e}^{-x}L_k^{(3/2)}(x)$$
$$= \frac{4}{3\sqrt{\pi}}\frac{eEn_e}{m_e}\times \delta_{k0}\frac{3\sqrt{\pi}}{4} = \frac{eEn_e}{m_e}\delta_{k0}. \qquad \text{(IX.1.19)}$$

The right-hand side of (IX.1.18)? Not so much. The electron–ion operator isn't too bad (see (VIII.4.11)):

$$\int \mathrm{d}\boldsymbol{v}\, v_\parallel L_k^{(3/2)}(x)\, C[f_e, f_i] = \int \mathrm{d}\boldsymbol{v}\, v_\parallel L_k^{(3/2)}(x)\, \frac{3\sqrt{\pi}}{4\tau_{ei}}\left(\frac{v_{\mathrm{th}e}}{v}\right)^3 \mathcal{L}\left[f_{e1} + 2\boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{u}_i\frac{\partial f_{\mathrm{M},e}}{\partial v^2}\right]$$
$$= \frac{3\sqrt{\pi}}{4\tau_{ei}}\, 2\pi v_{\mathrm{th}e}^3 \int_{-1}^{+1}\mathrm{d}\xi\, \xi \int_0^\infty \mathrm{d}v\, L_k^{(3/2)}(x)\,\frac{1}{2}\frac{\partial}{\partial\xi}(1-\xi^2)\frac{\partial}{\partial\xi}$$
$$\times \left[\frac{2v_\parallel}{v_{\mathrm{th}e}^2} f_{\mathrm{M},e}\sum_{\ell=0}^{N} u_{e,\ell}L_\ell^{(3/2)}(x) - \frac{2v_\parallel u_i}{v_{\mathrm{th}e}^2}f_{\mathrm{M},e}\right].$$

Let's do each term in brackets separately. The first term is

$$-\frac{3n_e}{2\tau_{ei}}\int_{-1}^{+1}\mathrm{d}\xi\,\xi^2\int_0^\infty \mathrm{d}x\,\mathrm{e}^{-x}L_k^{(3/2)}(x)\sum_{\ell=0}^{N} u_{e,\ell}L_\ell^{(3/2)}(x)$$
$$= -\frac{n_e}{\tau_{ei}}\int_0^\infty \mathrm{d}x\,\mathrm{e}^{-x}L_k^{(3/2)}(x)\left[u_{e,0}L_0^{(3/2)}(x) + u_{e,1}L_1^{(3/2)}(x) + u_{e,2}L_2^{(3/2)}(x) + \dots\right]$$
$$= -\frac{n_e}{\tau_{ei}}\begin{bmatrix} 1 & 3/2 & 15/8 \\ 3/2 & 13/4 & 69/16 \\ 15/8 & 69/16 & 433/64 \end{bmatrix}\begin{bmatrix} u_{e,0} \\ u_{e,1} \\ u_{e,2}\end{bmatrix} \quad \text{for } k = 0,1,2. \qquad \text{(IX.1.20)}$$

The second term is

$$\frac{3n_e}{2\tau_{ei}}\int_{-1}^{+1}\mathrm{d}\xi\,\xi^2\int_0^\infty \mathrm{d}x\,\mathrm{e}^{-x}L_k^{(3/2)}(x)\,u_i = \frac{n_e}{\tau_{ei}}\begin{bmatrix} 1 & 0 & 0 \\ 3/2 & 0 & 0 \\ 15/8 & 0 & 0\end{bmatrix}\begin{bmatrix} u_i \\ 0 \\ 0\end{bmatrix} \quad \text{for } k = 0,1,2. \qquad \text{(IX.1.21)}$$

Now the electron–electron piece: we need the perturbed Rosenbluth potentials written in terms of Laguerre polynomials. I'm not going to do this. See Hirshman (1977) and Hinton (1983); they exploit certain properties of the linearized collision operator, like it being self-adjoint and momentum conserving, to calculate the coefficients. The resulting contribution to the right-hand side of (IX.1.18) is

$$-\frac{n_e}{\tau_{ee}}\sqrt{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 3/4 \\ 0 & 3/4 & 45/16\end{bmatrix}\begin{bmatrix} u_{e,0} \\ u_{e,1} \\ u_{e,2}\end{bmatrix}; \quad \tau_{ee} = Z\tau_{ei}. \qquad \text{(IX.1.22)}$$

Combining (IX.1.20)–(IX.1.22) to obtain the right-hand side of (IX.1.18) leaves us with a matrix equation to be solved for the $u_{e,k}$ coefficients:

$$\frac{eE\tau_{ee}}{m_e}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = -\begin{bmatrix} Z & \frac{3}{2}Z & \frac{15}{8}Z \\ \frac{3}{2}Z & \sqrt{2}+\frac{13}{4}Z & \frac{3\sqrt{2}}{4}+\frac{69}{16}Z \\ \frac{15}{8}Z & \frac{3\sqrt{2}}{4}+\frac{69}{16}Z & \frac{45\sqrt{2}}{16}+\frac{433}{64}Z \end{bmatrix}\begin{bmatrix} u_{e,0}-u_i \\ u_{e,1} \\ u_{e,2} \end{bmatrix}. \tag{IX.1.23}$$

The solution to (IX.1.23) is obtained after inverting the 3×3 matrix:

$$\begin{bmatrix} u_{e,0}-u_i \\ u_{e,1} \\ u_{e,2} \end{bmatrix} = -\frac{eE\tau_{ee}}{m_e}\begin{bmatrix} \frac{9}{2Z}\left(1+\frac{151\sqrt{2}}{72}Z+\frac{217}{288}Z^2\right) & -\frac{45\sqrt{2}}{16}-\frac{33}{16}Z & -\frac{3\sqrt{2}}{4}+\frac{3}{8}Z \\ -\frac{45\sqrt{2}}{16}-\frac{33}{16}Z & \frac{45\sqrt{2}}{16}+\frac{13}{4}Z & -\frac{3\sqrt{2}}{4}+\frac{3}{2}Z \\ -\frac{3\sqrt{2}}{4}+\frac{3}{8}Z & -\frac{3\sqrt{2}}{4}+\frac{3}{2}Z & \sqrt{2}+Z \end{bmatrix}$$

$$\times\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\frac{1}{\frac{9}{2}\left(1+\frac{61\sqrt{2}}{72}Z+\frac{2}{9}Z^2\right)}. \tag{IX.1.24}$$

Knowing that $u_{e,0} = u_{\|e}$ (see (IX.1.15)), we find

$$\boxed{j_{\|} = -en_e\left(u_{\|e}-u_{\|i}\right) = \sigma E_{\|} \times \frac{1+\frac{151\sqrt{2}}{72}Z+\frac{217}{288}Z^2}{1+\frac{61\sqrt{2}}{72}Z+\frac{2}{9}Z^2},} \tag{IX.1.25}$$

where σ is given by (IX.1.2). The factor in parentheses takes on the following values for varying Z:

Z	1	2	3	...	∞
#	1.9499	2.3210	2.5292	...	3.3906

Note that 3.3906 corresponds to the $\sigma_{\mathcal{L}}$ value from the calculation using the Lorentz operator (see (IX.1.11)). These values match the full numerical result to within 0.03% (not that we know $\ln \lambda_{ei}$ to that accuracy!). The solution for f_{e1} to this order is (see (IX.1.14))

$$f_{e1} = \frac{2v_{\|}}{v_{the}^2} f_{M,e}\left[u_{e,0} + u_{e,1}\left(\frac{5}{2} - \frac{v^2}{v_{the}^2}\right) + u_{e,2}\left(\frac{35}{8} - \frac{7}{2}\frac{v^2}{v_{the}^2} + \frac{1}{2}\frac{v^4}{v_{the}^4}\right)\right] \tag{IX.1.26}$$

with

$$u_{e,0} = u_{\|e} = u_i - \frac{eE\tau_{ei}}{m_e} \times \underbrace{\frac{1 + \dfrac{151\sqrt{2}}{72}Z + \dfrac{217}{288}Z^2}{1 + \dfrac{61\sqrt{2}}{72}Z + \dfrac{2}{9}Z^2}}_{=\,1.9499 \text{ for } Z\,=\,1},$$

$$u_{e,1} = -\frac{2}{5}\frac{\widehat{q}_{\|e}}{p_e} = \frac{eE\tau_{ei}}{m_e} \times \underbrace{\frac{\dfrac{5\sqrt{2}}{8}Z + \dfrac{11}{24}Z^2}{1 + \dfrac{61\sqrt{2}}{72}Z + \dfrac{2}{9}Z^2}}_{=\,0.5545 \text{ for } Z\,=\,1},$$

$$u_{e,2} = \begin{array}{c}\text{energy-}\\\text{weighted}\\\text{heat flow}\end{array} = \frac{eE\tau_{ei}}{m_e} \times \underbrace{\frac{\dfrac{\sqrt{2}}{6}Z - \dfrac{1}{12}Z^2}{1 + \dfrac{61\sqrt{2}}{72}Z + \dfrac{2}{9}Z^2}}_{=\,0.0630 \text{ for } Z\,=\,1}.$$

Before proceeding to the next section, I want to point out something rather subtle and ask you to think about it. Go all the way back to (IX.1.10), where we found that $f_{e1} \propto v_\| v^3 f_{\mathrm{M,e}}$ for a Lorentz operator describing electron-ion collisions. The v^3 there was due to the velocity dependence of the Coulomb collisions. So where is that v^3 in (IX.1.26)? All I see there are even powers of v in the Laguerre sum. . . surely Coulomb collisions don't change their v dependence just because we've chosen to work in a Laguerre basis. Shouldn't I recover the Lorentz result by taking $Z \to \infty$ in the above expression for f_{e1}? The value of $j_\|$ certainly matches. What gives?

IX.2. The Chapman–Enskog expansion

The electrical conductivity is just one material property of a plasma, for which the kinetic theory provides a rigorous means of calculating. You should know by now that the moment equations derived by taking velocity moments of the kinetic equation are not closed, and that one must do analogous calculations to obtain the form of the pressure tensor and the heat flux in terms of lower-order moments of the distribution function in order to close the system. For a hydrodynamic system, the Chapman–Enskog expansion does just this. It results from establishing a hierarchy of temporal and spatial scales and asymptotically solving the kinetic equation order by order in this scale separation. Before establishing this hierarchy and carrying out the expansion, let's do a quick recapitulation of from whence the hydrodynamic equations came.

Start with the Vlasov–Landau kinetic equation,

$$\frac{\mathrm{D}f_\alpha}{\mathrm{D}t} \doteq \frac{\partial f_\alpha}{\partial t} + \boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla} f_\alpha + \frac{q_\alpha}{m_\alpha}\boldsymbol{E}\boldsymbol{\cdot}\frac{\partial f_\alpha}{\partial \boldsymbol{v}} = C[f_\alpha].$$

We could of course add additional forces on the charged particles to $q_\alpha \boldsymbol{E}$, such as that due to gravity, $m_\alpha \boldsymbol{g}$. Since we'll use quasi-neutrality to eliminate $\boldsymbol{E}$ at one point, let's do that:

$$\frac{\mathrm{D}f_\alpha}{\mathrm{D}t} \doteq \frac{\partial f_\alpha}{\partial t} + \boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla} f_\alpha + \left(\frac{q_\alpha}{m_\alpha}\boldsymbol{E} + \boldsymbol{g}\right)\boldsymbol{\cdot}\frac{\partial f_\alpha}{\partial \boldsymbol{v}} = C[f_\alpha]. \tag{IX.2.1}$$

Now, $\boldsymbol{v}$ contains both thermal and mean velocities. It is useful to split them apart (e.g., because they might have very different magnitudes):

$$\boldsymbol{w} \doteq \boldsymbol{v} - \boldsymbol{u}_\alpha(t, \boldsymbol{r}), \tag{IX.2.2}$$

where

$$\boldsymbol{u}_\alpha(t, \boldsymbol{r}) \doteq \frac{1}{n_\alpha} \int \mathrm{d}\boldsymbol{v}\, \boldsymbol{v} f_\alpha, \quad n_\alpha(t, \boldsymbol{r}) \doteq \int \mathrm{d}\boldsymbol{v}\, f_\alpha. \tag{IX.2.3}$$

Enacting this transformation of variables, $f_\alpha(t, \boldsymbol{r}, \boldsymbol{v}) \to f_\alpha(t, \boldsymbol{r}, \boldsymbol{w})$, through the use of

$$\left.\frac{\partial}{\partial t}\right|_{\boldsymbol{v}} = \left.\frac{\partial}{\partial t}\right|_{\boldsymbol{w}} + \left.\frac{\partial \boldsymbol{w}}{\partial t}\right|_{\boldsymbol{v}} \cdot \frac{\partial}{\partial \boldsymbol{w}} = \left.\frac{\partial}{\partial t}\right|_{\boldsymbol{w}} - \frac{\partial \boldsymbol{u}_\alpha}{\partial t} \cdot \frac{\partial}{\partial \boldsymbol{w}}, \tag{IX.2.4}$$

$$\left.\frac{\partial}{\partial \boldsymbol{r}}\right|_{\boldsymbol{v}} = \left.\frac{\partial}{\partial \boldsymbol{r}}\right|_{\boldsymbol{w}} + \left.\frac{\partial \boldsymbol{w}}{\partial \boldsymbol{r}}\right|_{\boldsymbol{v}} \cdot \frac{\partial}{\partial \boldsymbol{w}} = \left.\frac{\partial}{\partial \boldsymbol{r}}\right|_{\boldsymbol{w}} - \frac{\partial \boldsymbol{u}_\alpha}{\partial \boldsymbol{r}} \cdot \frac{\partial}{\partial \boldsymbol{w}}, \tag{IX.2.5}$$

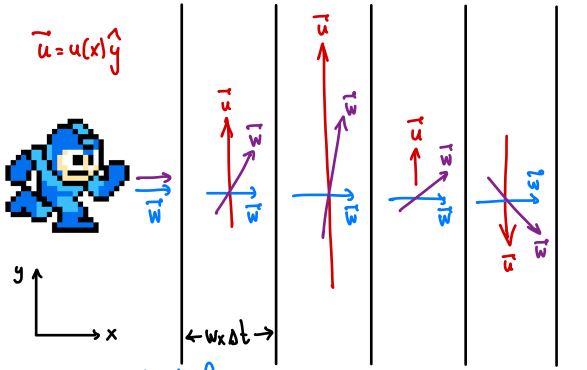
equation (IX.2.1) becomes

$$\frac{\mathrm{d} f_\alpha}{\mathrm{d} t_\alpha} + \boldsymbol{w} \cdot \boldsymbol{\nabla} f_\alpha + \Bigg(\underbrace{\frac{q_\alpha}{m_\alpha} \boldsymbol{E} + \boldsymbol{g} - \frac{\mathrm{d} \boldsymbol{u}_\alpha}{\mathrm{d} t_\alpha}}_{\doteq\, \boldsymbol{a}_\alpha(t, \boldsymbol{r})} - \boldsymbol{w} \cdot \boldsymbol{\nabla} \boldsymbol{u}_\alpha\Bigg) \cdot \frac{\partial f_\alpha}{\partial \boldsymbol{w}} = C[f_\alpha], \tag{IX.2.6}$$

where

$$\frac{\mathrm{d}}{\mathrm{d} t_\alpha} \doteq \frac{\partial}{\partial t} + \boldsymbol{u}_\alpha \cdot \boldsymbol{\nabla} \tag{IX.2.7}$$

is the Lagrangian time derivative taken in the frame comoving with the mean velocity $\boldsymbol{u}_\alpha$ of species α. The additional acceleration terms in (IX.2.6) that result from the frame transformation, *viz.* $\mathrm{d}\boldsymbol{u}_\alpha/\mathrm{d}t_\alpha$ and $\boldsymbol{w} \cdot \boldsymbol{\nabla} \boldsymbol{u}_\alpha$, are the result of boosting to a time- and space-dependent frame. The former term is fairly self-explanatory – particles must be accelerated so as to continue residing in the “fluid element” they comprise, which is itself being accelerated by various (magneto)hydrodynamic forces that result in $\mathrm{d}\boldsymbol{u}_\alpha/\mathrm{d}t_\alpha$– but the latter deserves some discussion. Imagine you are trying to walk at constant velocity $\boldsymbol{w} = w\hat{\boldsymbol{x}}$ across several layers of differentially moving conveyor belts with velocities $\boldsymbol{u} = u(x)\hat{\boldsymbol{y}}$, as in the figure below. In your frame (and the frame of the conveyor belts), your velocity will always be $w\hat{\boldsymbol{x}}$. But, in the lab frame, your velocity will include the velocity of the conveyor belts. This means that, every time you step onto a new conveyor belt that has some velocity oriented in the y direction that is different from that of the last conveyor belt, you will be accelerating in the lab frame. That is, your velocity in the lab frame will change over an interval of time from one conveyor belt to the next. Mathematically, the figure below corresponds to an acceleration $w\Delta u_y/\Delta x$ every time you step from one conveyor belt at position x with velocity $u\hat{\boldsymbol{y}}$ to another conveyor belt at position $x + \Delta x$ with velocity $(u + \Delta u)\hat{\boldsymbol{y}}$. The difference between these two points of view is enacted by adding $-\boldsymbol{w} \cdot \boldsymbol{\nabla} \boldsymbol{u}_\alpha$ to the acceleration term of (IX.2.6).



$\vec{w}$ in conveyor-belt frame is $\longrightarrow \vec{w}$, always
$\vec{w}$ in lab frame changes by $\Delta\vec{w} = (\vec{w}\cdot\vec{\nabla}\vec{u})\Delta t = \left(w_x \frac{du}{dx}\hat{y}\right)\Delta t$
with every timestep Δt.

Next, take those moments:

$$\int \mathrm{d}\boldsymbol{w}\,(\text{IX.2.6}):\quad \frac{\mathrm{d}}{\mathrm{d}t_\alpha}\cancelto{n_\alpha}{\int \mathrm{d}\boldsymbol{w}\, f_\alpha} + \cancelto{0}{\int \mathrm{d}\boldsymbol{w}\,\boldsymbol{w}\cdot\boldsymbol{\nabla} f_\alpha} + \boldsymbol{a}_\alpha\cdot \cancelto{0}{\int \mathrm{d}\boldsymbol{w}\,\frac{\partial f_\alpha}{\partial \boldsymbol{w}}}$$

$$- \underbrace{\int \mathrm{d}\boldsymbol{w}\,(\boldsymbol{w}\cdot\boldsymbol{\nabla}\boldsymbol{u}_\alpha)\cdot\frac{\partial f_\alpha}{\partial\boldsymbol{w}}}_{\overset{\text{bp}}{=}\, -(\boldsymbol{\nabla}\cdot\boldsymbol{u}_\alpha)\cancelto{n_\alpha}{\int \mathrm{d}\boldsymbol{w}\, f_\alpha}} = \cancelto{0}{\int \mathrm{d}\boldsymbol{w}\, C[f_\alpha]}$$

$$\Longrightarrow \quad \boxed{\frac{\mathrm{d}n_\alpha}{\mathrm{d}t_\alpha} + n_\alpha \boldsymbol{\nabla}\cdot\boldsymbol{u}_\alpha = 0} \quad \text{(continuity equation for species } \alpha) \tag{IX.2.8}$$

$$\int \mathrm{d}\boldsymbol{w}\, m_\alpha \boldsymbol{w}(\text{IX.2.6}):\quad \frac{\mathrm{d}}{\mathrm{d}t_\alpha}\cancelto{0}{\int \mathrm{d}\boldsymbol{w}\, m_\alpha\boldsymbol{w} f_\alpha} + \int \mathrm{d}\boldsymbol{w}\, m_\alpha \boldsymbol{w}\boldsymbol{w}\cdot\boldsymbol{\nabla} f_\alpha + m_\alpha\boldsymbol{a}_\alpha\cdot\underbrace{\int \mathrm{d}\boldsymbol{w}\,\boldsymbol{w}\frac{\partial f_\alpha}{\partial\boldsymbol{w}}}_{\overset{\text{bp}}{=}\, -n_\alpha \mathsf{I}}$$

$$- \cancelto{0}{\int \mathrm{d}\boldsymbol{w}\, m_\alpha\boldsymbol{w}(\boldsymbol{w}\cdot\boldsymbol{\nabla}\boldsymbol{u}_\alpha)\cdot\frac{\partial f_\alpha}{\partial\boldsymbol{w}}} = \int \mathrm{d}\boldsymbol{w}\, m_\alpha\boldsymbol{w}\,C[f_\alpha]$$

$$\Longrightarrow \quad \boxed{\boldsymbol{\nabla}\cdot\boldsymbol{P}_\alpha - m_\alpha n_\alpha \boldsymbol{a}_\alpha = \int \mathrm{d}\boldsymbol{w}\, m_\alpha\boldsymbol{w}\,C[f_\alpha] \doteq \boldsymbol{R}_\alpha} \quad \text{(force equation for species } \alpha) \tag{IX.2.9}$$

where

$$\boxed{\boldsymbol{P}_\alpha \doteq \int \mathrm{d}\boldsymbol{w}\, m_\alpha \boldsymbol{w}\boldsymbol{w} f_\alpha} \tag{IX.2.10}$$

is the thermal pressure tensor of species α and $\boldsymbol{R}_\alpha$ is the friction force on species α (recall Newton's third law, $\sum_\alpha \boldsymbol{R}_\alpha = 0$). Equation (IX.2.9) may of course be rewritten in the following, perhaps more familiar, form:

$$m_\alpha n_\alpha \frac{\mathrm{d}\boldsymbol{u}_\alpha}{\mathrm{d}t_\alpha} = m_\alpha n_\alpha \left(\frac{q_\alpha}{m_\alpha}\boldsymbol{E} + \boldsymbol{g}\right) - \boldsymbol{\nabla}\cdot\boldsymbol{P}_\alpha + \boldsymbol{R}_\alpha. \tag{IX.2.11}$$

If we sum (IX.2.11) over species, the electric-field term vanishes by quasineutrality, $\sum_\alpha q_\alpha n_\alpha = 0$. Then, defining the total mass density $\varrho \doteq \sum_\alpha m_\alpha n_\alpha$ and the mean center-of-mass velocity $\boldsymbol{u} \doteq \varrho^{-1} \sum_\alpha m_\alpha n_\alpha \boldsymbol{u}_\alpha$, equation (IX.2.11) implies

$$\varrho\biggl(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\biggr)\boldsymbol{u} = \varrho\boldsymbol{g} - \boldsymbol{\nabla}\cdot\bigl(\boldsymbol{P} + \boldsymbol{D}\bigr), \tag{IX.2.12}$$

where $\boldsymbol{P} \doteq \sum_\alpha \boldsymbol{P}_\alpha$ is the total pressure tensor and

$$\boldsymbol{D} \doteq \sum_\alpha m_\alpha n_\alpha \Delta\boldsymbol{u}_\alpha \Delta\boldsymbol{u}_\alpha \tag{IX.2.13}$$

is a tensor composed of species drifts relative to the center-of-mass velocity,

$$\Delta\boldsymbol{u}_\alpha \doteq \boldsymbol{u}_\alpha - \boldsymbol{u}. \tag{IX.2.14}$$

(Note that $\sum_\alpha m_\alpha n_\alpha \Delta\boldsymbol{u}_\alpha = 0$, by definition.) Returning to those moments...

$$\begin{aligned}\int \mathrm{d}\boldsymbol{w}\, m_\alpha w_i w_j(\text{IX.2.6}) :\quad & \frac{\mathrm{d}}{\mathrm{d}t_\alpha}\underbrace{\int \mathrm{d}\boldsymbol{w}\, m_\alpha w_i w_j}_{=P_{\alpha,ij}} + \int \mathrm{d}\boldsymbol{w}\, m_\alpha w_i w_j \boldsymbol{w}\cdot\boldsymbol{\nabla} f_\alpha \\ & + m_\alpha a_{\alpha,k} \cancelto{0}{\int \mathrm{d}\boldsymbol{w}\, w_i w_j \frac{\partial f_\alpha}{\partial w_k}} - \int \mathrm{d}\boldsymbol{w}\, m_\alpha w_i w_j (\boldsymbol{w}\cdot\boldsymbol{\nabla} u_{\alpha,\ell})\frac{\partial f_\alpha}{\partial w_\ell} \\ & = \int \mathrm{d}\boldsymbol{w}\, m_\alpha w_i w_j C[f_\alpha]. \end{aligned} \tag{IX.2.15}$$

Define the heat flux tensor for species α:

$$\boldsymbol{Q}_\alpha \doteq \int \mathrm{d}\boldsymbol{w}\, m_\alpha \boldsymbol{w}\boldsymbol{w}\boldsymbol{w} f_\alpha. \tag{IX.2.16}$$

Then, equation (IX.2.15) becomes, after integrating by parts the final term on its left-hand side,

$$\boxed{\frac{\mathrm{d}P_{\alpha,ij}}{\mathrm{d}t_\alpha} + (\boldsymbol{\nabla}\cdot\boldsymbol{Q}_\alpha)_{ij} + \bigl(\delta_{i\ell}P_{\alpha,jk} + \delta_{j\ell}P_{\alpha,ik} + \delta_{k\ell}P_{\alpha,ij}\bigr)\frac{\partial u_{\alpha,\ell}}{\partial r_k} = \int \mathrm{d}\boldsymbol{w}\, m_\alpha w_i w_j C[f_\alpha]} \tag{IX.2.17}$$

Usually the trace of this equation is taken, with

$$p_\alpha \doteq \frac{1}{3}\mathrm{tr}\,\boldsymbol{P}_\alpha. \tag{IX.2.18}$$

Then (IX.2.17) provides an evolutionary equation for the internal energy:

$$\frac{3}{2}\frac{\mathrm{d}p_\alpha}{\mathrm{d}t_\alpha} + \boldsymbol{\nabla}\cdot\boldsymbol{q}_\alpha + \frac{3}{2}p_\alpha\boldsymbol{\nabla}\cdot\boldsymbol{u}_\alpha + \boldsymbol{P}_\alpha : \boldsymbol{\nabla}\boldsymbol{u}_\alpha = Q_\alpha, \tag{IX.2.19}$$

where

$$\boldsymbol{q}_\alpha \doteq \int \mathrm{d}\boldsymbol{w}\, \frac{1}{2} m_\alpha w^2 \boldsymbol{w} f_\alpha \tag{IX.2.20}$$

is the conductive heat flux of species α and

$$Q_\alpha \doteq \int \mathrm{d}\boldsymbol{w}\, \frac{1}{2} m_\alpha w^2 C[f_\alpha] \tag{IX.2.21}$$

is the collisional energy exchange. Further writing

$$\boxed{\boldsymbol{P}_\alpha \doteq p_\alpha \boldsymbol{I} + \boldsymbol{\Pi}_\alpha,} \tag{IX.2.22}$$

where $\boldsymbol{\mathit{\Pi}}_\alpha$ is the viscous stress tensor of species α and using (IX.2.8) to replace $\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{u}_\alpha$ in (IX.2.19) by $\mathrm{d}\ln n_\alpha/\mathrm{d}t$, the internal energy equation (IX.2.19) provides an equation for the hydrodynamic entropy:

$$\boxed{\frac{3}{2}p_\alpha\frac{\mathrm{d}}{\mathrm{d}t_\alpha}\ln\frac{p_\alpha}{n_\alpha^{5/3}} = -\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{q}_\alpha - \boldsymbol{\mathit{\Pi}}_\alpha\boldsymbol{:}\boldsymbol{\nabla}\boldsymbol{u}_\alpha + Q_\alpha} \tag{IX.2.23}$$

Note that, in the absence of conductive heat fluxes, viscous stresses, and energy exchange amongst species, the hydrodynamic entropy of a fluid element is conserved (as it should be). Finally, using (IX.2.22), the force equation (IX.2.11) becomes

$$m_\alpha n_\alpha\frac{\mathrm{d}\boldsymbol{u}_\alpha}{\mathrm{d}t_\alpha} = m_\alpha n_\alpha\left(\frac{q_\alpha}{m_\alpha}\boldsymbol{E}+\boldsymbol{g}\right) - \boldsymbol{\nabla}p_\alpha - \boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\mathit{\Pi}}_\alpha + \boldsymbol{R}_\alpha. \tag{IX.2.24}$$

Clearly, to close the system of hydrodynamic equations (viz., (IX.2.8), (IX.2.23), and (IX.2.24)), we require $(\boldsymbol{\mathit{\Pi}}_\alpha, \boldsymbol{q}_\alpha, \boldsymbol{R}_\alpha, Q_\alpha)$ expressed in terms of the lower "fluid" moments $(n_\alpha, \boldsymbol{u}_\alpha, p_\alpha)$. This is the purpose of the Chapman–Enskog expansion, which is only possible when the collisional mean free path is much smaller than the lengthscales of interest (e.g., gradient scales) so that the distribution function f_α is nearly Maxwellian. This will give a tractable kinetic equation, without time variation, which will close the moment equations and allow evolution on a slow timescale. Let us proceed.

We adopt the ordering

$$\mathrm{Kn} \doteq \frac{\lambda_{\mathrm{mfp}}}{L} \doteq \epsilon \ll 1, \tag{IX.2.25}$$

which defines the Knudsen number Kn in terms of the mean free path and the characteristic gradient lengthscale L, and group the plasma timescales in the (extended) Bogoliubov hierarchy (§I) as follows:

$$\underset{\begin{pmatrix}\text{plasma freq.,}\\ \text{collision freq.}\end{pmatrix}}{\omega_{\mathrm{p}},\ \nu} \quad\gg\quad \underset{\begin{pmatrix}\text{sound-crossing}\\ \text{time}\end{pmatrix}^{-1}}{v_{\mathrm{th}}/L} \quad\gg\quad \underset{\begin{pmatrix}\text{diffusion}\\ \text{time}\end{pmatrix}^{-1}}{D/L^2}, \tag{IX.2.26}$$

where "$\gg$" means $\sim\epsilon^{-1}$. A few things to note before proceeding:

(1) ω_{p}^{-1} and ν^{-1} (denoted "t_0" below) are lumped together as the smallest timescales in the problem. Therefore, if you want to tell the difference between these scales, you'd have to do a subsidiary expansion in $\mathit{\Lambda}^{-1}$. But we don't, which will mean that Debye clouds are established instantaneously, and velocity-space irreversibility emerges instantaneously as well (even though $\nu^{-1}\sim\omega_{\mathrm{p}}^{-1}\mathit{\Lambda}/\ln\mathit{\Lambda}\gg\omega_{\mathrm{p}}^{-1}$). "Instantaneous" times a factor $\mathit{\Lambda}/\ln\mathit{\Lambda}$ is still considered "instantaneous".

(2) The timescale $L/v_{\mathrm{th}}\sim$ "t_1" (see below) is a non-dissipative timescale, and captures sound waves and the emergence of macroscopic force balance. This timescale is taken to be $\sim t_0/\epsilon$, since $t_1\sim L/v_{\mathrm{th}}\sim(L/\lambda_{\mathrm{mfp}})(\lambda_{\mathrm{mfp}}/v_{\mathrm{th}})\sim t_0/\epsilon$.

(3) The timescale $L^2/D\sim$ "t_2" (see below) captures processes leading to spatial diffusion. In a hydrodynamic plasma, $t_2\sim L^2/D\sim(L/\lambda_{\mathrm{mfp}})^2\nu^{-1}\sim t_0/\epsilon^2$.

Thus,

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_0} + \epsilon\frac{\partial}{\partial t_1} + \epsilon^2\frac{\partial}{\partial t_2}$$

is how time variations are measured – on three disparate timescales. The Chapman–Enskog expansion is a *multi-scale analysis* of the kinetic equation, which focuses on hydrodynamic (rather than kinetic-Vlasov) dynamics. We write

$$f_\alpha = f_0 + \epsilon f_1 + \epsilon^2 f_2 + \dots \tag{IX.2.27a}$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_0} + \epsilon \frac{\partial}{\partial t_1} + \epsilon^2 \frac{\partial}{\partial t_2} + \dots \tag{IX.2.27b}$$

and take the forces $q_\alpha \boldsymbol{E}$ and $m_\alpha \boldsymbol{g}$ to be $\mathcal{O}(\epsilon)$ – that is, they drive evolution on the hydrodynamic (i.e., sound-crossing) timescale. (This says that the electrostatic potential φ satisfies $e\varphi/T \lesssim 1$ and that the gravitational potential Φ satisfies $m\Phi/T \lesssim 1$.) Then, equation (IX.2.1) becomes

$$\boxed{\left(\frac{\mathrm{D}}{\mathrm{D}t_0} + \epsilon \frac{\mathrm{D}}{\mathrm{D}t_1} + \epsilon^2 \frac{\mathrm{D}}{\mathrm{D}t_2}\right)\left(f_0 + \epsilon f_1 + \epsilon^2 f_2 + \dots\right) = C\big[f_0 + \epsilon f_1 + \epsilon^2 f_2 + \dots\big]} \tag{IX.2.28}$$

Equation (IX.2.28) is analyzed order by order.

At zeroth order in ϵ, equation (IX.2.28) is simply

$$\frac{\mathrm{D}f_0}{\mathrm{D}t_0} = C[f_0]. \tag{IX.2.29}$$

This contains kinetic-scale physics that is "instantaneous". Thus, we drop the $\mathrm{D}/\mathrm{D}t_0$ term and obtain $C[f_0] = 0$. This states that the lowest-order distribution is a local Maxwellian that is stationary on the t_0 timescale:

$$f_0(t, \boldsymbol{x}) = \frac{\widetilde{n}(t, \boldsymbol{r})}{\pi^{3/2} \widetilde{v_{\mathrm{th}}}^3(t, \boldsymbol{r})} \exp\left[-\frac{|\boldsymbol{v} - \widetilde{\boldsymbol{u}}(t, \boldsymbol{r})|^2}{\widetilde{v_{\mathrm{th}}}^2(t, \boldsymbol{r})}\right], \tag{IX.2.30}$$

where $\widetilde{v_{\mathrm{th}}}^2(t, \boldsymbol{r}) \equiv 2\widetilde{T}(t, \boldsymbol{r})/m$ and the species label α has been dropped for economy of notation (it will be reintroduced when necessary). The reason for the tildes on $\widetilde{n}$, $\widetilde{\boldsymbol{u}}$, and $\widetilde{T}$ is that we only know that f_0 *looks* like a Maxwellian, *not* that these parameters correspond the *the* density, mean velocity, and temperature of the plasma. But, in fact, we have the freedom to set $\widetilde{n} = n$, $\widetilde{\boldsymbol{u}} = \boldsymbol{u}$, and $\widetilde{T} = T$. To prove that, look at the $\mathcal{O}(\epsilon)$ terms:

$$\frac{\mathrm{D}f_0}{\mathrm{D}t_1} = C[f_0, f_1] + C[f_1, f_0] \equiv \widehat{C} f_1. \tag{IX.2.31}$$

(Recall that the collision operator is bi-linear in its argument. The $\mathcal{O}(\epsilon)$ contributions correspond physically to a particle in f_0 colliding with a particle in f_1 and vice versa.) The solution to (IX.2.31) is

$$f_1 = \widehat{C}^{-1}\left[\frac{\mathrm{D}f_0}{\mathrm{D}t_1}\right], \tag{IX.2.32}$$

provided that we can invert the collision operator. Before discussing how to do that, note that f_1 can contain anything looking like a Maxwellian and the equation $\mathrm{D}f_0/\mathrm{D}t_1 = \widehat{C} f_1$ would not change. Put differently, the solution to $\widehat{C} f_1 = 0$ doesn't have to be $f_1 = 0$; it could be any additive combination of Maxwellians. (Mathematically, f_1 could be expanded in terms of the null eigenfunctions of $\widehat{C}$, which are just the basis functions for density, momentum, and temperature perturbations.) In other words, f_1 could simply adjust $\widetilde{n}$ until it's n, and likewise for $\widetilde{\boldsymbol{u}}$ and $\widetilde{T}$. By *choosing* $f_1 = 0$ as a solution to $\widehat{C} f_1 = 0$, we are packing all the information about the density, momentum, and temperature into f_0. In math speak, "$f_{n\geqslant 1}$ are constructed to be orthogonal to the hydrodynamic subspace" Krommes (2017). This affords the freedom to associated $\widetilde{n}$ with n, $\widetilde{\boldsymbol{u}}$ with $\boldsymbol{u}$, and $\widetilde{T}$ with

T. Thus,

$$\boxed{f_0(t,\boldsymbol{x}) = \frac{n(t,\boldsymbol{r})}{\pi^{3/2}v_{\rm th}^3(t,\boldsymbol{r})}\exp\left[-\frac{|\boldsymbol{v}-\boldsymbol{u}(t,\boldsymbol{r})|^2}{v_{\rm th}^2(t,\boldsymbol{r})}\right]} \tag{IX.2.33}$$

with n, $\boldsymbol{u}$, and T being governed by the fluid equations.

With f_0 given by (IX.2.33), we substitute it into (IX.2.31) and solve for f_1. To do so, first note that

$$\ln f_0 = \text{const} + \ln n - \frac{3}{2}\ln T - \frac{w^2}{v_{\rm th}^2}.$$

Then, with

$$\frac{\mathrm{D}}{\mathrm{D}t_1} = \underbrace{\frac{\partial}{\partial t_1} + \boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{\nabla}}_{\doteq\, \mathrm{d}/\mathrm{d}t_1} + \boldsymbol{w}\boldsymbol{\cdot}\boldsymbol{\nabla} + \bigl(\boldsymbol{a}_1 - \boldsymbol{w}\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{u}\bigr)\boldsymbol{\cdot}\frac{\partial}{\partial\boldsymbol{w}},$$

we have

$$\begin{aligned}\frac{\mathrm{D}\ln f_0}{\mathrm{D}t_1} &= \left(\frac{\mathrm{d}}{\mathrm{d}t_1} + \boldsymbol{w}\boldsymbol{\cdot}\boldsymbol{\nabla}\right)\ln n + \left(\frac{w^2}{v_{\rm th}^2} - \frac{3}{2}\right)\left(\frac{\mathrm{d}}{\mathrm{d}t_1} + \boldsymbol{w}\boldsymbol{\cdot}\boldsymbol{\nabla}\right)\ln T \\ &\quad + \bigl(\boldsymbol{a}_1 - \boldsymbol{w}\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{u}\bigr)\boldsymbol{\cdot}\left(-\frac{2\boldsymbol{w}}{v_{\rm th}^2}\right).\end{aligned} \tag{IX.2.34}$$

The quantities $\mathrm{d}\ln n/\mathrm{d}t_1$ and $\mathrm{d}\ln T/\mathrm{d}t_1$ can be obtained from the moment equations, appropriately ordered. At this point, the calculation could go in a few different directions. If we are interested in a *one-component plasma* (or a multi-component plasma whose species never interact with one another), then we know that the friction force $\boldsymbol{R} = 0$ and the energy exchange $Q = 0$, because the collision operator conserves momentum and energy. The moment equations to $\mathcal{O}(\epsilon)$ are then

$$\frac{\mathrm{d}\ln n}{\mathrm{d}t_1} = -\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{u}, \quad \frac{\mathrm{d}\ln T}{\mathrm{d}t_1} = -\frac{2}{3}\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{u}, \quad \boldsymbol{a}_1 = \frac{\boldsymbol{\nabla}p}{mn}. \tag{IX.2.35}$$

(We will explore the situation for an interacting multi-component plasma in §IX.3.) Substituting (IX.2.35) into (IX.2.34), we find

$$\frac{\mathrm{D}\ln f_0}{\mathrm{D}t_1} = \left(\frac{w^2}{v_{\rm th}^2} - \frac{5}{2}\right)\boldsymbol{w}\boldsymbol{\cdot}\boldsymbol{\nabla}\ln T + \left(\frac{\boldsymbol{w}\boldsymbol{w}}{v_{\rm th}^2} - \frac{\boldsymbol{I}}{3}\frac{w^2}{v_{\rm th}^2}\right)\boldsymbol{:}\Bigl[\boldsymbol{\nabla}\boldsymbol{u} + (\boldsymbol{\nabla}\boldsymbol{u})^T\Bigr],$$

where T denotes the transpose. Using this in (IX.2.31) leads to the *correction equation*

$$\boxed{\frac{\widehat{C}f_1}{f_0} = \left(\frac{w^2}{v_{\rm th}^2} - \frac{5}{2}\right)\boldsymbol{w}\boldsymbol{\cdot}\boldsymbol{\nabla}\ln T + \left(\frac{\boldsymbol{w}\boldsymbol{w}}{v_{\rm th}^2} - \frac{\boldsymbol{I}}{3}\frac{w^2}{v_{\rm th}^2}\right)\boldsymbol{:}\boldsymbol{W}} \tag{IX.2.36}$$

where the rate-of-strain tensor

$$\boldsymbol{W} \doteq \boldsymbol{\nabla}\boldsymbol{u} + (\boldsymbol{\nabla}\boldsymbol{u})^T - \frac{2}{3}(\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{u})\boldsymbol{I}. \tag{IX.2.37}$$

Note: If we were interested in a multi-species plasma of, say, ions and electrons, then we would have to decide how quickly their f_0 equilibrate. For example, how long does it take for $\boldsymbol{u}_e$ to become arbitrarily close to $\boldsymbol{u}_i$? How long does it take for T_e to become arbitrarily close to T_i? Depending on these choices (which really amount of a choice of mass ratio – see §IX.3), the friction force $\boldsymbol{R}$ and the energy exchange rate Q might appear in the correction equation.

For now, since it's our first time through this, let's suppose that we have a one-component plasma (OCP). The solution to (IX.2.36) is "simply"

$$f_1 = \widehat{C}^{-1}\left\{f_0\left[\left(\frac{w^2}{v_{\rm th}^2} - \frac{5}{2}\right)\boldsymbol{w}\cdot\boldsymbol{\nabla}\ln T + \left(\frac{\boldsymbol{ww}}{v_{\rm th}^2} - \frac{\mathsf{I}}{3}\frac{w^2}{v_{\rm th}^2}\right):\mathsf{W}\right]\right\}. \tag{IX.2.38}$$

Just as in the Spitzer–Härm problem (§IX.1), we must invert the collision operator. As a first pass, let's take $C[f] = -\nu(f - f_0)$, the Krook operator. Then

$$f_1 = -\frac{f_0}{\nu}\left[\left(\frac{w^2}{v_{\rm th}^2} - \frac{5}{2}\right)\boldsymbol{w}\cdot\boldsymbol{\nabla}\ln T + \left(\frac{\boldsymbol{ww}}{v_{\rm th}^2} - \frac{\mathsf{I}}{3}\frac{w^2}{v_{\rm th}^2}\right):\mathsf{W}\right]$$

and so

$$\boxed{\begin{aligned}\boldsymbol{\mathit{\Pi}} &\doteq \int {\rm d}\boldsymbol{w}\, m\left(\boldsymbol{ww} - \frac{\mathsf{I}}{3}w^2\right)f_1 = -\frac{p}{\nu}\mathsf{W},\\ \boldsymbol{q} &\doteq \int {\rm d}\boldsymbol{w}\,\frac{1}{2}mw^2\boldsymbol{w}f_1 = -\frac{5}{2}\frac{p}{\nu}\boldsymbol{\nabla}\frac{p}{mn} = -\frac{5}{4}\frac{nv_{\rm th}^2}{\nu}\boldsymbol{\nabla}T\end{aligned}} \tag{IX.2.39}$$

Easy! For the full Landau collision operator, not so easy. But you get the idea. For the general case, write

$$\boxed{\boldsymbol{\mathit{\Pi}} = -mn\mu\mathsf{W}, \quad \boldsymbol{q} = -n\kappa\boldsymbol{\nabla}T} \tag{IX.2.40}$$

with a viscous coefficient μ and heat diffusion coefficient κ (both $\propto v_{\rm th}^2/\nu$).

For your benefit, I include here the details of the calculations that lead to (IX.2.39). First, the heat flux $\boldsymbol{q}$. Orient $\boldsymbol{\nabla}\ln T$ along the z direction and write $\boldsymbol{w}$ in spherical coordinates: $\boldsymbol{w} = w\xi\hat{\boldsymbol{z}} + \sqrt{1-\xi^2}(\cos\vartheta\hat{\boldsymbol{x}} + \sin\vartheta\hat{\boldsymbol{y}})$. Then

$$\begin{aligned}\boldsymbol{q} &\equiv \int {\rm d}\boldsymbol{w}\,\frac{1}{2}mw^2\boldsymbol{w}f_1\\ &= -\frac{1}{\nu}\frac{{\rm d}\ln T}{{\rm d}z}\int_0^\infty {\rm d}w\,w^2\int_{-1}^{+1}{\rm d}\xi\int_0^{2\pi}{\rm d}\vartheta\,\frac{1}{2}mw^2\boldsymbol{w}\left(\frac{w^2}{v_{\rm th}^2} - \frac{5}{2}\right)w\xi f_0\\ &= -\frac{1}{\nu}\frac{{\rm d}\ln T}{{\rm d}z}\int_0^\infty {\rm d}w\,w^2\int_{-1}^{+1}{\rm d}\xi\,2\pi\,\frac{1}{2}mw^2w\xi\left(\frac{w^2}{v_{\rm th}^2} - \frac{5}{2}\right)w\xi\hat{\boldsymbol{z}}\,f_0 \quad \text{(doing } \vartheta \text{ integral)}\\ &= -\frac{mn}{\nu}\boldsymbol{\nabla}\ln T\,\frac{v_{\rm th}^4}{\sqrt{\pi}}\underbrace{\int_0^\infty \frac{{\rm d}w\,w^6}{v_{\rm th}^7}\left(\frac{w^2}{v_{\rm th}^2} - \frac{5}{2}\right){\rm e}^{-w^2/v_{\rm th}^2}}_{\substack{=\frac{1}{2}\int_0^\infty {\rm d}x\,x^{5/2}\left(x - \frac{5}{2}\right){\rm e}^{-x}\\ =\frac{1}{2}\left[\Gamma\left(\frac{9}{2}\right) - \frac{5}{2}\Gamma\left(\frac{7}{2}\right)\right]\\ =\frac{\sqrt{\pi}}{2}\left(\frac{105}{16} - \frac{5}{2}\frac{15}{8}\right) = \frac{15}{16}\sqrt{\pi}}}\underbrace{\int_{-1}^{+1}{\rm d}\xi\,\xi^2}_{=\frac{2}{3}} \quad \text{(inserting } f_0\text{, rearranging)}\\ &= -\frac{mn}{\nu}\boldsymbol{\nabla}\ln T\,\frac{5}{8}v_{\rm th}^4 = -\frac{5}{4}\frac{nv_{\rm th}^2}{\nu}\boldsymbol{\nabla}T.\end{aligned}$$

Next, the viscous stress $\boldsymbol{\mathit{\Pi}}$:

$$\boldsymbol{\mathit{\Pi}} \equiv \int {\rm d}\boldsymbol{w}\,m\left(\boldsymbol{ww} - \frac{\mathsf{I}}{3}w^2\right)f_1 = -\frac{1}{\nu}\int {\rm d}\boldsymbol{w}\,m\left(\boldsymbol{ww} - \frac{\mathsf{I}}{3}w^2\right)\left(\frac{\boldsymbol{ww}}{v_{\rm th}^2} - \frac{\mathsf{I}}{3}\frac{w^2}{v_{\rm th}^2}\right):\mathsf{W}\,f_0.$$

Before going any further, two simplifications can be made. First, note that $\mathsf{I}:\mathsf{W} = {\rm tr}(\mathsf{W}) = 0$.

Second, with $\boldsymbol{w}$ written in spherical coordinates as above,

$$\int_{-1}^{+1} \mathrm{d}\xi \int_0^{2\pi} \mathrm{d}\vartheta\, \boldsymbol{w}\boldsymbol{w} = \frac{4\pi}{3} w^2 \mathsfbfit{I} \quad \Longrightarrow \quad \int \mathrm{d}\boldsymbol{w}\, \boldsymbol{w}\boldsymbol{w} \,\boldsymbol{:}\, \mathsfbfit{W} \propto \mathsfbfit{I} \,\boldsymbol{:}\, \mathsfbfit{W} = 0.$$

Therefore,

$$\boldsymbol{\mathit{\Pi}} = -\frac{1}{\nu} \int \mathrm{d}\boldsymbol{w}\, m\boldsymbol{w}\boldsymbol{w}\, \frac{\boldsymbol{w}\boldsymbol{w}}{v_{\mathrm{th}}^2} \,\boldsymbol{:}\, \mathsfbfit{W} f_0 = -\frac{2T}{\nu} \int \mathrm{d}\boldsymbol{w}\, \frac{\boldsymbol{w}\boldsymbol{w}\boldsymbol{w}\boldsymbol{w}}{v_{\mathrm{th}}^4} \,\boldsymbol{:}\, \mathsfbfit{W} f_0 .$$

So we must compute $\int \mathrm{d}\xi \int \mathrm{d}\vartheta\, \boldsymbol{w}\boldsymbol{w}\boldsymbol{w}\boldsymbol{w}$. Writing $w_\parallel^2 = w^2\xi^2$ and $w_\perp^2 = w^2(1-\xi^2)$ makes things a bit more compact after performing the ϑ integral, which leaves

$$\begin{aligned}
2\pi \int_{-1}^{+1} \mathrm{d}\xi \biggl\{ & w_\parallel^4 \hat{\boldsymbol{z}}\hat{\boldsymbol{z}}\hat{\boldsymbol{z}}\hat{\boldsymbol{z}} + \frac{3}{8} w_\perp^4 \bigl(\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}}\hat{\boldsymbol{y}}\hat{\boldsymbol{y}}\hat{\boldsymbol{y}}\bigr) \\
& + \frac{w_\perp^2}{2} \left[w_\parallel^2 \bigl(\hat{\boldsymbol{z}}\hat{\boldsymbol{z}}\hat{\boldsymbol{x}}\hat{\boldsymbol{x}} + \hat{\boldsymbol{x}}\hat{\boldsymbol{x}}\hat{\boldsymbol{z}}\hat{\boldsymbol{z}} + \hat{\boldsymbol{z}}\hat{\boldsymbol{z}}\hat{\boldsymbol{y}}\hat{\boldsymbol{y}} + \hat{\boldsymbol{y}}\hat{\boldsymbol{y}}\hat{\boldsymbol{z}}\hat{\boldsymbol{z}}\bigr) + \frac{w_\perp^2}{4} \bigl(\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}\hat{\boldsymbol{y}}\hat{\boldsymbol{y}} + \hat{\boldsymbol{y}}\hat{\boldsymbol{y}}\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}\bigr) \right] \\
& + \frac{w_\perp^2}{2} \left[w_\parallel^2 \bigl(\hat{\boldsymbol{z}}\hat{\boldsymbol{x}} + \hat{\boldsymbol{x}}\hat{\boldsymbol{z}}\bigr)\bigl(\hat{\boldsymbol{z}}\hat{\boldsymbol{x}} + \hat{\boldsymbol{x}}\hat{\boldsymbol{z}}\bigr) + w_\parallel^2 \bigl(\hat{\boldsymbol{z}}\hat{\boldsymbol{y}} + \hat{\boldsymbol{y}}\hat{\boldsymbol{z}}\bigr)\bigl(\hat{\boldsymbol{z}}\hat{\boldsymbol{y}} + \hat{\boldsymbol{y}}\hat{\boldsymbol{z}}\bigr) + \frac{w_\perp^2}{4} \bigl(\hat{\boldsymbol{x}}\hat{\boldsymbol{y}} + \hat{\boldsymbol{y}}\hat{\boldsymbol{x}}\bigr)\bigl(\hat{\boldsymbol{x}}\hat{\boldsymbol{y}} + \hat{\boldsymbol{y}}\hat{\boldsymbol{x}}\bigr) \right] \biggr\}.
\end{aligned}$$

To perform the ξ integration, note that

$$\int_{-1}^{+1} \mathrm{d}\xi\, \xi^4 = \frac{2}{5}, \quad \int_{-1}^{+1} \mathrm{d}\xi\, (1-\xi^2)^2 = \frac{16}{15}, \quad \int_{-1}^{+1} \mathrm{d}\xi\, \xi^2(1-\xi^2) = \frac{4}{15}.$$

Then,

$$\begin{aligned}
\int_{-1}^{+1} \mathrm{d}\xi \int_0^{2\pi} \mathrm{d}\vartheta\, \boldsymbol{w}\boldsymbol{w}\boldsymbol{w}\boldsymbol{w} = 2\pi w^4 \times \frac{2}{15} \Bigl[& 2\bigl(\hat{\boldsymbol{z}}\hat{\boldsymbol{z}}\hat{\boldsymbol{z}}\hat{\boldsymbol{z}} + \hat{\boldsymbol{x}}\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}}\hat{\boldsymbol{y}}\hat{\boldsymbol{y}}\hat{\boldsymbol{y}}\bigr) + \mathsfbfit{I}\mathsfbfit{I} + \bigl(\hat{\boldsymbol{z}}\hat{\boldsymbol{x}} + \hat{\boldsymbol{x}}\hat{\boldsymbol{z}}\bigr)\bigl(\hat{\boldsymbol{z}}\hat{\boldsymbol{x}} + \hat{\boldsymbol{x}}\hat{\boldsymbol{z}}\bigr) \\
& + \bigl(\hat{\boldsymbol{z}}\hat{\boldsymbol{y}} + \hat{\boldsymbol{y}}\hat{\boldsymbol{z}}\bigr)\bigl(\hat{\boldsymbol{z}}\hat{\boldsymbol{y}} + \hat{\boldsymbol{y}}\hat{\boldsymbol{z}}\bigr) + \bigl(\hat{\boldsymbol{x}}\hat{\boldsymbol{y}} + \hat{\boldsymbol{y}}\hat{\boldsymbol{x}}\bigr)\bigl(\hat{\boldsymbol{x}}\hat{\boldsymbol{y}} + \hat{\boldsymbol{y}}\hat{\boldsymbol{x}}\bigr) \Bigr]
\end{aligned}$$

Inserting this expression into the equation for the viscous stress, and using the facts that $\mathsfbfit{W}$ is symmetric and $\mathsfbfit{I} \,\boldsymbol{:}\, \mathsfbfit{W} = 0$, gives

$$\boldsymbol{\mathit{\Pi}} = -\frac{2T}{\nu}\frac{8\pi}{15} \mathsfbfit{W} \int_0^\infty \mathrm{d}w\, w^2 \frac{w^4}{v_{\mathrm{th}}^4} f_0 = -\frac{p}{\nu} \mathsfbfit{W} \frac{16}{15\sqrt{\pi}} \underbrace{\int_0^\infty \frac{\mathrm{d}w\, w^6}{v_{\mathrm{th}}^7} \mathrm{e}^{-w^2/v_{\mathrm{th}}^2}}_{= \frac{1}{2}\Gamma\left(\frac{7}{2}\right) = \frac{15}{16}\sqrt{\pi}} = -\frac{p}{\nu} \mathsfbfit{W}.$$

Voila. Now, back to the expansion...

At $\mathcal{O}(\epsilon^2)$, equation (IX.2.28) is

$$\frac{\mathrm{D}f_0}{\mathrm{D}t_2} + \frac{\mathrm{D}f_1}{\mathrm{D}t_1} = C[f_0, f_2] + C[f_2, f_0] + C[f_1, f_1] \doteq \widehat{C} f_2 + C[f_1, f_1]. \qquad \text{(IX.2.41)}$$

This is called the "solvability condition"; we must ask whether this equation has a solution for f_2. To do so, take the following moments:

$$\int \mathrm{d}\boldsymbol{w}\,(\text{IX.2.41}) \quad\Longrightarrow\quad \int \mathrm{d}\boldsymbol{w} \left(\frac{\mathrm{D}f_0}{\mathrm{D}t_2} + \frac{\mathrm{D}f_1}{\mathrm{D}t_1}\right) = 0 \quad (C \text{ conserves number}) \tag{IX.2.42a}$$

$$\int \mathrm{d}\boldsymbol{w}\,\boldsymbol{v}(\text{IX.2.41}) \quad\Longrightarrow\quad \int \mathrm{d}\boldsymbol{w}\,\boldsymbol{v} \left(\frac{\mathrm{D}f_0}{\mathrm{D}t_2} + \frac{\mathrm{D}f_1}{\mathrm{D}t_1}\right) = 0 \quad (C \text{ conserves momentum}) \tag{IX.2.42b}$$

$$\int \mathrm{d}\boldsymbol{w}\,w^2(\text{IX.2.41}) \quad\Longrightarrow\quad \int \mathrm{d}\boldsymbol{w}\,w^2 \left(\frac{\mathrm{D}f_0}{\mathrm{D}t_2} + \frac{\mathrm{D}f_1}{\mathrm{D}t_1}\right) = 0 \quad (C \text{ conserves energy}) \tag{IX.2.42c}$$

Now,

$$\int \mathrm{d}\boldsymbol{w}\,(1, \boldsymbol{w}, w^2)\frac{\mathrm{d}f_1}{\mathrm{d}t_1} = \frac{\mathrm{d}}{\mathrm{d}t_1}\int \mathrm{d}\boldsymbol{w}\,(1, \boldsymbol{w}, w^2) f_1 = 0,$$

and

$$\begin{aligned}
&\int \mathrm{d}\boldsymbol{w}\,(1, \boldsymbol{w}, w^2)(\boldsymbol{a}_1 - \boldsymbol{w}\cdot\boldsymbol{\nabla}\boldsymbol{u})\cdot\frac{\partial f_1}{\partial \boldsymbol{w}} \\
&\overset{\text{bp}}{=} -\int \mathrm{d}\boldsymbol{w}\,(0, \boldsymbol{I}, 2\boldsymbol{w})\cdot\boldsymbol{a}_1 f_1 + \int \mathrm{d}\boldsymbol{w}\,(\boldsymbol{\nabla}\cdot\boldsymbol{u}, \boldsymbol{w}\boldsymbol{\nabla}\cdot\boldsymbol{u} + \boldsymbol{w}\cdot\boldsymbol{\nabla}\boldsymbol{u}, w^2\boldsymbol{\nabla}\cdot\boldsymbol{u} + 2\boldsymbol{w}\boldsymbol{w}:\boldsymbol{\nabla}\boldsymbol{u}) f_1 \\
&= 0.
\end{aligned}$$

So (IX.2.42) becomes

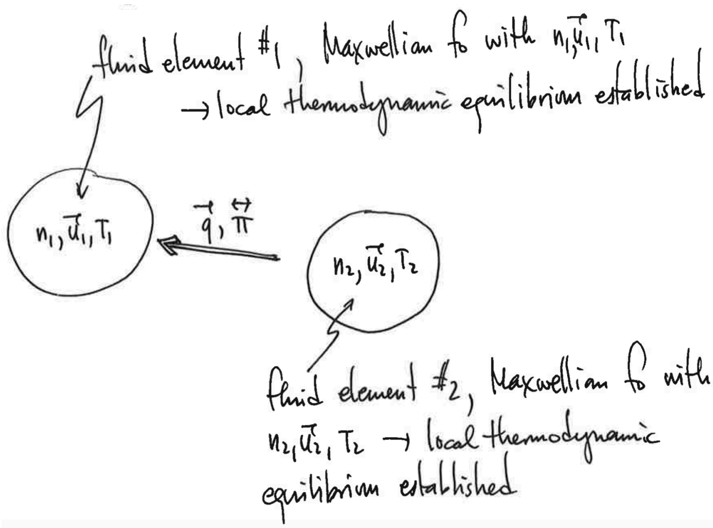
$$\underbrace{\int \mathrm{d}\boldsymbol{w}\,(1, \boldsymbol{w}, w^2)\left(\frac{\mathrm{D}f_0}{\mathrm{D}t}\right)}_{\substack{\text{hydrodynamic quantities} \\ \text{evolve without dissipation}}} = -\underbrace{\int \mathrm{d}\boldsymbol{w}\,(1, \boldsymbol{w}, w^2)\boldsymbol{w}\cdot\boldsymbol{\nabla}f_1}_{\substack{\text{dissipative fluxes} \\ = -\boldsymbol{\nabla}\cdot(0, \boldsymbol{\Pi}/m, 2\boldsymbol{q}/m)}}\,. \tag{IX.2.43}$$

The solvability condition (IX.2.41) therefore returns the transport equations! Good. From Krommes (2018), §23.2.7:

> Thus, in this kind of asymptotic expansion, one obtains successively more and more information about lower-order quantities from solvability conditions at higher and higher order. This behavior is not unique to this problem; it occurs in virtually all asymptotic problems in which a large parameter like ϵ^{-1} determines the lowest-order physics. The behavior of charged particles in a strong magnetic field is another such problem, and some derivations of particle drifts and gyrokinetic equations follow essentially this same asymptotic route. It is an important technique with which you should be familiar.

One could, of course, do better than what is done in this section by inverting a more accurate collision operator. But the purpose here was just to show you how Chapman–Enskog works. Our real goal is the Braginskii equations for an ion-electron plasma, which is next. There is where we'll be more sophisticated with our collision operator.

Pictorial summary:

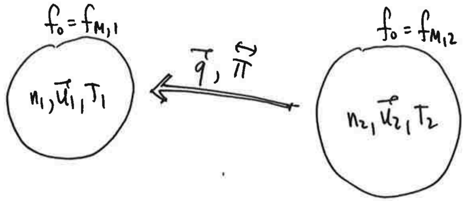


Different temperatures and velocities driven transport between these fluid elements, which is approximately linear in the gradients. This is connected with the slight departures from Maxwellian that result from the system trying to achieve global thermodynamic equilibrium by relaxing these gradients diffusively. What we'll see in the next section is that a strong magnetic field – "strong" meaning $\rho/\lambda_{\mathrm{mfp}} \ll 1$ – can impede this transport.

IX.3. The Braginskii-MHD equations

We now arrive at the crown jewel of classical transport: Braginskii's equations for a collisional, magnetized plasma. These are a great example of rewarding mathematics and rich physics, and we will explore both. Speaking of a good blend of rigorous presentation and pedagogical discussion, Braginskii's original article published in Rev. Mod. Phys. in 1965 is a must-read classic. Do yourself a favor and consult it alongside these notes. John Krommes also has an article on classical transport in a magnetized plasma, recently published in J. Plasma Phys., which employs a "modern" projection-operator technique to determine the transport equations. Same physics, different formalism... to each their own.

The basic idea can be seen by returning to the illustration at the end of the Chapman–Enskog notes:



If we put a magnetic field with $\rho/\lambda_{\mathrm{mfp}} \ll 1$ perpendicular to the white arrow, then those heat and momentum fluxes will be stifled since the particles cannot travel a collisional mean free path across the magnetic field. If we put a magnetic field along the white arrow, then heat and momentum fluxes can readily proceed since the particles can stream along the magnetic field unimpeded but for collisions. Thus, transport in a magnetized plasma is fundamentally *anisotropic*. Let's explore this, and other such effects, rigorously through a Chapman–Enskog expansion.

IX.4. Ordering of parameters

One difference between the following calculation and the Chapman–Enskog expansion detailed in §IX.2 is that we will focus on an ion-electron plasma, with collision rates ordered such that $f_{\rm i}$ and $f_{\rm e}$ are both Maxwellian to zeroth order. But, following Braginskii, we will exploit the smallness of $m_{\rm e}/m_{\rm i}$ to allow the Maxwellian parameters $\boldsymbol{u}_{\rm i}$ and $T_{\rm i}$ to be different than $\boldsymbol{u}_{\rm e}$ and $T_{\rm e}$, respectively. In Braginskii's words: "this feature makes it possible to obtain separate transport equations for the electrons and ions with different temperatures (and different velocities) and to uncouple the electron and ion kinetic equations."

There are three different small parameters in this problem:

$$\frac{\rho_{\rm i}}{L} \ll 1 \qquad \text{(the plasma is magnetized)}$$

$$\frac{\lambda_{\rm mfp}}{L} \ll 1 \qquad \text{(the plasma is collisional)}$$

$$\sqrt{\frac{m_{\rm e}}{m_{\rm i}}} \ll 1 \qquad \text{(temperature and mean velocity equilibration takes some time)}$$

We will take $T_{\rm i} \sim T_{\rm e}$ (i.e., ions and electrons have comparable temperatures, not differing by more than $\sim\sqrt{m_{\rm i}/m_{\rm e}}$) and a "high-flow" ordering:

$$\boldsymbol{u}_{\rm i} \sim \boldsymbol{u}_{\rm e} \sim v_{\rm thi} \sim \sqrt{\frac{m_{\rm e}}{m_{\rm i}}} v_{\rm the} \ll v_{\rm the}. \tag{IX.4.1}$$

"Low-flow" orderings with flow velocities of order the diamagnetic drift velocity introduce additional physics; see Catto & Simakov (2004), which follows Mikhailovskii & Tsypin (1971, 1984). Note that individual collisional mean free paths are all comparable,

$$\lambda_{\rm ee} \sim \lambda_{\rm ei} \sim \lambda_{\rm ii}, \tag{IX.4.2}$$

despite the collision frequencies being different: $\nu_{\rm ee} \sim \nu_{\rm ei} \sim \sqrt{m_{\rm i}/m_{\rm e}}\,\nu_{\rm ii}$.

Within the

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_0} + \epsilon \frac{\partial}{\partial t_1} + \epsilon^2 \frac{\partial}{\partial t_2} + \dots$$

timescale ordering in the Chapman–Enskog expansion, which we adopt here as well, the "t_1" dynamical timescale is $\sim v_{\rm thi}/L \sim \sqrt{m_e/m_i}\, v_{\rm the}/L$, while the "$t_0$" collisional timescale is $\sim\nu_{ee}^{-1} \sim \nu_{ei}^{-1}$. The timescale "$t_2$" again denotes the diffusion timescale. To simplify the derivation of the Braginskii equations, we use the ordering

$$\boxed{\frac{\rho_i}{L} \ll \frac{\lambda_{ii}}{L} \sim \sqrt{\frac{m_e}{m_i}} \ll 1} \tag{IX.4.3}$$

so that the plasma is collisional, but not so collisional that Larmor orbits are disturbed.

To achieve this, it is common to first expand using the orderings

$$\frac{\rho_i}{L} \sim \frac{\lambda_{ii}}{L} \sim \sqrt{\frac{m_e}{m_i}} \ll 1 \quad \text{and} \quad \frac{\rho_e}{L} \sim \frac{\lambda_{ee}}{L} \sim \frac{\lambda_{ei}}{L} \sim \sqrt{\frac{m_e}{m_i}} \ll 1, \tag{IX.4.4}$$

and *then* perform a subsidiary expansion to achieve undisrupted Larmor orbits:

$$\frac{\rho_i}{\lambda_{ii}} \sim \frac{\nu_{ii}}{\Omega_i} \ll 1 \sim \frac{L}{\lambda_{ii}}\sqrt{\frac{m_e}{m_i}} \quad \text{and} \quad \frac{\rho_e}{\lambda_{ee}} \sim \frac{\nu_{ee}}{\Omega_e} \ll 1 \sim \frac{L}{\lambda_{ee}}\sqrt{\frac{m_e}{m_i}}. \tag{IX.4.5}$$

Note: This isn't the only way of obtaining the Braginskii equations. One could first expand the kinetic equation in $\rho_i/L \ll 1$ with $\lambda_{ii}/L \sim 1$ to get drift kinetics (which becomes "kinetic MHD" in the high-flow ordering; Kulsrud (1964, 1983)) and then perform subsidiary expansions in $\lambda_{ii}/L \ll 1$ and $m_e/m_i \ll 1$.

What follows is a derivation of the Braginskii-MHD equations using the full Landau collision operator, just like Braginskii (1965). I admit that it is certainly not the most pedagogical exercise to work through Braginskii's calculation for the first time while retaining the full Landau operator. So if you find yourself getting lost, you should probably skip to the hand-written pages appended to these typed lecture notes, where simplified collision operators are used to perform the Braginskii–Chapman–Enskog procedure in a simpler fashion. It's the same general procedure, but the inversion of the collision operator there is a trivial task; here, it's more complicated, involving the projection of the collision operator onto a Laguerre–Legendre basis.

IX.5. Expansion of the kinetic equation

With the inclusion of the Lorentz force, the kinetic equation in $(t, \boldsymbol{r}, \boldsymbol{w} \doteq \boldsymbol{v} - \boldsymbol{u}_\alpha)$ variables for species α (see (IX.2.6)) is

$$\frac{\mathrm{d}f_\alpha}{\mathrm{d}t_\alpha} + \boldsymbol{w}\boldsymbol{\cdot}\boldsymbol{\nabla} f_\alpha + \left[\frac{q_\alpha}{m_\alpha}\left(\boldsymbol{E} + \frac{\boldsymbol{u}_\alpha \times \boldsymbol{B}}{c}\right) + \boldsymbol{g} - \frac{\mathrm{d}\boldsymbol{u}_\alpha}{\mathrm{d}t_\alpha} - \boldsymbol{w}\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{u}_\alpha\right]\boldsymbol{\cdot}\frac{\partial f_\alpha}{\partial \boldsymbol{w}} = -\frac{q_\alpha}{m_\alpha}\frac{\boldsymbol{w}\times\boldsymbol{B}}{c}\boldsymbol{\cdot}\frac{\partial f_\alpha}{\partial \boldsymbol{w}} + C[f_\alpha]. \tag{IX.5.1}$$

(Recall that $\mathrm{d}/\mathrm{d}t_\alpha \doteq \partial/\partial t_\alpha + \boldsymbol{u}_\alpha\boldsymbol{\cdot}\boldsymbol{\nabla}$, with the temporal and spatial derivatives both taken at fixed $\boldsymbol{w}$.) Using the force equation for species α (viz., (IX.2.11) with (IX.2.22)),

$$\frac{\mathrm{d}\boldsymbol{u}_\alpha}{\mathrm{d}t_\alpha} = \frac{q_\alpha}{m_\alpha}\left(\boldsymbol{E} + \frac{\boldsymbol{u}_\alpha\times\boldsymbol{B}}{c}\right) + \boldsymbol{g} - \frac{\boldsymbol{\nabla} p_\alpha}{m_\alpha n_\alpha} - \frac{\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\mathit{\Pi}}_\alpha}{m_\alpha n_\alpha} + \frac{\boldsymbol{R}_\alpha}{m_\alpha n_\alpha},$$

to replace $\mathrm{d}\boldsymbol{u}_\alpha/\mathrm{d}t_\alpha$ in (IX.5.1) leads to

$$\boxed{\frac{\mathrm{d}f_\alpha}{\mathrm{d}t_\alpha} + \boldsymbol{w}\boldsymbol{\cdot}\boldsymbol{\nabla} f_\alpha + \left(\frac{\boldsymbol{\nabla} p_\alpha}{m_\alpha n_\alpha} + \frac{\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\mathit{\Pi}}_\alpha}{m_\alpha n_\alpha} - \frac{\boldsymbol{R}_\alpha}{m_\alpha n_\alpha}\right)\boldsymbol{\cdot}\frac{\partial f_\alpha}{\partial\boldsymbol{w}} - (\boldsymbol{w}\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{u}_\alpha)\boldsymbol{\cdot}\frac{\partial f_\alpha}{\partial\boldsymbol{w}} = -\Omega_\alpha(\boldsymbol{w}\times\hat{\boldsymbol{b}})\boldsymbol{\cdot}\frac{\partial f_\alpha}{\partial\boldsymbol{w}} + C[f_\alpha]} \tag{IX.5.2}$$

where $\Omega_\alpha \doteq q_\alpha B/m_\alpha c$ and $\hat{\boldsymbol{b}} \doteq \boldsymbol{B}/B$ is the unit vector in the direction of the magnetic field. The Braginskii ordering will be applied to (IX.5.2) for each species to solve for f_α and thus obtain the transport coefficients.

IX.5.1. *Expansion of the electron kinetic equation*

Let's do the electrons first. Set

$$\epsilon \sim \frac{\rho_e}{L} \sim \frac{\lambda_{ee}}{L} \sim \frac{\lambda_{ei}}{L} \sim \sqrt{\frac{m_e}{m_i}} \ll 1$$

and write

$$f_e = f_{e0} + \epsilon f_{e1} + \epsilon f_{e2} + \dots,$$

like we did in the Chapman–Enskog expansion (see (IX.2.27)). Then (IX.5.2) becomes

$$\underbrace{\frac{\mathrm{d}f_e}{\mathrm{d}t_e}}_{②} + \underbrace{\boldsymbol{w}\cdot\boldsymbol{\nabla} f_e}_{①} + \left(\underbrace{\frac{\boldsymbol{\nabla} p_e}{m_e n_e}}_{①} + \underbrace{\frac{\boldsymbol{\nabla}\cdot\boldsymbol{\Pi}_e}{m_e n_e}}_{①} - \underbrace{\frac{\boldsymbol{R}_{ei}}{m_e n_e}}_{①}\right)\cdot\frac{\partial f_e}{\partial \boldsymbol{w}} - \underbrace{(\boldsymbol{w}\cdot\boldsymbol{\nabla}\boldsymbol{u}_e)\cdot\frac{\partial f_e}{\partial \boldsymbol{w}}}_{②}$$

$$= \underbrace{-\,\Omega_e(\boldsymbol{w}\times\hat{\boldsymbol{b}})\cdot\frac{\partial f_e}{\partial \boldsymbol{w}}}_{⓪} + \underbrace{C_{ee}[f_e, f_e]}_{⓪} + \underbrace{C_{ei}[f_e, f_i]}_{⓪}, \qquad \text{(IX.5.3)}$$

where the circled numbers indicate the lowest order in ϵ at which each term appears. (Recall that $t_0 \sim \nu_{ee}^{-1} \sim \nu_{ei}^{-1} \sim \Omega_e^{-1}$ and $t_1 \sim (v_{\mathrm{th}i}/L)^{-1}$. The term involving the electron viscous stress tensor $\boldsymbol{\Pi}_e$ will eventually be shown to be $\sim\epsilon^2$, rather than $\sim\epsilon$, after f_{e0} is found to be an isotropic Maxwellian.)

At zeroth order in ϵ, equation (IX.5.3) is

$$0 = -\Omega_e(\boldsymbol{w}\times\hat{\boldsymbol{b}})\cdot\frac{\partial f_{e0}}{\partial \boldsymbol{w}} + C_{ee}[f_{e0}, f_{e0}] + C_{ei}[f_{e0}, f_{i0}], \qquad \text{(IX.5.4)}$$

with the condition that $\int \mathrm{d}\boldsymbol{w}\,\boldsymbol{w} f_{e0} = 0$. Because of the mass-ratio expansion, we can take

$$C_{ei} \simeq \frac{3\sqrt{\pi}}{4\tau_{ei}}\left(\frac{v_{\mathrm{th}e}}{w}\right)^3 \mathcal{L},$$

where $\mathcal{L}$ is the Lorentz operator (VIII.4.4). Then we multiply (IX.5.4) by $\ln f_{e0}$ and integrate over velocity space to obtain

$$0 = \int \mathrm{d}\boldsymbol{w}\, \ln f_{e0}\, C_{ee}[f_{e0}, f_{e0}] + \int \mathrm{d}\boldsymbol{w}\, \ln f_{e0}\, \frac{3\sqrt{\pi}}{4\tau_{ei}}\left(\frac{v_{\mathrm{th}e}}{w}\right)^3 \mathcal{L}[f_{e0}].$$

Both of these are positive quantities, and so each must vanish in order to satisfy this equation. We know that C_{ee} vanishes if f_{e0} is Maxwellian, and that $\mathcal{L}$ vanishes if f_{e0} is isotropic in velocity space. Thus,

$$\boxed{f_{e0} = f_{\mathrm{M}e} \doteq \frac{n_e(t,\boldsymbol{r})}{\pi^{3/2} v_{\mathrm{th}e}^3(t,\boldsymbol{r})} \exp\left[-\frac{|\boldsymbol{v}-\boldsymbol{u}_e(t,\boldsymbol{r})|^2}{v_{\mathrm{th}e}^2(t,\boldsymbol{r})}\right]} \qquad \text{(IX.5.5)}$$

with f_{e1} adjusted so that n_e, $\boldsymbol{u}_e$, and T_e are the true electron density, flow velocity, and temperature (recall the discussion in §IX.2). To ensure this, we require

$$\int \mathrm{d}\boldsymbol{w}\, f_{e1} = \int \mathrm{d}\boldsymbol{w}\, \boldsymbol{v} f_{e1} = \int \mathrm{d}\boldsymbol{w}\, w^2 f_{e1} = 0$$

in order that (IX.5.5) hold for the true moments. In other words, f_{e1} is purely kinetic.

At first order in ϵ, equation (IX.5.3) is

$$\boldsymbol{w}\cdot\boldsymbol{\nabla} f_{\mathrm{M}e} + \left(\frac{\boldsymbol{\nabla} p_e}{m_e n_e} + \frac{\boldsymbol{\nabla}\cdot\boldsymbol{\mathit{\Pi}}_e}{m_e n_e} + \frac{\boldsymbol{R}_{ei}}{m_e n_e}\right)\cdot\frac{\partial f_{\mathrm{M}e}}{\partial \boldsymbol{w}}$$
$$= -\mathit{\Omega}_e(\boldsymbol{w}\times\hat{\boldsymbol{b}})\cdot\frac{\partial f_{e,1}}{\partial\boldsymbol{w}} + C^{\ell}_{ee}[f_{e1}] + \frac{3\pi}{4\tau_{ei}}\left(\frac{v_{\mathrm{th}e}}{w}\right)^3\mathcal{L}[\dots], \tag{IX.5.6}$$

where

$$C^{\ell}_{ee}[f_{e1}] = C_{ee}[f_{e1}, f_{\mathrm{M}e}] + C_{ee}[f_{\mathrm{M}e}, f_{e1}]$$

is the linearized electron–electron collision operator (see (VIII.7.6)) and the $[\dots]$ argument of the Lorentz operator is

$$f_{e1} - \frac{m_e\boldsymbol{w}\cdot(\boldsymbol{u}_i - \boldsymbol{u}_e)}{T_e} f_{\mathrm{M}e},$$

the $\mathcal{O}(\sqrt{m_e/m_i})$ piece of the Lorentz operator for electron–ion collisions (VIII.4.4). We can simplify some terms in (IX.5.6) by using the following:

- $\mathcal{L}\big[\boldsymbol{w}\cdot(\boldsymbol{u}_i - \boldsymbol{u}_e)\big] = -\boldsymbol{w}\cdot(\boldsymbol{u}_i - \boldsymbol{u}_e)$;
- $\boldsymbol{w}\cdot\boldsymbol{\nabla} f_{\mathrm{M}e} = \left[\boldsymbol{w}\cdot\boldsymbol{\nabla}\ln p_e + \left(\dfrac{w^2}{v^2_{\mathrm{th}e}} - \dfrac{5}{2}\right)\boldsymbol{w}\cdot\boldsymbol{\nabla}\ln T_e\right] f_{\mathrm{M}e}$;
- $\dfrac{\partial f_{\mathrm{M}e}}{\partial\boldsymbol{w}} = -\dfrac{2\boldsymbol{w}}{v^2_{\mathrm{th}e}} f_{\mathrm{M}e}$;
- $\boldsymbol{R}_{ei} = \dfrac{m_e n_e}{\tau_{ei}}(\boldsymbol{u}_i - \boldsymbol{u}_e) - \dfrac{3\sqrt{\pi}}{4\tau_{ei}}\, m_e v^3_{\mathrm{th}e}\displaystyle\int \mathrm{d}\boldsymbol{w}'\,\frac{\boldsymbol{w}'}{w'^3} f_{e1}(\boldsymbol{w}')$;
- $\boldsymbol{\mathit{\Pi}}_e \simeq \displaystyle\int \mathrm{d}\boldsymbol{w}\, m_e\left(\boldsymbol{w}\boldsymbol{w} - \frac{\mathsf{I}}{3}w^2\right) f_{\mathrm{M}e} = 0$.

Thus, equation (IX.5.6) becomes

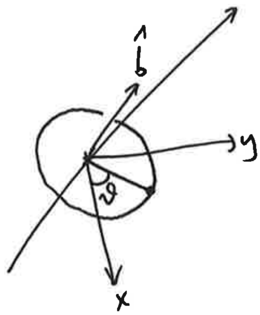
$$f_{\mathrm{M}e}\left(\frac{w^2}{v^2_{\mathrm{th}e}} - \frac{5}{2}\right)\boldsymbol{w}\cdot\boldsymbol{\nabla}\ln T_e + f_{\mathrm{M}e}\left[1 - \frac{3\sqrt{\pi}}{4}\left(\frac{v_{\mathrm{th}e}}{w}\right)^3\right]\frac{2\boldsymbol{w}\cdot(\boldsymbol{u}_i - \boldsymbol{u}_e)}{v^2_{\mathrm{th}e}\tau_{ei}}$$
$$= -\mathit{\Omega}_e(\boldsymbol{w}\times\hat{\boldsymbol{b}})\cdot\frac{\partial f_{e1}}{\partial\boldsymbol{w}} + C^{\ell}_{ee}[f_{e1}] + \frac{3\sqrt{\pi}}{4\tau_{ei}}\left(\frac{v_{\mathrm{th}e}}{w}\right)^3\mathcal{L}[f_{e1}]$$
$$+ \frac{3\sqrt{\pi}}{2\tau_{ei}}\frac{f_{\mathrm{M}e}}{n_e}\, v_{\mathrm{th}e}\boldsymbol{w}\cdot\int \mathrm{d}\boldsymbol{w}'\,\frac{\boldsymbol{w}'}{w'^3} f_{e1}(\boldsymbol{w}'). \tag{IX.5.7}$$

Believe it or not, we can solve this in a way similar to the Spitzer–Härm problem by using Laguerre polynomials. But first, let us cast $(\boldsymbol{w}\times\hat{\boldsymbol{b}})\cdot\partial/\partial\boldsymbol{w}$ in a more mathematically useful and physically revealing form.

Anticipating that the material properties of the plasma are biased with respect to the magnetic-field direction, decompose the peculiar velocity of the particle as follows:

$$\boldsymbol{w} = \boldsymbol{w}_{\parallel} + \boldsymbol{w}_{\perp} \doteq w_{\parallel}\hat{\boldsymbol{b}} + w_{\perp}(\cos\vartheta\,\hat{\boldsymbol{x}} + \sin\vartheta\,\hat{\boldsymbol{y}}), \tag{IX.5.8}$$

where $w_{\parallel} \doteq w\xi$ $(w_{\perp})$ is the component of $\boldsymbol{w}$ oriented along (across) the magnetic field. The angle ϑ is the "gyrophase", which tracks the angular position of a particle as it gyrates about the magnetic-field line in the $\hat{\boldsymbol{x}}$–$\hat{\boldsymbol{y}}$ plane:



Using (IX.5.8), it is straightforward to show that

$$\frac{\partial}{\partial \boldsymbol{w}} = \hat{\boldsymbol{b}}\frac{\partial}{\partial w_\parallel}\bigg|_{w_\perp,\vartheta} + \frac{\boldsymbol{w}_\perp}{w_\perp}\frac{\partial}{\partial w_\perp}\bigg|_{w_\parallel,\vartheta} - \frac{\boldsymbol{w}\times\hat{\boldsymbol{b}}}{w_\perp^2}\frac{\partial}{\partial\vartheta}\bigg|_{w_\parallel,w_\perp}$$

and that $\mathrm{d}\boldsymbol{w} = \mathrm{d}\vartheta\, w_\perp \mathrm{d}w_\perp \mathrm{d}w_\parallel$. Also useful will be

$$\langle \boldsymbol{w}\rangle_\vartheta = w_\parallel \hat{\boldsymbol{b}},$$
$$\langle \boldsymbol{w}\boldsymbol{w}\rangle_\vartheta = \langle \boldsymbol{w}\rangle_\vartheta\langle \boldsymbol{w}\rangle_\vartheta + \langle \boldsymbol{w}_\perp\boldsymbol{w}_\perp\rangle_\vartheta = w_\parallel^2\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} + \frac{w_\perp^2}{2}\left(\boldsymbol{I} - \hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\right),$$

where

$$\big\langle \dots \big\rangle_\vartheta \doteq \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\vartheta\,(\dots) \tag{IX.5.9}$$

denotes the gyro-average. It will be helpful in what follows to note the following identities:

$$\frac{\partial \boldsymbol{w}_\perp}{\partial\vartheta} = -\boldsymbol{w}_\perp\times\hat{\boldsymbol{b}} \quad\text{and}\quad \frac{\partial(\boldsymbol{w}_\perp\times\hat{\boldsymbol{b}})}{\partial\vartheta} = \boldsymbol{w}_\perp. \tag{IX.5.10}$$

With these results, equation (IX.5.7) becomes

$$\begin{aligned} f_{\mathrm{M}e}\left(\frac{w^2}{v_{\mathrm{th}e}^2} - \frac{5}{2}\right)\boldsymbol{w}\cdot\boldsymbol{\nabla}\ln T_e + f_{\mathrm{M}e}\left[1 - \frac{3\sqrt{\pi}}{4}\left(\frac{v_{\mathrm{th}e}}{w}\right)^3\right]\frac{2\boldsymbol{w}\cdot(\boldsymbol{u}_i - \boldsymbol{u}_e)}{v_{\mathrm{th}e}^2\tau_{ei}} \\ = \varOmega_e\frac{\partial f_{e1}}{\partial\vartheta} + C_{ee}^\ell[f_{e1}] + \frac{3\sqrt{\pi}}{4\tau_{ei}}\left(\frac{v_{\mathrm{th}e}}{w}\right)^3\mathcal{L}[f_{e1}] \\ + \frac{3\sqrt{\pi}}{2\tau_{ei}}\frac{f_{\mathrm{M}e}}{n_e}v_{\mathrm{th}e}\boldsymbol{w}\cdot\int\mathrm{d}\boldsymbol{w}'\,\frac{\boldsymbol{w}'}{w'^3}f_{e1}(\boldsymbol{w}'). \end{aligned} \tag{IX.5.11}$$

The solution to (IX.5.11) must have the form

$$f_{e1} = \underbrace{\langle f_{e1}\rangle_\vartheta}_{\substack{\text{gyrophase-}\\\text{independent}\\\text{piece}}} + \underbrace{\widetilde{f}_{e1}}_{\substack{\text{gyrophase-}\\\text{dependent}\\\text{piece}}}. \tag{IX.5.12}$$

Note that $\langle C_{ee}^\ell[f_{e1}]\rangle_\vartheta = C_{ee}^\ell[\langle f_{e1}\rangle_\vartheta]$ and $\langle\mathcal{L}[f_{e1}]\rangle_\vartheta = \mathcal{L}[\langle f_{e1}\rangle_\vartheta]$. Then, using $\langle\boldsymbol{w}\rangle_\vartheta = w_\parallel\hat{\boldsymbol{b}}$, the gyro-average of (IX.5.11) is

$$f_{\mathrm{Me}}\left(\frac{w^2}{v_{\mathrm{the}}^2}-\frac{5}{2}\right)w_\parallel\hat{\boldsymbol{b}}\boldsymbol{\cdot}\boldsymbol{\nabla}\ln T_e+f_{\mathrm{Me}}\left[1-\frac{3\sqrt{\pi}}{4}\left(\frac{v_{\mathrm{the}}}{w}\right)^3\right]\frac{2w_\parallel(u_{\parallel i}-u_{\parallel e})}{v_{\mathrm{the}}^2\tau_{ei}}$$
$$=C_{ee}^{\ell}[\langle f_{e1}\rangle_\vartheta]+\frac{3\sqrt{\pi}}{4\tau_{ei}}\left(\frac{v_{\mathrm{the}}}{w}\right)^3\mathcal{L}[\langle f_{e1}\rangle_\vartheta]+\frac{3\sqrt{\pi}}{2\tau_{ei}}\frac{f_{\mathrm{Me}}}{n_e}v_{\mathrm{the}}w_\parallel\int\mathrm{d}\boldsymbol{w}'\,\frac{w'_\parallel}{w'^3}f_{e1}(\boldsymbol{w}'). \tag{IX.5.13}$$

(Note that the $f_{e1}(\boldsymbol{w}')$ in the final integral term of (IX.5.13) may be replaced with impunity by its gyro-average, $\langle f_{e1}\rangle_\vartheta$.) Following (IX.1.14), write

$$\langle f_{e1}\rangle_\vartheta=\frac{2w_\parallel}{v_{\mathrm{the}}^2}f_{\mathrm{Me}}\sum_{k=1}^{N}u_{e,k}L_k^{(3/2)}\left(\frac{w^2}{v_{\mathrm{the}}^2}\right). \tag{IX.5.14}$$

Here, the Laguerre sum starts at $k=1$ since $u_{e,0}=0$; i.e., there is no parallel mean flow in f_{e1}, since it's in f_{e0} by construction. The right-hand side of (IX.5.13) was already calculated in §IX.1 (see (IX.1.23)):

$$-\frac{n_e}{\tau_{ee}}\begin{bmatrix}\sqrt{2}+\dfrac{13}{4}Z & \dfrac{3\sqrt{2}}{4}+\dfrac{69}{16}Z\\[2ex] \dfrac{3\sqrt{2}}{4}+\dfrac{69}{16}Z & \dfrac{45\sqrt{2}}{16}+\dfrac{433}{64}Z\end{bmatrix}\begin{bmatrix}u_{e,1}\\[2ex] u_{e,2}\end{bmatrix}.$$

The transformed left-hand side of (IX.5.13) is (with $x\doteq w^2/v_{\mathrm{the}}^2$)

$$\int\mathrm{d}\boldsymbol{w}\,w_\parallel L_k^{(3/2)}(x)f_{\mathrm{Me}}\left\{\left(\frac{w^2}{v_{\mathrm{the}}^2}-\frac{5}{2}\right)w_\parallel\hat{\boldsymbol{b}}\boldsymbol{\cdot}\boldsymbol{\nabla}\ln T_e+\left[1-\frac{3\sqrt{\pi}}{4}\left(\frac{v_{\mathrm{the}}}{w}\right)^3\right]\frac{2w_\parallel(u_{\parallel i}-u_{\parallel e})}{v_{\mathrm{the}}^2\tau_{ei}}\right\}$$
$$=\int\mathrm{d}\boldsymbol{w}\,w^2\xi^2L_k^{(3/2)}(x)f_{\mathrm{Me}}\left[\left(x^2-\frac{5}{2}\right)\hat{\boldsymbol{b}}\boldsymbol{\cdot}\boldsymbol{\nabla}\ln T_e+\left(1-\frac{3\sqrt{\pi}}{4x^{3/2}}\right)\frac{2(u_{\parallel i}-u_{\parallel e})}{v_{\mathrm{the}}^2\tau_{ei}}\right]$$
$$=\frac{4\pi}{3}\frac{n_e}{\pi\sqrt{\pi}}\frac{v_{\mathrm{the}}^2}{2}\int_0^\infty\mathrm{d}x\,x^{3/2}\mathrm{e}^{-x}L_k^{(3/2)}(x)\left[-L_1^{(3/2)}(x)\hat{\boldsymbol{b}}\boldsymbol{\cdot}\boldsymbol{\nabla}\ln T_e\right.$$
$$\left.+L_0^{(3/2)}(x)\frac{2(u_{\parallel i}-u_{\parallel e})}{v_{\mathrm{the}}^2\tau_{ei}}-\frac{3\sqrt{\pi}}{2x^{3/2}}\frac{u_{\parallel i}-u_{\parallel e}}{v_{\mathrm{the}}^2\tau_{ei}}\right]$$
$$=-\frac{4}{3\sqrt{\pi}}\frac{p_e}{m_e}\hat{\boldsymbol{b}}\boldsymbol{\cdot}\boldsymbol{\nabla}T_e\times\delta_{k1}\frac{15\sqrt{\pi}}{8}+\cancel{\frac{4}{3\sqrt{\pi}}\frac{p_e}{m_e}\frac{2(u_{\parallel i}-u_{\parallel e})}{v_{\mathrm{the}}^2\tau_{ei}}\times\delta_{k0}\frac{3\sqrt{\pi}}{4}}$$
$$-\frac{p_e}{m_e}\frac{2(u_{\parallel i}-u_{\parallel e})}{v_{\mathrm{the}}^2\tau_{ei}}\times\left(\cancel{\delta_{k0}}+\frac{3}{2}\delta_{k1}+\frac{15}{8}\delta_{k2}\right)$$

$$\Longrightarrow\quad\frac{n_e}{\tau_{ei}}\begin{bmatrix}\dfrac{5}{2}\dfrac{\tau_{ei}}{m_e}\hat{\boldsymbol{b}}\boldsymbol{\cdot}\boldsymbol{\nabla}T_e+\dfrac{3}{2}(u_{\parallel i}-u_{\parallel e})\\[2ex] \dfrac{15}{8}(u_{\parallel i}-u_{\parallel e})\end{bmatrix}$$
$$=\frac{n_e}{\tau_{ee}}\begin{bmatrix}\sqrt{2}+\dfrac{13}{4}Z & \dfrac{3\sqrt{2}}{4}+\dfrac{69}{16}Z\\[2ex] \dfrac{3\sqrt{2}}{4}+\dfrac{69}{16}Z & \dfrac{45\sqrt{2}}{16}+\dfrac{433}{64}Z\end{bmatrix}\begin{bmatrix}u_{e,1}\\[2ex] u_{e,2}\end{bmatrix}. \tag{IX.5.15}$$

Inverting, we find

$$u_{e,1} = \frac{5}{2}\frac{\tau_{ei}}{m_e}\hat{\boldsymbol{b}}\cdot\boldsymbol{\nabla}T_e \times \underbrace{\frac{\dfrac{45\sqrt{2}}{16}Z + \dfrac{433}{64}Z^2}{\dfrac{72}{16} + \dfrac{151\sqrt{2}}{16}Z + \dfrac{217}{64}Z^2}}_{=\,0.506 \text{ for } Z=1}$$
$$+ (u_{\|i} - u_{\|e}) \times \underbrace{\frac{\dfrac{45\sqrt{2}}{16}Z + \dfrac{33}{16}Z^2}{\dfrac{72}{16} + \dfrac{151\sqrt{2}}{16}Z + \dfrac{217}{64}Z^2}}_{=\,0.284 \text{ for } Z=1}, \tag{IX.5.16}$$

$$u_{e,2} = -\frac{5}{2}\frac{\tau_{ei}}{m_e}\hat{\boldsymbol{b}}\cdot\boldsymbol{\nabla}T_e \times \underbrace{\frac{\dfrac{3\sqrt{2}}{4}Z + \dfrac{69}{16}Z^2}{\dfrac{72}{16} + \dfrac{151\sqrt{2}}{16}Z + \dfrac{217}{64}Z^2}}_{=\,0.253 \text{ for } Z=1}$$
$$+ (u_{\|i} - u_{\|e}) \times \underbrace{\frac{\dfrac{3\sqrt{2}}{4}Z - \dfrac{3}{8}Z^2}{\dfrac{72}{16} + \dfrac{151\sqrt{2}}{16}Z + \dfrac{217}{64}Z^2}}_{=\,0.032 \text{ for } Z=1}, \tag{IX.5.17}$$

Thus, for $Z=1$,

$$\boxed{\begin{aligned}\langle f_{e1}\rangle_\vartheta = {} & w_\| f_{\mathrm{M}e} L_1^{(3/2)}\left(\frac{w^2}{v_{\mathrm{th}e}^2}\right)\left[0.506 \times \frac{5}{2}\tau_{ei}\hat{\boldsymbol{b}}\cdot\boldsymbol{\nabla}\ln T_e + 0.284 \times \frac{m_e(u_{\|i}-u_{\|e})}{T_e}\right] \\ & - w_\| f_{\mathrm{M}e} L_2^{(3/2)}\left(\frac{w^2}{v_{\mathrm{th}e}^2}\right)\left[0.253 \times \frac{5}{2}\tau_{ei}\hat{\boldsymbol{b}}\cdot\boldsymbol{\nabla}\ln T_e - 0.032 \times \frac{m_e(u_{\|i}-u_{\|e})}{T_e}\right]\end{aligned}} \tag{IX.5.18}$$

Next we obtain $\widetilde{f}_{e1}$, the gyrophase-dependent piece. Do (IX.5.11) minus (IX.5.13):

$$\begin{aligned} f_{\mathrm{M}e}\left(\frac{w^2}{v_{\mathrm{th}e}^2} - \frac{5}{2}\right)\boldsymbol{w}_\perp\cdot\boldsymbol{\nabla}\ln T_e + f_{\mathrm{M}e}\left[1 - \frac{3\sqrt{\pi}}{4}\left(\frac{v_{\mathrm{th}e}}{w}\right)^3\right]\frac{2\boldsymbol{w}_\perp\cdot(\boldsymbol{u}_i - \boldsymbol{u}_e)}{v_{\mathrm{th}e}^2\tau_{ei}} \\ = \Omega_e\frac{\partial\widetilde{f}_{e1}}{\partial\vartheta} + C_{ee}^\ell[\widetilde{f}_{e1}] + \frac{3\sqrt{\pi}}{4\tau_{ei}}\left(\frac{v_{\mathrm{th}e}}{w}\right)^3\mathcal{L}[\widetilde{f}_{e1}] \\ + \frac{3\sqrt{\pi}}{2\tau_{ei}}\frac{f_{\mathrm{M}e}}{n_e}v_{\mathrm{th}e}\boldsymbol{w}_\perp\cdot\int \mathrm{d}\boldsymbol{w}'\,\frac{\boldsymbol{w}'}{w'^3}\widetilde{f}_{e1}(\boldsymbol{w}'), \end{aligned} \tag{IX.5.19}$$

where $\boldsymbol{w}_\perp \doteq \boldsymbol{w} - w_\|\hat{\boldsymbol{b}}$. Now we impose our subsidiary expansion (IX.4.5) on (IX.5.19). To lowest order, the only terms in (IX.5.19) that survive are

$$f_{\mathrm{M}e}\left(\frac{w^2}{v_{\mathrm{th}e}^2} - \frac{5}{2}\right)\boldsymbol{w}_\perp\cdot\boldsymbol{\nabla}\ln T_e = \Omega_e\frac{\partial\widetilde{f}_{e1}^{(0)}}{\partial\vartheta}, \tag{IX.5.20}$$

where we have written $\widetilde{f}_{e1} = \widetilde{f}^{(0)}_{e1} + \widetilde{f}^{(1)}_{e1} + \dots$. To next order,

$$f_{\mathrm{M}e}\left[1 - \frac{3\sqrt{\pi}}{4}\left(\frac{v_{\mathrm{th}e}}{w}\right)^3\right]\frac{2\boldsymbol{w}_\perp \cdot (\boldsymbol{u}_i - \boldsymbol{u}_e)}{v^2_{\mathrm{th}e}\tau_{ei}} = \Omega_e \frac{\partial \widetilde{f}^{(1)}_{e1}}{\partial \vartheta} + C^\ell_{ee}[\widetilde{f}^{(0)}_{e1}] + \frac{3\sqrt{\pi}}{4\tau_{ei}}\left(\frac{v_{\mathrm{th}e}}{w}\right)^3 \mathcal{L}[\widetilde{f}^{(0)}_{e1}] \\ + \frac{3\sqrt{\pi}}{2\tau_{ei}}\frac{f_{\mathrm{M}e}}{n_e} v_{\mathrm{th}e}\boldsymbol{w}_\perp \cdot \int \mathrm{d}\boldsymbol{w}' \, \frac{\boldsymbol{w}'}{w'^3}\widetilde{f}^{(0)}_{e1}(\boldsymbol{w}'). \tag{IX.5.21}$$

Equation (IX.5.20) becomes, upon integrating over ϑ,

$$\boxed{\widetilde{f}^{(0)}_{e1} = -f_{\mathrm{M}e}\, L^{(3/2)}_1(x)\frac{\boldsymbol{w}_\perp \times \hat{\boldsymbol{b}}}{\Omega_e} \cdot \boldsymbol{\nabla} \ln T_e} \tag{IX.5.22}$$

There is no need to solve (IX.5.21) for $\widetilde{f}^{(1)}_{e1}$; ultimately we'll only need to know $\partial \widetilde{f}^{(1)}_{e1}/\partial\vartheta$. It will pay, however, to insert (IX.5.22) into (IX.5.21) and simplify where possible. "Where possible" means that pesky integral over $\boldsymbol{w}'$ on the right-hand side of (IX.5.21):

$$\begin{aligned}\int \mathrm{d}\boldsymbol{w}' \, \frac{\boldsymbol{w}'}{w'^3}\widetilde{f}^{(0)}_{e1}(\boldsymbol{w}') &= \int \mathrm{d}\boldsymbol{w}' \, \frac{\boldsymbol{w}'}{w'^3} f_{\mathrm{M}e}\, L^{(3/2)}_1(x')\frac{\boldsymbol{w}'_\perp \times \hat{\boldsymbol{b}}}{\Omega_e} \cdot \boldsymbol{\nabla} \ln T_e \\ &= \frac{1}{\Omega_e}\int \mathrm{d}\boldsymbol{w}' \, \frac{\langle \boldsymbol{w}'_\perp \boldsymbol{w}'_\perp \rangle}{w'^3} f_{\mathrm{M}e}\, L^{(3/2)}_1(x') \times \hat{\boldsymbol{b}} \cdot \boldsymbol{\nabla} \ln T_e \\ &= \frac{1}{\Omega_e}\frac{4\pi}{3}\left(\mathsf{I} - \hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\right) \times \hat{\boldsymbol{b}} \cdot \boldsymbol{\nabla} T_e \int \mathrm{d}w' \, w' f_{\mathrm{M}e}\, L^{(3/2)}_1(x') \\ &= \frac{1}{\Omega_e}\frac{4\pi}{3}\frac{n_e}{\pi\sqrt{\pi}}\frac{1}{2v_{\mathrm{th}e}}\hat{\boldsymbol{b}} \times \boldsymbol{\nabla} \ln T_e \cancelto{0}{\int \mathrm{d}x' \, \mathrm{e}^{-x'} L^{(3/2)}_1(x')} \\ &= 0. \end{aligned}$$

Well, that's nice. Equation (IX.5.21) becomes

$$\Omega_e \frac{\partial \widetilde{f}^{(1)}_{e1}}{\partial \vartheta} = f_{\mathrm{M}e}\left[1 - \frac{3\sqrt{\pi}}{4}\left(\frac{v_{\mathrm{th}e}}{w}\right)^3\right]\frac{2\boldsymbol{w}_\perp \cdot (\boldsymbol{u}_i - \boldsymbol{u}_e)}{v^2_{\mathrm{th}e}\tau_{ei}} \\ - C^\ell_{ee}[\widetilde{f}^{(0)}_{e1}] - \frac{3\sqrt{\pi}}{4\tau_{ei}}\left(\frac{v_{\mathrm{th}e}}{w}\right)^3 \mathcal{L}[\widetilde{f}^{(0)}_{e1}]. \tag{IX.5.23}$$

Now that we have f_{e1}, let's compute some moments.

First, the electron heat flux:

$$\begin{aligned}\boldsymbol{q}_e &= \int \mathrm{d}\boldsymbol{w} \, \boldsymbol{w}\frac{1}{2}m_e w^2 f_e \simeq \int \mathrm{d}\boldsymbol{w} \, \boldsymbol{w}\frac{1}{2}m_e w^2 f_{e1} \\ &= T_e \int \mathrm{d}\boldsymbol{w} \, \boldsymbol{w}\left(\frac{w^2}{v^2_{\mathrm{th}e}} - \frac{5}{2}\right) f_{e1} \quad \left(\text{since } \textstyle\int \mathrm{d}\boldsymbol{w} \, \boldsymbol{w} f_{e1} = 0\right) \\ &= -T_e \int \mathrm{d}\boldsymbol{w} \, \boldsymbol{w} L^{(3/2)}_1\left(\frac{w^2}{v^2_{\mathrm{th}e}}\right)\Big(\underbrace{\langle f_{e1}\rangle_\vartheta}_{\substack{\text{gives}\\ \boldsymbol{q}_{\parallel e}}} + \underbrace{\widetilde{f}^{(0)}_{e1}}_{\substack{\text{gives}\\ \boldsymbol{q}_{\times e}}} + \underbrace{\widetilde{f}^{(1)}_{e1}}_{\substack{\text{gives}\\ \boldsymbol{q}_{\perp e}}} + \underbrace{\;\dots\;}_{\substack{\text{higher-}\\ \text{order}\\ \text{stuff}}}\Big) \end{aligned} \tag{IX.5.24}$$

Using (IX.5.18), the *parallel electron heat flux* for $Z = 1$ is (with $x \doteq w^2/v_{\rm the}^2$)

$$\begin{aligned}
\boldsymbol{q}_{\parallel e} &\doteq -T_e \int \mathrm{d}\boldsymbol{w}\, \boldsymbol{w} L_1^{(3/2)}(x)\langle f_{e1}\rangle_\vartheta \\
&= -T_e \int \mathrm{d}\boldsymbol{w}\, \langle \boldsymbol{w}\rangle_\vartheta\, L_1^{(3/2)}(x)\langle f_{e1}\rangle_\vartheta = -T_e \int \mathrm{d}\boldsymbol{w}\, w_\parallel \hat{\boldsymbol{b}}\, L_1^{(3/2)}(x)\langle f_{e1}\rangle_\vartheta \\
&= -\frac{4\pi}{3}\int \mathrm{d}w\, w^4 \left[L_1^{(3/2)}(x)\right]^2 f_{\rm Me}\left[0.506\,\frac{5}{2}\tau_{ei}\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\boldsymbol{\cdot}\boldsymbol{\nabla}T_e + 0.284\, m_e(u_{\parallel i} - u_{\parallel e})\hat{\boldsymbol{b}}\right] \\
&= -\frac{4\pi}{3}\frac{n_e}{\pi\sqrt{\pi}}\frac{v_{\rm the}^2}{2}\left[0.506\,\frac{5}{2}\tau_{ei}\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\boldsymbol{\cdot}\boldsymbol{\nabla}T_e + 0.284\, m_e(u_{\parallel i} - u_{\parallel e})\hat{\boldsymbol{b}}\right]\int_0^\infty \mathrm{d}x\, x^{3/2}\mathrm{e}^{-x}\left[L_1^{(3/2)}(x)\right]^2 \\
&= -\frac{4}{3\sqrt{\pi}}\frac{p_e}{m_e}\left[0.506\,\frac{5}{2}\tau_{ei}\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\boldsymbol{\cdot}\boldsymbol{\nabla}T_e + 0.284\, m_e(u_{\parallel i} - u_{\parallel e})\hat{\boldsymbol{b}}\right] \times \frac{15\sqrt{\pi}}{8} \\
&= -3.16\,\frac{p_e\tau_{ei}}{m_e}\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\boldsymbol{\cdot}\boldsymbol{\nabla}T_e - 0.71\, p_e(u_{\parallel i} - u_{\parallel e})\hat{\boldsymbol{b}}.
\end{aligned} \tag{IX.5.25}$$

Using (IX.5.22), the *diamagnetic electron heat flux* for any Z is

$$\begin{aligned}
\boldsymbol{q}_{\times e} &= -T_e \int \mathrm{d}\boldsymbol{w}\, \boldsymbol{w}_\perp L_1^{(3/2)}(x)\widetilde{f}_{e1}^{(0)} \\
&= \int \mathrm{d}\boldsymbol{w}\, \boldsymbol{w}_\perp \left[L_1^{(3/2)}(x)\right]^2 f_{\rm Me}\frac{\boldsymbol{w}_\perp\times\hat{\boldsymbol{b}}}{\Omega_e}\boldsymbol{\cdot}\boldsymbol{\nabla}T_e \\
&= \frac{1}{\Omega_e}\int \mathrm{d}\boldsymbol{w}\, \langle \boldsymbol{w}_\perp\boldsymbol{w}_\perp\rangle_\vartheta \left[L_1^{(3/2)}(x)\right]^2 f_{\rm Me}\times\hat{\boldsymbol{b}}\boldsymbol{\cdot}\boldsymbol{\nabla}T_e \\
&= \frac{1}{\Omega_e}\frac{4\pi}{3}\frac{n_e}{\pi\sqrt{\pi}}\frac{v_{\rm the}^2}{2}\left(\mathsf{I} - \hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\right)\times\hat{\boldsymbol{b}}\boldsymbol{\cdot}\boldsymbol{\nabla}T_e\int \mathrm{d}x\, x^{3/2}\mathrm{e}^{-x}\left[L_1^{(3/2)}(x)\right]^2 \\
&= \frac{1}{\Omega_e}\frac{4}{3\sqrt{\pi}}\frac{p_e}{m_e}\hat{\boldsymbol{b}}\times\boldsymbol{\nabla}T_e\times\frac{15\sqrt{\pi}}{8} \\
&= \frac{5}{2}\frac{p_e}{m_e\Omega_e}\hat{\boldsymbol{b}}\times\boldsymbol{\nabla}T_e. \qquad \text{(NB: } \Omega_e < 0\text{!)}
\end{aligned} \tag{IX.5.26}$$

Finally, using (IX.5.23), the *perpendicular electron heat flux* is

$$\begin{aligned}
\boldsymbol{q}_{\perp e} &= -T_e \int \mathrm{d}\boldsymbol{w}\, \boldsymbol{w}_\perp L_1^{(3/2)}(x)\widetilde{f}_{e1}^{(1)} \\
&= -T_e \int \mathrm{d}\boldsymbol{w}\, \frac{\partial(\boldsymbol{w}_\perp\times\hat{\boldsymbol{b}})}{\partial\vartheta} L_1^{(3/2)}(x)\widetilde{f}_{e1}^{(1)} \\
&\overset{\rm bp}{=} T_e \int \mathrm{d}\boldsymbol{w}\, (\boldsymbol{w}_\perp\times\hat{\boldsymbol{b}}) L_1^{(3/2)}(x)\frac{\partial\widetilde{f}_{e1}^{(1)}}{\partial\vartheta} \\
&= T_e \int \mathrm{d}\boldsymbol{w}\, \frac{\boldsymbol{w}_\perp\times\hat{\boldsymbol{b}}}{\Omega_e} L_1^{(3/2)}(x)\, f_{\rm Me}\left[1 - \frac{3\sqrt{\pi}}{4}\left(\frac{v_{\rm the}}{w}\right)^3\right]\frac{2\boldsymbol{w}_\perp\boldsymbol{\cdot}(\boldsymbol{u}_i - \boldsymbol{u}_e)}{v_{\rm the}^2\tau_{ei}} \\
&\quad - T_e \int \mathrm{d}\boldsymbol{w}\, \frac{\boldsymbol{w}_\perp\times\hat{\boldsymbol{b}}}{\Omega_e} L_1^{(3/2)}(x)\left\{C_{ee}^\ell[\widetilde{f}_{e1}^{(0)}] + \frac{3\sqrt{\pi}}{4\tau_{ei}}\left(\frac{v_{\rm the}}{w}\right)^3\mathcal{L}[\widetilde{f}_{e1}^{(0)}]\right\} \\
&= \ldots\text{crunch, crunch, crunch}\ldots \\
&= \frac{3}{2}\frac{p_e\tau_{ei}^{-1}}{\Omega_e}\hat{\boldsymbol{b}}\times(\boldsymbol{u}_i - \boldsymbol{u}_e) - 4.66\,\frac{p_e\tau_{ee}^{-1}}{m_e\Omega_e^2}\boldsymbol{\nabla}_\perp T_e.
\end{aligned} \tag{IX.5.27}$$

Next, the *electron viscosity*. Simple: $\boldsymbol{\Pi}_e = 0$, at least to the order at which we're working, since $f_{e1} \propto P_1(\xi)$ and non-zero viscosity requires a contribution from $P_2(\xi)$.

Finally, the friction terms. Recall the friction force for electron–ion collisions (see (VIII.4.17))

$$\boldsymbol{R}_{ei} = \frac{m_e n_e}{\tau_{ei}}(\boldsymbol{u}_i - \boldsymbol{u}_e) - \frac{3\sqrt{\pi}}{4\tau_{ei}} m_e v_{\mathrm{the}}^3 \int \mathrm{d}\boldsymbol{w}\, \frac{\boldsymbol{w}}{w^3} f_{e1}(\boldsymbol{w})$$

Then the *parallel friction force* for $Z = 1$ is

$$\begin{aligned}
R_{\parallel ei} &= \frac{m_e n_e}{\tau_{ei}}(u_{\parallel i} - u_{\parallel e}) - \frac{3\sqrt{\pi}}{4\tau_{ei}} m_e v_{\mathrm{the}}^3 \int \mathrm{d}\boldsymbol{w}\, \frac{w_\parallel}{w^3} \langle f_{e1} \rangle_\vartheta \\
&= \frac{m_e n_e}{\tau_{ei}}(u_{\parallel i} - u_{\parallel e}) \left\{ 1 - \frac{3\sqrt{\pi}}{2} \frac{v_{\mathrm{the}}}{n_e} \int \mathrm{d}\boldsymbol{w}\, \frac{w_\parallel^2}{w^3} f_{\mathrm{M}e} \left[0.0284\, L_1^{(3/2)}(x) + 0.032\, L_2^{(3/2)}(x) \right] \right\} \\
&\quad - \frac{5}{2} n_e \hat{\boldsymbol{b}} \boldsymbol{\cdot} \boldsymbol{\nabla} T_e \times \frac{3\sqrt{\pi}}{2} \frac{v_{\mathrm{the}}}{n_e} \int \mathrm{d}\boldsymbol{w}\, \frac{w_\parallel^2}{w^3} f_{\mathrm{M}e} \left[0.506\, L_1^{(3/2)}(x) - 0.253\, L_2^{(3/2)}(x) \right] \\
&= \frac{m_e n_e}{\tau_{ei}}(u_{\parallel i} - u_{\parallel e}) \left[1 - 0.0284 \int_0^\infty \mathrm{d}x\, \mathrm{e}^{-x} L_1^{(3/2)}(x) - 0.032 \int_0^\infty \mathrm{d}x\, \mathrm{e}^{-x} L_2^{(3/2)}(x) \right] \\
&\quad - \frac{5}{2} n_e \hat{\boldsymbol{b}} \boldsymbol{\cdot} \boldsymbol{\nabla} T_e \left[0.506 \underbrace{\int_0^\infty \mathrm{d}x\, \mathrm{e}^{-x} L_1^{(3/2)}(x)}_{=3/2} - 0.253 \underbrace{\int_0^\infty \mathrm{d}x\, \mathrm{e}^{-x} L_2^{(3/2)}(x)}_{=15/8} \right] \\
&= 0.51\, \frac{m_e n_e}{\tau_{ei}}(u_{\parallel i} - u_{\parallel e}) - 0.71\, n_e \hat{\boldsymbol{b}} \boldsymbol{\cdot} \boldsymbol{\nabla} T_e.
\end{aligned} \tag{IX.5.28}$$

Likewise, the *perpendicular friction force* is

$$\begin{aligned}
\boldsymbol{R}_{\perp ei} &= \frac{m_e n_e}{\tau_{ei}}(\boldsymbol{u}_{\perp i} - \boldsymbol{u}_{\perp e}) - \frac{3\sqrt{\pi}}{4\tau_{ei}} m_e v_{\mathrm{the}}^3 \int \mathrm{d}\boldsymbol{w}\, \frac{\boldsymbol{w}_\perp}{w^3} \widetilde{f}_{e1} \\
&= \frac{m_e n_e}{\tau_{ei}}(\boldsymbol{u}_{\perp i} - \boldsymbol{u}_{\perp e}) + \frac{3\sqrt{\pi}}{4\Omega_e \tau_{ei}} m_e v_{\mathrm{the}}^3 \int \mathrm{d}\boldsymbol{w}\, \frac{\boldsymbol{w}_\perp}{w^3} f_{\mathrm{M}e}\, L_1^{(3/2)}(x)\, \boldsymbol{w}_\perp \boldsymbol{\cdot} \hat{\boldsymbol{b}} \times \boldsymbol{\nabla} \ln T_e \\
&= \frac{m_e n_e}{\tau_{ei}}(\boldsymbol{u}_{\perp i} - \boldsymbol{u}_{\perp e}) + \frac{3\sqrt{\pi}}{4\Omega_e \tau_{ei}} \frac{n_e}{\sqrt{\pi}} \underbrace{\int_{-1}^1 \mathrm{d}\xi\, (1 - \xi^2)}_{=4/3} \underbrace{\int_0^\infty \mathrm{d}x\, \mathrm{e}^{-x}\, L_1^{(3/2)}(x)}_{=3/2} \hat{\boldsymbol{b}} \times \boldsymbol{\nabla} T_e \\
&= \frac{m_e n_e}{\tau_{ei}}(\boldsymbol{u}_{\perp i} - \boldsymbol{u}_{\perp e}) + \frac{3}{2} \frac{n_e}{\Omega_e \tau_{ei}}\, \hat{\boldsymbol{b}} \times \boldsymbol{\nabla} T_e.
\end{aligned} \tag{IX.5.29}$$

The *collisional energy exchange* was already obtained in (VIII.5.16) using Maxwellian electrons and ions; it is given by

$$Q_{ie} = \frac{3 m_e n_e}{m_i \tau_{ei}}(T_e - T_i). \tag{IX.5.30}$$

With f_{e0} Maxwellian and (as we will show) f_{i0} Maxwellian with $T_i \neq T_e$, the difference between the temperatures of f_{e1} and f_{i1} will only give small corrections to the collisional energy exchange rate, and so (IX.5.30) remains valid to leading order. This energy exchange is, indeed, on a long timescale, and our use of a small mass ratio captures this.

A summary of electron transport is as follows (NB: $\Omega_e < 0$ and $Z = 1$):

$$\boxed{\begin{aligned}
\boldsymbol{q}_{\parallel e} &= -3.16\frac{p_e\tau_{ee}}{m_e}\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\boldsymbol{\cdot}\boldsymbol{\nabla}T_e - 0.71p_e(u_{\parallel i} - u_{\parallel e})\hat{\boldsymbol{b}}\\
\boldsymbol{q}_{\times e} &= \frac{5}{2}\frac{p_e}{m_e\Omega_e}\hat{\boldsymbol{b}}\times\boldsymbol{\nabla}T_e\\
\boldsymbol{q}_{\perp e} &= -4.66\frac{p_e\tau_{ee}^{-1}}{m_e\Omega_e^2}\boldsymbol{\nabla}_\perp T_e + \frac{3}{2}\frac{p_e\tau_{ei}^{-1}}{\Omega_e}\hat{\boldsymbol{b}}\times(\boldsymbol{u}_i - \boldsymbol{u}_e)\\
\boldsymbol{\Pi}_e &= 0\\
\boldsymbol{R}_{\parallel ei} &= 0.51\frac{m_e n_e}{\tau_{ei}}(u_{\parallel i} - u_{\parallel e})\hat{\boldsymbol{b}} - 0.71n_e\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\boldsymbol{\cdot}\boldsymbol{\nabla}T_e\\
\boldsymbol{R}_{\perp ei} &= \frac{m_e n_e}{\tau_{ei}}(\boldsymbol{u}_{\perp i} - \boldsymbol{u}_{\perp e}) + \frac{3}{2}\frac{n_e}{\Omega_e\tau_{ei}}\hat{\boldsymbol{b}}\times\boldsymbol{\nabla}T_e\\
Q_{ie} &= 3\frac{m_e n_e}{m_i\tau_{ei}}(T_e - T_i)
\end{aligned}}$$

The physics captured by these expressions is thoroughly discussed in §IX.6.

IX.5.2. *Expansion of the ion kinetic equation*

Now the ions. Following the expansion procedure used for the electrons, set

$$\epsilon \sim \frac{\rho_i}{L} \sim \frac{\lambda_{ii}}{L} \sim \sqrt{\frac{m_e}{m_i}} \ll 1$$

and write

$$f_i = f_{i0} + \epsilon f_{i1} + \epsilon^2 f_{i2} + \dots.$$

Then (IX.5.2) becomes

$$\underbrace{\frac{\mathrm{d}f_i}{\mathrm{d}t_i}}_{①} + \underbrace{\boldsymbol{w}\boldsymbol{\cdot}\boldsymbol{\nabla}f_i}_{①} + \left(\underbrace{\frac{\boldsymbol{\nabla}p_i}{m_i n_i}}_{①} + \underbrace{\frac{\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\Pi}_i}{m_i n_i}}_{②} - \underbrace{\frac{\boldsymbol{R}_{ie}}{m_i n_i}}_{①}\right)\boldsymbol{\cdot}\frac{\partial f_i}{\partial\boldsymbol{w}} - \underbrace{(\boldsymbol{w}\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{u}_i)\boldsymbol{\cdot}\frac{\partial f_i}{\partial\boldsymbol{w}}}_{①}$$

$$= \underbrace{-\,\Omega_i(\boldsymbol{w}\times\hat{\boldsymbol{b}})\boldsymbol{\cdot}\frac{\partial f_i}{\partial\boldsymbol{w}}}_{⓪} + \underbrace{C_{ii}[f_i, f_i]}_{⓪} + \underbrace{C_{ie}[f_i, f_e]}_{①}, \qquad \text{(IX.5.31)}$$

where the circled numbers indicate the lowest order in ϵ at which each term appears. (Note that $C_{ie} \sim \nu_{ie} \sim \nu_{ii}\sqrt{m_e/m_i} \sim \epsilon\, C_{ii}$.)

At zeroth order in ϵ, equation (IX.5.31) is

$$0 = -\Omega_i(\boldsymbol{w}\times\hat{\boldsymbol{b}})\boldsymbol{\cdot}\frac{\partial f_{i0}}{\partial\boldsymbol{w}} + C_{ii}[f_{i0}, f_{i0}], \qquad \text{(IX.5.32)}$$

with the condition that $\int \mathrm{d}\boldsymbol{w}\,\boldsymbol{w} f_{i0} = 0$. The solution to (IX.5.32) is a Maxwellian:

$$\boxed{f_{i0} = f_{\mathrm{M}i} \doteq \frac{n_i(t,\boldsymbol{r})}{\pi^{3/2}v_{\mathrm{th}i}^3(t,\boldsymbol{r})}\exp\left[-\frac{|\boldsymbol{v} - \boldsymbol{u}_i(t,\boldsymbol{r})|^2}{v_{\mathrm{th}i}^2(t,\boldsymbol{r})}\right]} \qquad \text{(IX.5.33)}$$

with f_{i1} adjusted so that n_i, $\boldsymbol{u}_i$, and T_i are the true ion density, flow velocity, and temperature (recall the discussion in §IX.2). To ensure this, we require

$$\int \mathrm{d}\boldsymbol{w}\, f_{i1} = \int \mathrm{d}\boldsymbol{w}\, \boldsymbol{w} f_{i1} = \int \mathrm{d}\boldsymbol{w}\, w^2 f_{i1} = 0$$

in order that (IX.5.33) hold for the true moments. In other words, f_{i1} is purely kinetic (as in the electron case).

At first order in ϵ, equation (IX.5.31) is

$$\frac{\mathrm{d}f_{\mathrm{M}i}}{\mathrm{d}t_i} + \boldsymbol{w}\cdot\boldsymbol{\nabla} f_{\mathrm{M}i} + \left(\frac{\boldsymbol{\nabla}p_i}{m_i n_i} - \frac{\boldsymbol{R}_{ie}}{m_i n_i} - \boldsymbol{w}\cdot\boldsymbol{\nabla}\boldsymbol{u}_i\right)\cdot\frac{\partial f_{\mathrm{M}i}}{\partial \boldsymbol{w}}$$
$$= -\Omega_i(\boldsymbol{w}\times\hat{\boldsymbol{b}})\cdot\frac{\partial f_{i,1}}{\partial\boldsymbol{w}} + C^{\ell}_{ii}[f_{i1}] + C_{ie}[f_{\mathrm{M}i}, f_{\mathrm{M}e}], \tag{IX.5.34}$$

where

$$C^{\ell}_{ii}[f_{i1}] = C_{ii}[f_{i1}, f_{\mathrm{M}i}] + C_{ii}[f_{\mathrm{M}i}, f_{i1}]$$

is the linearized ion–ion collision operator (see (VIII.7.6)) and $C_{ie}[f_{\mathrm{M}i}, f_{\mathrm{M}e}]$ is given by (VIII.5.12) written to $\mathcal{O}(\epsilon)$:

$$C_{ie}[f_{\mathrm{M}i}, f_{\mathrm{M}e}] = -\frac{\boldsymbol{R}_{ei}}{p_i}\cdot\boldsymbol{w} f_{\mathrm{M}i} - \frac{2m_e n_e}{m_i n_i}\frac{1}{\tau_{ei}}\left(1-\frac{T_e}{T_i}\right)\left(\frac{w^2}{v^2_{\mathrm{th}i}} - \frac{3}{2}\right) f_{\mathrm{M}i}$$
$$= -\frac{\boldsymbol{R}_{ie}}{m_i n_i}\cdot\frac{\partial f_{\mathrm{M}i}}{\partial\boldsymbol{w}} + \frac{2}{3}\frac{Q_{ie}}{p_i}\left(\frac{w^2}{v^2_{\mathrm{th}i}} - \frac{3}{2}\right) f_{\mathrm{M}i}, \tag{IX.5.35}$$

where we have used $\boldsymbol{R}_{ie} = -\boldsymbol{R}_{ei}$ and (VIII.5.16) to obtain the final line. We can simplify some terms in (IX.5.34) by using the following:

- $\displaystyle \frac{\mathrm{d}f_{\mathrm{M}i}}{\mathrm{d}t_i} = \left[\frac{\mathrm{d}\ln n_i}{\mathrm{d}t_i} + \left(\frac{w^2}{v^2_{\mathrm{th}i}} - \frac{3}{2}\right)\frac{\mathrm{d}\ln T_i}{\mathrm{d}t_i}\right] f_{\mathrm{M}i}\ ;$
- $\displaystyle \frac{\mathrm{d}\ln n_i}{\mathrm{d}t_i} = -\boldsymbol{\nabla}\cdot\boldsymbol{u}_i, \quad \frac{\mathrm{d}\ln T_i}{\mathrm{d}t_i} = -\frac{2}{3}\boldsymbol{\nabla}\cdot\boldsymbol{u}_i + \frac{2}{3}\frac{Q_{ie}}{p_i}\ ;$
- $\displaystyle \boldsymbol{w}\cdot\boldsymbol{\nabla} f_{\mathrm{M}i} = \left[\boldsymbol{w}\cdot\boldsymbol{\nabla}\ln p_i + \left(\frac{w^2}{v^2_{\mathrm{th}i}} - \frac{5}{2}\right)\boldsymbol{w}\cdot\boldsymbol{\nabla}\ln T_i\right] f_{\mathrm{M}i}\ ;$
- $\displaystyle \frac{\partial f_{\mathrm{M}i}}{\partial\boldsymbol{w}} = -\frac{2\boldsymbol{w}}{v^2_{\mathrm{th}i}} f_{\mathrm{M}i}\ .$

Thus, equation (IX.5.34) becomes

$$f_{\mathrm{M}i}\left[\left(\frac{w^2}{v^2_{\mathrm{th}i}} - \frac{5}{2}\right)\boldsymbol{w}\cdot\boldsymbol{\nabla}\ln T_i + \left(\frac{\boldsymbol{w}\boldsymbol{w}}{v^2_{\mathrm{th}i}} - \frac{\mathsf{I}}{3}\frac{w^2}{v^2_{\mathrm{th}i}}\right) : \mathsf{W}_i\right] = \Omega_i\frac{\partial f_{i1}}{\partial\vartheta} + C^{\ell}_{ii}[f_{i1}]. \tag{IX.5.36}$$

As before, the solution to (IX.5.36) must have the form

$$f_{i1} = \underbrace{\langle f_{i1}\rangle_\vartheta}_{\substack{\text{gyrophase-}\\ \text{independent}\\ \text{piece}}} + \underbrace{\widetilde{f}_{i1}}_{\substack{\text{gyrophase-}\\ \text{dependent}\\ \text{piece}}}\ . \tag{IX.5.37}$$

Note that $\langle C^{\ell}_{ii}[f_{i1}]\rangle_\vartheta = C^{\ell}_{ii}[\langle f_{i1}\rangle_\vartheta]$. Then, using $\langle\boldsymbol{w}\rangle_\vartheta = w_\parallel\hat{\boldsymbol{b}}$ and $\langle\boldsymbol{w}\boldsymbol{w}\rangle_\vartheta = w^2_\parallel\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} +$

$(w_\perp^2/2)(\mathsf{I} - \hat{\boldsymbol{b}}\hat{\boldsymbol{b}})$, the gyro-average of (IX.5.36) is

$$f_{\mathrm{M}i}\left[\left(\frac{w^2}{v_{\mathrm{th}i}^2} - \frac{5}{2}\right)w_\parallel \hat{\boldsymbol{b}}\cdot\boldsymbol{\nabla}\ln T_i + \left(\frac{w_\parallel^2 - w_\perp^2/2}{v_{\mathrm{th}i}^2}\right)\left(\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} - \frac{\mathsf{I}}{3}\right) : \mathsf{W}_i\right] = C_{ii}^\ell[\langle f_{i1}\rangle_\vartheta]. \quad \text{(IX.5.38)}$$

To solve this equation, we must invert the collision operator on its right-hand side.

Before doing so, note that the difference between (IX.5.38) and its electron equivalent, equation (IX.5.13), is the second (viscous) term on its left-hand side. The addition of this term requires the use of both $L_k^{(3/2)}$ and $L_k^{(5/2)}$ as our basis functions, where the latter satisfy the orthogonality relation

$$\int_0^\infty \mathrm{d}x\, x^{5/2}\mathrm{e}^{-x}L_p^{(5/2)}(x)L_q^{(5/2)}(x) = \frac{\Gamma(p+7/2)}{\Gamma(p+1)}\delta_{pq}.$$

Thus

$$\langle f_{i1}\rangle_\vartheta = \frac{2w_\parallel}{v_{\mathrm{th}i}^2}f_{\mathrm{M}i}\sum_{k=1}^N u_{i,k}L_k^{(3/2)}\left(\frac{w^2}{v_{\mathrm{th}i}^2}\right) + \frac{2}{3}\left(\frac{w_\parallel^2 - w_\perp^2/2}{p_i v_{\mathrm{th}i}^2}\right)f_{\mathrm{M}i}\sum_{k=0}^N \pi_{i,k}L_k^{(5/2)}\left(\frac{w^2}{v_{\mathrm{th}i}^2}\right) \quad \text{(IX.5.39}a\text{)}$$

$$\doteq \langle f_{i1}\rangle_\vartheta^q + \langle f_{i1}\rangle_\vartheta^\pi. \quad \text{(IX.5.39}b\text{)}$$

(Once again, note that the 3/2-Laguerre sum starts at $k = 1$ since $u_{i,0} = 0$; i.e., there is no parallel mean flow in f_{i1}, since it's in f_{i0} by construction.) This expansion works because the viscous stress associated with $\langle f_{i1}\rangle_\vartheta^\pi$ is

$$\begin{aligned}
\int \mathrm{d}\boldsymbol{w}\, m_i\left(\boldsymbol{w}\boldsymbol{w} - \frac{\mathsf{I}}{3}w^2\right)\langle f_{i1}\rangle_\vartheta^\pi &= \int \mathrm{d}\boldsymbol{w}\, m_i\left\langle \boldsymbol{w}\boldsymbol{w} - \frac{\mathsf{I}}{3}w^2\right\rangle_\vartheta \langle f_{i1}\rangle_\vartheta^\pi \\
&= \left(\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} - \frac{\mathsf{I}}{3}\right)\int \mathrm{d}\boldsymbol{w}\, m_i\left(w_\parallel^2 - \frac{w_\perp^2}{2}\right)\langle f_{i1}\rangle_\vartheta^\pi \\
&= \left(\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} - \frac{\mathsf{I}}{3}\right)\int \mathrm{d}\boldsymbol{w}\, m_i\left(w_\parallel^2 - \frac{w_\perp^2}{2}\right)\frac{2}{3}\left(\frac{w_\parallel^2 - w_\perp^2/2}{p_i v_{\mathrm{th}i}^2}\right)f_{\mathrm{M}i}\sum_{k=0}^N \pi_{i,k}L_k^{(5/2)}\left(\frac{w^2}{v_{\mathrm{th}i}^2}\right) \\
&= \left(\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} - \frac{\mathsf{I}}{3}\right)\frac{4}{3\sqrt{\pi}}\int_{-1}^{+1}\mathrm{d}\xi\left(\frac{3\xi^2 - 1}{2}\right)^2\sum_{k=0}^N \pi_{i,k}\int_0^\infty \mathrm{d}x\, x^{5/2}\mathrm{e}^{-x}\underbrace{L_0^{(5/2)}(x)}_{\substack{\text{can insert}\\ \text{here, since}\\ \text{it equals 1}}}L_k^{(5/2)}(x) \\
&= \left(\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} - \frac{\mathsf{I}}{3}\right)\frac{4}{3\sqrt{\pi}}\times\frac{2}{5}\sum_{k=0}^N \pi_{i,k}\,\frac{15\sqrt{\pi}}{8}\delta_{k0} \\
&= \left(\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} - \frac{\mathsf{I}}{3}\right)\pi_{i,0},
\end{aligned}$$

i.e., the zeroth Laguerre coefficient is the parallel component of the viscous stress tensor. Good. Before proceeding, it is also worth noting that

$$w_\parallel^2 - \frac{w_\perp^2}{2} = \frac{3\xi^2 - 1}{2}w^2 = P_2(\xi)w^2,$$

where P_2 is the second Legendre polynomial, so that

$$\langle f_{i1}\rangle_\vartheta^\pi = \frac{2}{3}xP_2(\xi)f_{\mathrm{M}i}\sum_{k=0}^N \frac{\pi_{i,k}}{p_i}L_k^{(5/2)}(x),$$

The factor of 2/3 provides the proper normalization of the second Legendre polynomial: recall that

$$\int_{-1}^{+1} \mathrm{d}\xi\, P_p(\xi) P_q(\xi) = \frac{2}{2p+1}\delta_{pq}.$$

Thus, equation (IX.5.39) may be written

$$\langle f_{i1}\rangle_\vartheta^q + \langle f_{i1}\rangle_\vartheta^\pi = 2xP_1(\xi) f_{\mathrm{M}i} \sum_{k=1}^N \frac{u_{i,k}}{v_{\mathrm{th}i}} L_k^{(3/2)}(x) + \frac{2}{3}xP_2(\xi) f_{\mathrm{M}i} \sum_{k=0}^N \frac{\pi_{i,k}}{p_i} L_k^{(5/2)}(x). \tag{IX.5.40}$$

If you're seeing a pattern here, you might be interested to read Grad (1949*a*,*b*), in which the distribution function is expanded in terms of Laguerre-polynomial energy moments and Legendre-polynomial angular moments of the departures of the lowest-order distribution function from a Maxwellian. (His expansion was in terms of Hermite polynomials, but this is equivalent.)

The expansion (IX.5.39) means that we need the linearized collision operator written out in terms of $L_k^{(5/2)}$ polynomials. Before doing so, let's handle the heat-flux term first ($\langle f_{i1}\rangle_\vartheta^q$) and then worry about the viscous term ($\langle f_{i1}\rangle_\vartheta^\pi$). In order to calculate the ion heat flux from (IX.5.38), we must solve

$$\int \mathrm{d}\boldsymbol{w}\, w_\parallel L_k^{(3/2)}(x) f_{\mathrm{M}i}\left(\frac{w^2}{v_{\mathrm{th}i}^2} - \frac{5}{2}\right) w_\parallel \hat{\boldsymbol{b}}\cdot\boldsymbol{\nabla}\ln T_i = \frac{n_i}{\tau_{ii}} \begin{bmatrix} \sqrt{2} & \dfrac{3\sqrt{2}}{4} \\ \dfrac{3\sqrt{2}}{4} & \dfrac{45\sqrt{2}}{16} \end{bmatrix} \begin{bmatrix} u_{i,1} \\ u_{i,2} \end{bmatrix}. \tag{IX.5.41}$$

The left-hand side of this equation is equal to

$$-\frac{4}{3\sqrt{\pi}}\frac{p_i}{m_i}\hat{\boldsymbol{b}}\cdot\boldsymbol{\nabla}\ln T_i \times \delta_{k1}\frac{15\sqrt{\pi}}{8},$$

and so, inverting the 2×2 matrix on the right-hand side of (IX.5.41), we find

$$u_{i,1} = \frac{5}{4}\frac{\sqrt{2}\tau_{ii}}{m_i}\hat{\boldsymbol{b}}\cdot\boldsymbol{\nabla}T_i \times \left(\frac{5}{4}\right), \tag{IX.5.42}$$

$$u_{i,2} = \frac{5}{4}\frac{\sqrt{2}\tau_{ii}}{m_i}\hat{\boldsymbol{b}}\cdot\boldsymbol{\nabla}T_i \times \left(-\frac{1}{3}\right). \tag{IX.5.43}$$

Thus,

$$\boxed{\langle f_{i1}\rangle_\vartheta^q = w_\parallel f_{\mathrm{M}i}\tau_i \frac{5}{4}\hat{\boldsymbol{b}}\cdot\boldsymbol{\nabla}\ln T_i \left[\frac{5}{4}L_1^{(3/2)}\left(\frac{w^2}{v_{\mathrm{th}i}^2}\right) - \frac{1}{3}L_2^{(3/2)}\left(\frac{w^2}{v_{\mathrm{th}i}^2}\right)\right],} \tag{IX.5.44}$$

where $\tau_i \doteq \sqrt{2}\tau_{ii}$.

Now the viscous term. Consulting Hirshman & Sigmar (1981, equations (4.31)–(4.40)), the linearized ion–ion collision operator written in terms of the $L_k^{(5/2)}(x)$ polynomials is (to the order we're seeking)

$$-\frac{6}{5}\frac{n_i}{\tau_i}\begin{bmatrix} 1 & 3/4 \\ 3/4 & 205/48 \end{bmatrix}\begin{bmatrix} \pi_{i,1} \\ \pi_{i,2} \end{bmatrix}. \tag{IX.5.45}$$

So, we must solve

$$-\frac{6}{5}\frac{n_i}{\tau_i}\begin{bmatrix}1 & 3/4\\ 3/4 & 205/48\end{bmatrix}\begin{bmatrix}\pi_{i,1}\\ \pi_{i,2}\end{bmatrix} = m_i n_i \int \mathrm{d}\boldsymbol{w}\left(w_\parallel^2 - \frac{w_\perp^2}{2}\right)L_k^{(5/2)}(x) \times f_{\mathrm{M}i}\left(\frac{w_\parallel^2 - w_\perp^2/2}{v_{\mathrm{th}i}^2}\right)\left(\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} - \frac{\mathsf{I}}{3}\right) : \boldsymbol{W}_i. \tag{IX.5.46}$$

Following through with the velocity-space integration, the right-hand side of (IX.5.46) is

$$= \frac{3}{2}p_i n_i \left(\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} - \frac{\mathsf{I}}{3}\right) : \boldsymbol{W}_i\,\delta_{k0}.$$

Inverting the 2×2 matrix on the left-hand side of (IX.5.46) thus obtains

$$\pi_{i,0} = -\frac{1025}{712}p_i\tau_i\left(\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} - \frac{\mathsf{I}}{3}\right) : \boldsymbol{W}_i \tag{IX.5.47}$$

$$\pi_{i,1} = \frac{45}{178}p_i\tau_i\left(\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} - \frac{\mathsf{I}}{3}\right) : \boldsymbol{W}_i, \tag{IX.5.48}$$

so that

$$\boxed{\langle f_{i1}\rangle_\vartheta^\pi = -\left(\frac{w_\parallel^2 - w_\perp^2/2}{v_{\mathrm{th}i}^2}\right)f_{\mathrm{M}i}\tau_i\left(\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} - \frac{\mathsf{I}}{3}\right) : \boldsymbol{W}_i\left[\frac{1025}{1068}L_0^{(5/2)}(x) - \frac{45}{267}L_1^{(5/2)}(x)\right]} \tag{IX.5.49}$$

Next we calculate $\widetilde{f}_{i1}$, the gyrophase-dependent piece. Do (IX.5.36) minus (IX.5.38):

$$f_{\mathrm{M}i}\left(\frac{w^2}{v_{\mathrm{th}i}^2} - \frac{5}{2}\right)\boldsymbol{w}_\perp\cdot\boldsymbol{\nabla}\ln T_i + f_{\mathrm{M}i}\left[\frac{\boldsymbol{w}_\parallel\boldsymbol{w}_\perp + \boldsymbol{w}_\perp\boldsymbol{w}_\parallel + \boldsymbol{w}_\perp\boldsymbol{w}_\perp - (w_\perp^2/2)(\mathsf{I} - \hat{\boldsymbol{b}}\hat{\boldsymbol{b}})}{v_{\mathrm{th}i}^2}\right] : \boldsymbol{W}_i = \Omega_i\frac{\partial\widetilde{f}_{i1}}{\partial\vartheta} + C_{ii}^\ell[\widetilde{f}_{i1}]. \tag{IX.5.50}$$

Now we impose our subsidiary expansion (IX.4.5) on (IX.5.50). To lowest order, the only terms in (IX.5.50) that survive are

$$f_{\mathrm{M}i}\left(\frac{w^2}{v_{\mathrm{th}i}^2} - \frac{5}{2}\right)\boldsymbol{w}_\perp\cdot\boldsymbol{\nabla}\ln T_i + f_{\mathrm{M}i}\Big[\dots\Big] : \boldsymbol{W}_i = \Omega_i\frac{\partial\widetilde{f}_{i1}^{(0)}}{\partial\vartheta}, \tag{IX.5.51}$$

where we have written $\widetilde{f}_{i1} = \widetilde{f}_{i1}^{(0)} + \widetilde{f}_{i1}^{(1)} + \dots$. To next order,

$$0 = \Omega_i\frac{\partial\widetilde{f}_{i1}^{(1)}}{\partial\vartheta} + C_{ii}^\ell[\widetilde{f}_{i1}^{(0)}]. \tag{IX.5.52}$$

Equation (IX.5.51) becomes, upon integrating over ϑ,

$$\boxed{\begin{aligned}\widetilde{f}_{i1}^{(0)} &= f_{\mathrm{M}i}\,L_1^{(3/2)}(x)\frac{\boldsymbol{w}_\perp\times\hat{\boldsymbol{b}}}{\Omega_i}\cdot\boldsymbol{\nabla}\ln T_i + f_{\mathrm{M}i}\frac{\boldsymbol{w}_\parallel\boldsymbol{w}_\perp\times\hat{\boldsymbol{b}} + \boldsymbol{w}_\perp\times\hat{\boldsymbol{b}}\boldsymbol{w}_\parallel}{v_{\mathrm{th}i}^2} : \frac{\boldsymbol{W}_i}{\Omega_i}\\ &\quad + f_{\mathrm{M}i}\frac{w_\perp^2}{4v_{\mathrm{th}i}^2}\big[\sin 2\vartheta(\hat{\boldsymbol{x}}\hat{\boldsymbol{x}} - \hat{\boldsymbol{y}}\hat{\boldsymbol{y}}) - \cos 2\vartheta(\hat{\boldsymbol{x}}\hat{\boldsymbol{y}} + \hat{\boldsymbol{y}}\hat{\boldsymbol{x}})\big] : \frac{\boldsymbol{W}_i}{\Omega_i}.\end{aligned}} \tag{IX.5.53}$$

As before, we need not solve (IX.5.52) for $\widetilde{f}_{i1}^{(1)}$, because an integration by parts will move a $\partial/\partial\vartheta$ onto $\widetilde{f}_{i1}^{(1)}$ when computing moments. Now that we have f_{i1}, let's compute some moments.

First, the ion heat flux:

$$\begin{aligned}
\boldsymbol{q}_i &= \int \mathrm{d}\boldsymbol{w}\, \boldsymbol{w} \frac{1}{2} m_i w^2 f_i \simeq \int \mathrm{d}\boldsymbol{w}\, \boldsymbol{w} \frac{1}{2} m_i w^2 f_{i1} \\
&= T_i \int \mathrm{d}\boldsymbol{w}\, \boldsymbol{w} \left(\frac{w^2}{v_{\mathrm{th}i}^2} - \frac{5}{2} \right) f_{i1} \quad \left(\text{since } \textstyle\int \mathrm{d}\boldsymbol{w}\, \boldsymbol{w} f_{i1} = 0\right) \\
&= -T_i \int \mathrm{d}\boldsymbol{w}\, \boldsymbol{w} L_1^{(3/2)}\left(\frac{w^2}{v_{\mathrm{th}i}^2} \right) \Bigl(\underbrace{\langle f_{i1} \rangle_\vartheta}_{\substack{\text{gives}\\ \boldsymbol{q}_{\parallel i}}} + \underbrace{\widetilde{f}_{i1}^{(0)}}_{\substack{\text{gives}\\ \boldsymbol{q}_{\times i}}} + \underbrace{\widetilde{f}_{i1}^{(1)}}_{\substack{\text{gives}\\ \boldsymbol{q}_{\perp i}}} + \underbrace{\quad \dots \quad}_{\substack{\text{higher-}\\ \text{order}\\ \text{stuff}}} \Bigr)
\end{aligned} \tag{IX.5.54}$$

Using (IX.5.44), the *parallel ion heat flux* is (with $x \doteq w^2/v_{\mathrm{th}i}^2$)

$$\begin{aligned}
\boldsymbol{q}_{\parallel i} &\doteq -T_i \int \mathrm{d}\boldsymbol{w}\, \boldsymbol{w} L_1^{(3/2)}(x) \langle f_{i1} \rangle_\vartheta \\
&= -T_i \int \mathrm{d}\boldsymbol{w}\, \langle \boldsymbol{w} \rangle_\vartheta\, L_1^{(3/2)}(x) \langle f_{i1} \rangle_\vartheta = -T_i \int \mathrm{d}\boldsymbol{w}\, w_\parallel \hat{\boldsymbol{b}}\, L_1^{(3/2)}(x) \langle f_{i1} \rangle_\vartheta \\
&= -\frac{4\pi}{3} \int \mathrm{d}w\, w^4 \left[L_1^{(3/2)}(x) \right]^2 f_{\mathrm{M}i} \left[\frac{25}{16} \tau_i \hat{\boldsymbol{b}}\hat{\boldsymbol{b}} \boldsymbol{\cdot} \boldsymbol{\nabla} T_i \right] \\
&= -\frac{4\pi}{3} \frac{n_e}{\pi\sqrt{\pi}} \frac{v_{\mathrm{th}e}^2}{2} \left[\frac{25}{16} \tau_i \hat{\boldsymbol{b}}\hat{\boldsymbol{b}} \boldsymbol{\cdot} \boldsymbol{\nabla} T_i \right] \int_0^\infty \mathrm{d}x\, x^{3/2} \mathrm{e}^{-x} \left[L_1^{(3/2)}(x) \right]^2 \\
&= -\frac{4}{3\sqrt{\pi}} \frac{p_e}{m_e} \left[\frac{25}{16} \tau_i \hat{\boldsymbol{b}}\hat{\boldsymbol{b}} \boldsymbol{\cdot} \boldsymbol{\nabla} T_i \right] \times \frac{15\sqrt{\pi}}{8} \\
&= -\underbrace{\frac{125}{32}}_{\simeq 3.91} \frac{p_i \tau_i}{m_i} \hat{\boldsymbol{b}}\hat{\boldsymbol{b}} \boldsymbol{\cdot} \boldsymbol{\nabla} T_i .
\end{aligned} \tag{IX.5.55}$$

Using (IX.5.53), the *diamagnetic ion heat flux* is

$$\begin{aligned}
\boldsymbol{q}_{\times i} &= -T_i \int \mathrm{d}\boldsymbol{w}\, \boldsymbol{w}_\perp L_1^{(3/2)}(x) \widetilde{f}_{i1}^{(0)} \\
&= \int \mathrm{d}\boldsymbol{w}\, \boldsymbol{w}_\perp \left[L_1^{(3/2)}(x) \right]^2 f_{\mathrm{M}i} \frac{\boldsymbol{w}_\perp \times \hat{\boldsymbol{b}}}{\Omega_i} \boldsymbol{\cdot} \boldsymbol{\nabla} T_i \\
&= \frac{1}{\Omega_i} \int \mathrm{d}\boldsymbol{w}\, \langle \boldsymbol{w}_\perp \boldsymbol{w}_\perp \rangle_\vartheta \left[L_1^{(3/2)}(x) \right]^2 f_{\mathrm{M}i} \times \hat{\boldsymbol{b}} \boldsymbol{\cdot} \boldsymbol{\nabla} T_i \\
&= \frac{1}{\Omega_i} \frac{4\pi}{3} \frac{n_i}{\pi\sqrt{\pi}} \frac{v_{\mathrm{th}i}^2}{2} \left(\boldsymbol{I} - \hat{\boldsymbol{b}}\hat{\boldsymbol{b}} \right) \times \hat{\boldsymbol{b}} \boldsymbol{\cdot} \boldsymbol{\nabla} T_i \int \mathrm{d}x\, x^{3/2} \mathrm{e}^{-x} \left[L_1^{(3/2)}(x) \right]^2 \\
&= \frac{1}{\Omega_i} \frac{4}{3\sqrt{\pi}} \frac{p_i}{m_i} \hat{\boldsymbol{b}} \times \boldsymbol{\nabla} T_i \times \frac{15\sqrt{\pi}}{8} \\
&= \frac{5}{2} \frac{p_i}{m_i \Omega_i} \hat{\boldsymbol{b}} \times \boldsymbol{\nabla} T_i .
\end{aligned} \tag{IX.5.56}$$

Finally, using (IX.5.52), the *perpendicular ion heat flux* is

$$
\begin{aligned}
\boldsymbol{q}_{\perp i} &= -T_i \int \mathrm{d}\boldsymbol{w}\, \boldsymbol{w}_\perp L_1^{(3/2)}(x) \widetilde{f}_{i1}^{(1)} \\
&= -T_i \int \mathrm{d}\boldsymbol{w}\, \frac{\partial(\boldsymbol{w}_\perp \times \hat{\boldsymbol{b}})}{\partial \vartheta} L_1^{(3/2)}(x) \widetilde{f}_{i1}^{(1)} \\
&\overset{\text{bp}}{=} T_i \int \mathrm{d}\boldsymbol{w}\, (\boldsymbol{w}_\perp \times \hat{\boldsymbol{b}}) L_1^{(3/2)}(x) \frac{\partial \widetilde{f}_{i1}^{(1)}}{\partial \vartheta} \\
&= -T_i \int \mathrm{d}\boldsymbol{w}\, \frac{\boldsymbol{w}_\perp \times \hat{\boldsymbol{b}}}{\Omega_i} L_1^{(3/2)}(x)\, C_{ii}^{\ell}[\widetilde{f}_{i1}^{(0)}] \\
&= \dots \text{crunch, crunch, crunch} \dots \\
&= -2\, \frac{p_i \tau_i^{-1}}{m_i \Omega_i^2} \boldsymbol{\nabla}_\perp T_i .
\end{aligned}
\tag{IX.5.57}
$$

Next, the ion viscosity:

$$
\begin{aligned}
\boldsymbol{\Pi}_i &= \int \mathrm{d}\boldsymbol{w}\, m_i \left(\boldsymbol{w}\boldsymbol{w} - \frac{\mathsf{I}}{3} w^2\right) f_i \simeq \int \mathrm{d}\boldsymbol{w}\, m_i \left(\boldsymbol{w}\boldsymbol{w} - \frac{\mathsf{I}}{3} w^2\right) f_{i1} \\
&= \left(\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} - \frac{\mathsf{I}}{3}\right) \underbrace{\int \mathrm{d}\boldsymbol{w}\, m_i \left(w_\parallel^2 - \frac{w_\perp^2}{2}\right) \langle f_{i1} \rangle_\vartheta}_{=\,\Pi_{\parallel i}} \\
&\quad + \int \mathrm{d}\boldsymbol{w}\, m_i \left(\boldsymbol{w}\boldsymbol{w} - \frac{\mathsf{I}}{3} w^2\right) \Big(\underbrace{\widetilde{f}_{i1}^{(0)}}_{\substack{\text{gives}\\ \boldsymbol{\Pi}_{\times i}}} + \underbrace{\widetilde{f}_{i1}^{(1)}}_{\substack{\text{gives}\\ \boldsymbol{\Pi}_{\perp i}}} + \underbrace{\quad\dots\quad}_{\substack{\text{higher-}\\ \text{order}\\ \text{stuff}}} \Big)
\end{aligned}
\tag{IX.5.58}
$$

We already found the *parallel ion viscosity* $\Pi_{\parallel i} = \pi_{i,0}$:

$$
\Pi_{\parallel i} = -\underbrace{\frac{1025}{1068}}_{\simeq 0.96} p_i \tau_i \frac{3}{2} \left(\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} - \frac{\mathsf{I}}{3}\right) : \mathsf{W}_i .
\tag{IX.5.59}
$$

Using (IX.5.53), the *diamagnetic ion viscosity* is

$$
\begin{aligned}
\boldsymbol{\Pi}_{\times i} &= \int \mathrm{d}\boldsymbol{w}\, m_i \left(\boldsymbol{w}\boldsymbol{w} - \frac{\boldsymbol{\mathsf{I}}}{3} w^2 \right) \widetilde{f}_{i1}^{(0)} \\
&= \int \mathrm{d}\boldsymbol{w}\, m_i \big(\boldsymbol{w}_\parallel \boldsymbol{w}_\perp + \boldsymbol{w}_\perp \boldsymbol{w}_\parallel + \boldsymbol{w}_\perp \boldsymbol{w}_\perp \big) f_{\mathrm{M}i} \\
&\quad \times \left\{ \frac{\boldsymbol{w}_\parallel \boldsymbol{w}_\perp \times \hat{\boldsymbol{b}} + \boldsymbol{w}_\perp \times \hat{\boldsymbol{b}} \boldsymbol{w}_\parallel}{v_{\mathrm{th}i}^2} + \frac{w_\perp^2}{4 v_{\mathrm{th}i}^2} \big[\sin 2\vartheta (\hat{\boldsymbol{x}}\hat{\boldsymbol{x}} - \hat{\boldsymbol{y}}\hat{\boldsymbol{y}}) - \cos 2\vartheta (\hat{\boldsymbol{x}}\hat{\boldsymbol{y}} + \hat{\boldsymbol{y}}\hat{\boldsymbol{x}}) \big] \right\} : \frac{\boldsymbol{\mathsf{W}}_i}{\Omega_i} \\
&= \ldots \text{crunch, crunch, crunch} \ldots \\
&= \frac{p_i}{4\Omega_i} \Big[2 \big(\hat{\boldsymbol{b}}\hat{\boldsymbol{y}} + \hat{\boldsymbol{y}}\hat{\boldsymbol{b}} \big) \big(\hat{\boldsymbol{b}}\hat{\boldsymbol{x}} + \hat{\boldsymbol{x}}\hat{\boldsymbol{b}} \big) - 2 \big(\hat{\boldsymbol{b}}\hat{\boldsymbol{x}} + \hat{\boldsymbol{x}}\hat{\boldsymbol{b}} \big) \big(\hat{\boldsymbol{b}}\hat{\boldsymbol{y}} + \hat{\boldsymbol{y}}\hat{\boldsymbol{b}} \big) \\
&\qquad\quad + \big(\hat{\boldsymbol{x}}\hat{\boldsymbol{x}} - \hat{\boldsymbol{y}}\hat{\boldsymbol{y}} \big) \big(\hat{\boldsymbol{x}}\hat{\boldsymbol{y}} + \hat{\boldsymbol{y}}\hat{\boldsymbol{x}} \big) + \big(\hat{\boldsymbol{x}}\hat{\boldsymbol{y}} + \hat{\boldsymbol{y}}\hat{\boldsymbol{x}} \big) \big(\hat{\boldsymbol{x}}\hat{\boldsymbol{x}} - \hat{\boldsymbol{y}}\hat{\boldsymbol{y}} \big) \Big] : \boldsymbol{\mathsf{W}}_i \\
&= \frac{p_i}{4\Omega_i} \begin{bmatrix} \mathsf{W}_{xy} + \mathsf{W}_{yx} & \mathsf{W}_{xx} - \mathsf{W}_{yy} & -2(\mathsf{W}_{zy} + \mathsf{W}_{yz}) \\ \mathsf{W}_{xx} - \mathsf{W}_{yy} & -(\mathsf{W}_{xy} + \mathsf{W}_{yx}) & 2(\mathsf{W}_{zx} + \mathsf{W}_{xz}) \\ -2(\mathsf{W}_{zy} + \mathsf{W}_{yz}) & 2(\mathsf{W}_{zx} + \mathsf{W}_{xz}) & 0 \end{bmatrix} \\
&= \frac{p_i}{4\Omega_i} \Big[\hat{\boldsymbol{b}} \times \boldsymbol{\mathsf{W}}_i \boldsymbol{\cdot} \big(\boldsymbol{\mathsf{I}} + 3\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} \big) - \big(\boldsymbol{\mathsf{I}} + 3\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} \big) \boldsymbol{\cdot} \boldsymbol{\mathsf{W}}_i \times \hat{\boldsymbol{b}} \Big]. \qquad \text{(IX.5.60)}
\end{aligned}
$$

Finally, using (IX.5.52), the *perpendicular ion viscosity* is

$$
\begin{aligned}
\boldsymbol{\Pi}_{\perp i} &= \int \mathrm{d}\boldsymbol{w}\, m_i \left(\boldsymbol{w}\boldsymbol{w} - \frac{\boldsymbol{\mathsf{I}}}{3} w^2 \right) \widetilde{f}_{i1}^{(1)} \\
&= \int \mathrm{d}\boldsymbol{w}\, m_i \big(\boldsymbol{w}_\parallel \boldsymbol{w}_\perp + \boldsymbol{w}_\perp \boldsymbol{w}_\parallel + \boldsymbol{w}_\perp \boldsymbol{w}_\perp \big) \widetilde{f}_{i1}^{(1)} \\
&= \int \mathrm{d}\boldsymbol{w}\, m_i \frac{\partial}{\partial \vartheta} \big(\boldsymbol{w}_\parallel \boldsymbol{w}_\perp \times \hat{\boldsymbol{b}} + \boldsymbol{w}_\perp \times \hat{\boldsymbol{b}} \boldsymbol{w}_\parallel \big) \widetilde{f}_{i1}^{(1)} \\
&\quad + \int \mathrm{d}\boldsymbol{w}\, m_i \frac{w_\perp^2}{4} \frac{\partial}{\partial \vartheta} \big[\sin 2\vartheta (\hat{\boldsymbol{x}}\hat{\boldsymbol{x}} - \hat{\boldsymbol{y}}\hat{\boldsymbol{y}}) - \cos 2\vartheta (\hat{\boldsymbol{x}}\hat{\boldsymbol{y}} + \hat{\boldsymbol{y}}\hat{\boldsymbol{x}}) \big] \widetilde{f}_{i1}^{(1)} \\
&\overset{\text{bp}}{=} - \int \mathrm{d}\boldsymbol{w}\, m_i \big(\boldsymbol{w}_\parallel \boldsymbol{w}_\perp \times \hat{\boldsymbol{b}} + \boldsymbol{w}_\perp \times \hat{\boldsymbol{b}} \boldsymbol{w}_\parallel \big) \frac{\partial \widetilde{f}_{i1}^{(1)}}{\partial \vartheta} \\
&\quad - \int \mathrm{d}\boldsymbol{w}\, m_i \frac{w_\perp^2}{4} \big[\sin 2\vartheta (\hat{\boldsymbol{x}}\hat{\boldsymbol{x}} - \hat{\boldsymbol{y}}\hat{\boldsymbol{y}}) - \cos 2\vartheta (\hat{\boldsymbol{x}}\hat{\boldsymbol{y}} + \hat{\boldsymbol{y}}\hat{\boldsymbol{x}}) \big] \frac{\partial \widetilde{f}_{i1}^{(1)}}{\partial \vartheta} \\
&= \int \mathrm{d}\boldsymbol{w}\, m_i \left(\boldsymbol{w}_\parallel \frac{\boldsymbol{w}_\perp \times \hat{\boldsymbol{b}}}{\Omega_i} + \frac{\boldsymbol{w}_\perp \times \hat{\boldsymbol{b}}}{\Omega_i} \boldsymbol{w}_\parallel \right) C_{ii}^{\ell} [\widetilde{f}_{i1}^{(0)}] \\
&\quad \int \mathrm{d}\boldsymbol{w}\, m_i \frac{w_\perp^2}{4\Omega_i} \big[\sin 2\vartheta (\hat{\boldsymbol{x}}\hat{\boldsymbol{x}} - \hat{\boldsymbol{y}}\hat{\boldsymbol{y}}) - \cos 2\vartheta (\hat{\boldsymbol{x}}\hat{\boldsymbol{y}} + \hat{\boldsymbol{y}}\hat{\boldsymbol{x}}) \big] C_{ii}^{\ell} [\widetilde{f}_{i1}^{(0)}].
\end{aligned}
$$

Insert (IX.5.53) for $\widetilde{f}_{i1}^{(0)}$ to obtain

$$\begin{aligned}
\boldsymbol{\mathit{\Pi}}_{\perp i} &= -\int \mathrm{d}\boldsymbol{w}\, m_i \left(\boldsymbol{w}_\parallel \frac{\boldsymbol{w}_\perp \times \hat{\boldsymbol{b}}}{\mathit{\Omega}_i} + \frac{\boldsymbol{w}_\perp \times \hat{\boldsymbol{b}}}{\mathit{\Omega}_i} \boldsymbol{w}_\parallel \right) \frac{6}{5} \frac{f_{\mathrm{M}i}}{\tau_i} \left(\frac{\boldsymbol{w}_\parallel \boldsymbol{w}_\perp \times \hat{\boldsymbol{b}} + \boldsymbol{w}_\perp \times \hat{\boldsymbol{b}} \boldsymbol{w}_\parallel}{v_{\mathrm{th}i}^2} \right) : \frac{\boldsymbol{W}_i}{\mathit{\Omega}_i} \\
&\quad - \int \mathrm{d}\boldsymbol{w}\, m_i \frac{w_\perp^2}{4\mathit{\Omega}_i} \left[\sin 2\vartheta (\hat{\boldsymbol{x}}\hat{\boldsymbol{x}} - \hat{\boldsymbol{y}}\hat{\boldsymbol{y}}) - \cos 2\vartheta (\hat{\boldsymbol{x}}\hat{\boldsymbol{y}} + \hat{\boldsymbol{y}}\hat{\boldsymbol{x}}) \right] \\
&\quad \times \frac{6}{5} \frac{f_{\mathrm{M}i}}{\tau_i} \frac{w_\perp^2}{4 v_{\mathrm{th}i}^2} \left[\sin 2\vartheta (\hat{\boldsymbol{x}}\hat{\boldsymbol{x}} - \hat{\boldsymbol{y}}\hat{\boldsymbol{y}}) - \cos 2\vartheta (\hat{\boldsymbol{x}}\hat{\boldsymbol{y}} + \hat{\boldsymbol{y}}\hat{\boldsymbol{x}}) \right] : \frac{\boldsymbol{W}_i}{\mathit{\Omega}_i} \\
&= \dots \text{crunch, crunch, crunch} \dots \\
&= -\frac{3}{5} \frac{p_i \tau_i^{-1}}{\mathit{\Omega}_i^2} \Big[\big(\hat{\boldsymbol{b}}\hat{\boldsymbol{y}} + \hat{\boldsymbol{y}}\hat{\boldsymbol{b}}\big)\big(\hat{\boldsymbol{b}}\hat{\boldsymbol{y}} + \hat{\boldsymbol{y}}\hat{\boldsymbol{b}}\big) + \big(\hat{\boldsymbol{b}}\hat{\boldsymbol{x}} + \hat{\boldsymbol{x}}\hat{\boldsymbol{b}}\big)\big(\hat{\boldsymbol{b}}\hat{\boldsymbol{x}} + \hat{\boldsymbol{x}}\hat{\boldsymbol{b}}\big) \Big] : \boldsymbol{W}_i \\
&\quad - \frac{3}{20} \frac{p_i \tau_i^{-1}}{\mathit{\Omega}_i^2} \Big[\big(\hat{\boldsymbol{x}}\hat{\boldsymbol{y}} + \hat{\boldsymbol{y}}\hat{\boldsymbol{x}}\big)\big(\hat{\boldsymbol{x}}\hat{\boldsymbol{y}} + \hat{\boldsymbol{y}}\hat{\boldsymbol{x}}\big) + \big(\hat{\boldsymbol{x}}\hat{\boldsymbol{x}} - \hat{\boldsymbol{y}}\hat{\boldsymbol{y}}\big)\big(\hat{\boldsymbol{x}}\hat{\boldsymbol{x}} - \hat{\boldsymbol{y}}\hat{\boldsymbol{y}}\big) \Big] : \boldsymbol{W}_i \\
&= -\frac{3}{10} \frac{p_i \tau_i^{-1}}{\mathit{\Omega}_i^2} \begin{bmatrix} \frac{1}{2}(W_{xx} - W_{yy}) & \frac{1}{2}(W_{xy} + W_{yx}) & 2(W_{xz} + W_{zx}) \\ \frac{1}{2}(W_{xy} + W_{yx}) & -\frac{1}{2}(W_{xx} + W_{yy}) & 2(W_{yz} + W_{zy}) \\ 2(W_{zx} + W_{xz}) & 2(W_{zy} + W_{yz}) & 0 \end{bmatrix} \\
&= -\frac{3}{10} \frac{p_i \tau_i^{-1}}{\mathit{\Omega}_i^2} \Big[\big(\boldsymbol{I} - \hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\big) \cdot \boldsymbol{W}_i \cdot \big(\boldsymbol{I} + 3\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\big) + \big(\boldsymbol{I} + 3\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\big) \cdot \boldsymbol{W}_i \cdot \big(\boldsymbol{I} - \hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\big) \Big]. \qquad \text{(IX.5.61)}
\end{aligned}$$

A summary of ion transport is as follows:

$$\boxed{\begin{aligned}
\boldsymbol{q}_{\parallel i} &= -3.91 \frac{p_i \tau_i}{m_i} \hat{\boldsymbol{b}}\hat{\boldsymbol{b}} \cdot \boldsymbol{\nabla} T_i \\
\boldsymbol{q}_{\times i} &= \frac{5}{2} \frac{p_i}{m_i \mathit{\Omega}_i} \hat{\boldsymbol{b}} \times \boldsymbol{\nabla} T_i \\
\boldsymbol{q}_{\perp i} &= -2 \frac{p_i \tau_i^{-1}}{m_i \mathit{\Omega}_i^2} \boldsymbol{\nabla}_\perp T_i \\
\boldsymbol{\mathit{\Pi}}_{\parallel i} &= -0.96 p_i \tau_i \frac{3}{2} \left(\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} - \frac{\boldsymbol{I}}{3} \right) \left(\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} - \frac{\boldsymbol{I}}{3} \right) : \boldsymbol{W}_i \\
\boldsymbol{\mathit{\Pi}}_{\times i} &= \frac{p_i}{4\mathit{\Omega}_i} \Big[\hat{\boldsymbol{b}} \times \boldsymbol{W}_i \cdot \big(\boldsymbol{I} + 3\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\big) - \big(\boldsymbol{I} + 3\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\big) \cdot \boldsymbol{W}_i \times \hat{\boldsymbol{b}} \Big] \\
\boldsymbol{\mathit{\Pi}}_{\perp i} &= -\frac{3}{10} \frac{p_i \tau_i^{-1}}{\mathit{\Omega}_i^2} \Big[\big(\boldsymbol{I} - \hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\big) \cdot \boldsymbol{W}_i \cdot \big(\boldsymbol{I} + 3\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\big) + \big(\boldsymbol{I} + 3\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\big) \cdot \boldsymbol{W}_i \cdot \big(\boldsymbol{I} - \hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\big) \Big]
\end{aligned}}$$

where $\tau_i \doteq \sqrt{2}\tau_{ii}$ and

$$\boldsymbol{W} \doteq \boldsymbol{\nabla}\boldsymbol{u} + (\boldsymbol{\nabla}\boldsymbol{u})^T - \frac{2}{3}(\boldsymbol{\nabla} \cdot \boldsymbol{u})\boldsymbol{I}.$$

is the rate-of-strain tensor.[19]

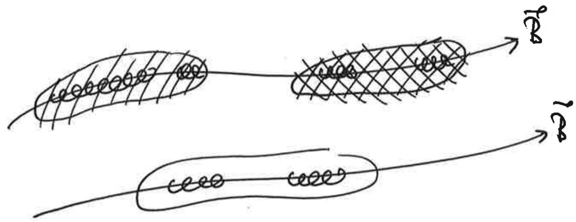
---

[19] If you'd like to read more about transport coefficients, heat fluxes, and viscous stresses, consult Catto & Simakov (2004). Those authors talk about so-called Mikhailovskii–Tsypin contributions to these transport terms when a drift ordering is assumed (rather than Braginskii's "high-flow" ordering, which takes the Mach number of be of order unity).

IX.6. Discussion of Braginskii transport

First, read §3 of Braginskii (1965)!

That there should be different transport efficiencies in different directions in a magnetized plasma should not be surprising: the smallness of Larmor radii puts constraints on the motions of particles comprising the fluid elements, and these constraints manifest quite clearly in all the $\hat{\boldsymbol{b}}\,\boldsymbol{\cdot}$, $\hat{\boldsymbol{b}}\times$, $\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\,\boldsymbol{:}$, etc. operations. The parallel transport is the easiest to understand: when $\Omega_\alpha\tau_\alpha \gg 1$, particles are constrained to move along field lines, and so a collisional mean free path can easily be transversed along the field, but *not* across it. Here's an illustration:



 and  can viscously transport momentum and conductively transport heat between one another. But these two cannot easily do so with , because of the smallness of the particles' Larmor radii. It is very difficult for a particle to travel a collisional mean free path across a field line. In fact, in Braginskii's ordering, to leading order in $\rho/\lambda_{\rm mfp}$, such particles *cannot* do so. The result is that transport is along field lines, and only gradients of quantities oriented along field lines are adequately sampled. If we relax $\Omega_\alpha\tau_\alpha \gg 1$, then there is some cross-field transport, which is a bit more difficult to understand. We'll get to that after a discussion of parallel transport.

IX.6.1. *Discussion of parallel transport*

Collecting together all the terms related to collisional transport occurring along the magnetic field, we have

$$\boldsymbol{q}_{\parallel e} = -3.16\frac{p_e\tau_e}{m_e}\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\boldsymbol{\cdot}\boldsymbol{\nabla}T_e - 0.71 n_e(\boldsymbol{u}_{\parallel i} - \boldsymbol{u}_{\parallel e}), \tag{IX.6.1a}$$

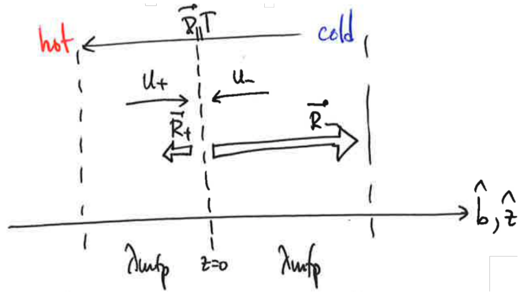
$$\boldsymbol{q}_{\parallel i} = -3.91\frac{p_i\tau_i}{m_i}\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\boldsymbol{\cdot}\boldsymbol{\nabla}T_i, \tag{IX.6.1b}$$

$$\boldsymbol{\Pi}_{\parallel i} = -0.96 p_i\tau_i\frac{3}{2}\left(\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} - \frac{\boldsymbol{I}}{3}\right)\left(\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} - \frac{\boldsymbol{I}}{3}\right)\boldsymbol{:}\boldsymbol{W}_i, \tag{IX.6.1c}$$

$$\boldsymbol{R}_{\parallel ei} = -0.71 n_e\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\boldsymbol{\cdot}\boldsymbol{\nabla}T_e + 0.51\frac{m_e n_e}{\tau_e}(\boldsymbol{u}_{\parallel i} - \boldsymbol{u}_{\parallel e}) \tag{IX.6.1d}$$

We have already discussed $\boldsymbol{R}_{\parallel ei}$ in the context of the electron–ion collision operator. But why does a temperature gradient provide a friction force, and why are ion–electron drifts giving a heat flux?

The former, the $-0.71 n_e\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\boldsymbol{\cdot}\boldsymbol{\nabla}T_e$ term in the parallel electron heat flux, is due to the fact that, in the presence of a temperature gradient, the energy of a particle is correlated with its direction: particles from high-temperature regions will have more energy than those from low-temperature regions. The difference in energy gives a friction force because particles coming from the low-temperature regions will collide more often (recall $\tau \propto T^{3/2}$) and will lose more momentum than particles going in the opposite direction, coming from high-temperature regions. Figure 1 in Braginskii (1965) shows this:



Flows arriving at $z = 0$ from the left ($\boldsymbol{u}_+$) come from regions with higher T and thus experience a weaker friction force $\boldsymbol{R}_+$. Flows arriving at $z = 0$ from the right ($\boldsymbol{u}_-$) come from regions with lower T and thus experience a stronger friction force $\boldsymbol{R}_-$. The unbalanced part of the friction will be $\sim -\lambda_{\rm mfp}(\mathrm{d}\ln T/\mathrm{d}z) \times mnv_{\rm th}/\tau$, which is just a Taylor expansion of the friction force about $z = 0$. While the form $-n_e\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\cdot\boldsymbol{\nabla}T_e$ makes no explicit reference to the collision timescale, the force *is* due to collisions: the dependence on the temperature gradient actually comes via $\mathrm{d}\ln\tau/\mathrm{d}z = (3/2)\mathrm{d}\ln T/\mathrm{d}z$.

The latter, the $-0.71n_e(\boldsymbol{u}_{\parallel i} - \boldsymbol{u}_{\parallel e})$ term in the parallel electron heat flux, has a similar origin: slow electrons are more likely to collide with ions and so are more apt to acquire the mean ion velocity than are fast electrons. This gives a heat flow, even if $\boldsymbol{u}_{\parallel e} = 0$. Again, even though this term does not explicitly make reference to the collision frequency, is it physically caused by the temperature dependence of τ_{ei}.

The fact that both electron-heat-flux terms have a 0.71 prefactor is not an accident. This is related to something called "Onsager symmetry", a symmetry of the off-diagonal terms in the transport laws. You can read about it in, e.g., Chapter 26 of Krommes (2018).

Regarding the $\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\cdot\boldsymbol{\nabla}T$ heat fluxes, note that (i) electron heat transport is a factor $\sim\sqrt{m_i/m_e}$ faster than is ion heat transport (i.e., electrons dominate the thermal conductivity) and (ii) one must be very careful in a system in which $\hat{\boldsymbol{b}}$ is a fluctuating quantity, since the heat flux is not only directed along $\hat{\boldsymbol{b}}$ but also is proportional to the projection of $\boldsymbol{\nabla}T$ along $\hat{\boldsymbol{b}}$. (This causes interesting effects in a weakly collisional, stratified plasma; see Balbus (2000, 2001); Quataert (2008); Kunz (2011) for a Braginskii calculation and Xu & Kunz (2016) for a more general approach starting from the Vlasov equation.)

Finally, the parallel ion viscous stress $\boldsymbol{\mathit{\Pi}}_{\parallel i}$. First, note that the ions dominate the parallel viscosity (by a factor $\sim\sqrt{m_i/m_e}$). Secondly, while the illustration of field-line-separated fluid elements can give some intuition for why momentum would be difficult to transport across field lines, consider the following alternative interpretation of $\boldsymbol{\mathit{\Pi}}_\parallel\dots$

Recall (IX.5.8): $\boldsymbol{w} = w_\parallel\hat{\boldsymbol{b}} + w_\perp(\cos\vartheta\hat{\boldsymbol{x}} + \sin\vartheta\hat{\boldsymbol{y}})$. Using this in the definition of the pressure tensor $\mathsf{\boldsymbol{P}}$ (see (IX.2.22)), we find

$$\begin{aligned}\mathsf{\boldsymbol{P}} &= \int \mathrm{d}\boldsymbol{w}\, mw_\parallel^2 f\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} + \int \mathrm{d}\boldsymbol{w}\, mw_\perp^2\left(\cos^2\vartheta\hat{\boldsymbol{x}}\hat{\boldsymbol{x}} + \sin^2\vartheta\hat{\boldsymbol{y}}\hat{\boldsymbol{y}}\right)f \\ &+ \int \mathrm{d}\boldsymbol{w}\,\frac{1}{2}mw_\perp^2\sin 2\vartheta(\hat{\boldsymbol{x}}\hat{\boldsymbol{y}} + \hat{\boldsymbol{y}}\hat{\boldsymbol{x}})f + \int \mathrm{d}\boldsymbol{w}\, mw_\parallel w_\perp\left[\cos\vartheta(\hat{\boldsymbol{x}}\hat{\boldsymbol{b}} + \hat{\boldsymbol{b}}\hat{\boldsymbol{x}}) + \sin\vartheta(\hat{\boldsymbol{y}}\hat{\boldsymbol{b}} + \hat{\boldsymbol{b}}\hat{\boldsymbol{y}})\right]f.\end{aligned} \tag{IX.6.2}$$

To lowest order in $\omega/\mathit{\Omega}$, we have shown that f is gyrotropic, $f = \langle f\rangle_\vartheta = f(w_\parallel, w_\perp)$. Thus, the integrals over $\sin\vartheta$, $\cos\vartheta$, and $\sin 2\vartheta$ are all zero and the entire second line of

(IX.6.2) vanishes. The result is that

$$\boldsymbol{P} = \underbrace{\int \mathrm{d}\boldsymbol{w}\, mw_\parallel^2 f}_{\doteq\, p_\parallel} \hat{\boldsymbol{b}}\hat{\boldsymbol{b}} + \underbrace{\int \mathrm{d}\boldsymbol{w}\, \frac{1}{2} mw_\perp^2 f}_{\doteq\, p_\perp} (\boldsymbol{I} - \hat{\boldsymbol{b}}\hat{\boldsymbol{b}}) = p_\parallel \hat{\boldsymbol{b}}\hat{\boldsymbol{b}} + p_\perp (\boldsymbol{I} - \hat{\boldsymbol{b}}\hat{\boldsymbol{b}}), \tag{IX.6.3}$$

where $p_\parallel$ ($p_\perp$) is the parallel (perpendicular) pressure; i.e., the pressure tensor is diagonal in a frame aligned with the magnetic field. Just to remind you: if ν^{-1} and λ_{mfp} were the smallest interesting scales, then the dominant term in the kinetic equation would be $C[f] = 0$, and so f would be Maxwellian and the pressure would be isotropic. Thus, we are considering deviations from a Maxwellian. Now then, with

$$p \doteq \frac{1}{3} \mathrm{tr}\, \boldsymbol{P} = \frac{2}{3} p_\perp + \frac{1}{3} p_\parallel, \tag{IX.6.4}$$

equation (IX.6.3) may be written as

$$\boldsymbol{P} = p\boldsymbol{I} - \left(\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} - \frac{\boldsymbol{I}}{3} \right) (p_\perp - p_\parallel) \doteq p\boldsymbol{I} + \boldsymbol{\Pi}. \tag{IX.6.5}$$

And so we see that the viscous stress is related to the pressure anisotropy, $p_\perp - p_\parallel$.

Why would there be a bias in the thermal pressure with respect to the magnetic-field direction? Recall adiabatic invariance in a magnetized plasma:

$$\mu \doteq \frac{mw_\perp^2}{2B} \simeq \mathrm{const}, \tag{IX.6.6}$$

$$J \doteq \oint \mathrm{d}\ell\, mw_\parallel \simeq \mathrm{const}. \tag{IX.6.7}$$

Averaging (IX.6.6) over the particle distribution function gives $\langle \mu \rangle = T_\perp / B \simeq \mathrm{const}$; i.e., if the magnetic-field strength changes in a fluid element, so too must $T_\perp$. Similarly averaging the square of (IX.6.7) over the particle distribution function gives $\langle J^2 \rangle \propto T_\parallel \ell^2 \simeq \mathrm{const}$. Taking the length of the mirroring field ℓ to be $\propto B/n$, this implies $T_\parallel (B/n)^2 \simeq \mathrm{const}$; i.e., if the ratio of magnetic-field strength and plasma density change in a fluid element, so too must $T_\parallel$. Thus, we obtain the double-adiabatic laws discussed in Chew *et al.* (1956),

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{T_\perp}{B} \right) = \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{p_\perp}{nB} \right) \simeq 0, \tag{IX.6.8}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{T_\parallel B^2}{n^2} \right) = \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{p_\parallel B^2}{n^3} \right) \simeq 0. \tag{IX.6.9}$$

You should understand the "$\simeq$" as holding true on timescales smaller than those on which the two adiabatic invariants are broken (e.g., collisional timescales or, in the case of the second adiabatic invariant J, acoustic timescales). Note that combining (IX.6.11) and

(IX.6.12) yields an equation for the evolution of the total pressure p (see (IX.6.4)):

$$\begin{aligned}\frac{\mathrm{d}p}{\mathrm{d}t} &= \frac{2}{3}\frac{\mathrm{d}p_\perp}{\mathrm{d}t} + \frac{1}{3}\frac{\mathrm{d}p_\parallel}{\mathrm{d}t} \\ &= \underbrace{\frac{2}{3}p_\perp\frac{\mathrm{d}}{\mathrm{d}t}\ln Bn}_{\substack{\text{use } p_\perp = p \\ +(p_\perp - p_\parallel)/3}} - \frac{2}{3}\nu(p_\perp - p) + \underbrace{\frac{1}{3}p_\parallel\frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{n^3}{B^2}}_{\substack{\text{use } p_\parallel = p \\ -2(p_\perp - p_\parallel)/3}} - \frac{1}{3}\nu(p_\parallel - p) \\ &= \frac{5}{3}p\frac{\mathrm{d}\ln n}{\mathrm{d}t} + \frac{2}{3}(p_\perp - p_\parallel)\frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{B}{n^{2/3}}.\end{aligned}$$

Multiplying through by 3/2 and moving the $\mathrm{d}\ln n/\mathrm{d}t$ term to the left-hand side results in an evolutionary equation for the hydrodynamic entropy, $\ln pn^{-5/3}$:

$$\boxed{\frac{3}{2}p\frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{p}{n^{5/3}} = (p_\perp - p_\parallel)\frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{B}{n^{2/3}}} \qquad \text{(IX.6.10)}$$

Note that if $p_\perp = p_\parallel$, entropy is conserved in a fluid element. This will come in handy.

Now, if our interest is in weakly collisional plasmas – which, in this class, it is – then the difference between $p_\perp$ and $p_\parallel$ will remain small. One way to model this (correctly, in fact) is to modify (IX.6.8) and (IX.6.9) to include the isotropizing effect of collisions:

$$\frac{\mathrm{d}p_\perp}{\mathrm{d}t} = p_\perp\frac{\mathrm{d}}{\mathrm{d}t}\ln Bn - \nu(p_\perp - p), \qquad \text{(IX.6.11)}$$

$$\frac{\mathrm{d}p_\parallel}{\mathrm{d}t} = p_\parallel\frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{n^3}{B^2} - \nu(p_\parallel - p). \qquad \text{(IX.6.12)}$$

Note that, if $\nu \gg \mathrm{d}/\mathrm{d}t$, these equations effectively push $p_\perp, p_\parallel \to p$. Subtracting (IX.6.12) from (IX.6.11) provides an equation for the time evolution of the pressure anisotropy:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(p_\perp - p_\parallel) &= p_\perp\frac{\mathrm{d}}{\mathrm{d}t}\ln Bn - p_\parallel\frac{\mathrm{d}}{\mathrm{d}}\ln\frac{n^3}{B^2} - \nu(p_\perp - p_\parallel) \\ &\simeq 3p\frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{B}{n^{2/3}} - \nu(p_\perp - p_\parallel),\end{aligned} \qquad \text{(IX.6.13)}$$

where in the second equality we have used $|p_\perp - p_\parallel| \ll p$. In this case, the left-hand side of (IX.6.13) is much smaller than the collisional term (recall Braginskii's ordering), so that

$$p_\perp - p_\parallel \simeq \frac{3p}{\nu}\frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{B}{n^{2/3}}. \qquad \text{(IX.6.14)}$$

This states that pressure anisotropy in a weakly collisional, magnetized plasma is set via a balance between adiabatic production and collisional relaxation. In this case, equation (IX.6.14) becomes

$$\frac{3}{2}p\frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{p}{n^{5/3}} = \frac{3p}{\nu}\left(\frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{B}{n^{2/3}}\right)^2 \geqslant 0. \qquad \text{(IX.6.15)}$$

Entropy cannot decrease. Good. Namely, collisional isotropization of the pressure increases entropy.

How does all this relate to the form of $\boldsymbol{\Pi}_\parallel$ that we derived by using the Braginskii ordering in the Chapman–Enskog expansion? Recall the continuity equation

$$\frac{\mathrm{d}\ln n}{\mathrm{d}t} = -\boldsymbol{\nabla}\cdot\boldsymbol{u},$$

and, from the induction equation,

$$\frac{\mathrm{d}\ln B}{\mathrm{d}t} = (\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} - \mathsf{\boldsymbol{I}}) \boldsymbol{:} \boldsymbol{\nabla}\boldsymbol{u}$$

(to be proven in HW07). Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{B}{n^{2/3}} = \left[(\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} - \mathsf{\boldsymbol{I}}) - \frac{2}{3}(-\mathsf{\boldsymbol{I}})\right] \boldsymbol{:} \boldsymbol{\nabla}\boldsymbol{u} = \left(\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} - \frac{\mathsf{\boldsymbol{I}}}{3}\right) \boldsymbol{:} \boldsymbol{\nabla}\boldsymbol{u}. \tag{IX.6.16}$$

Inserting this expression into (IX.6.14) obtains the *Braginskii pressure anisotropy*

$$\boxed{p_\perp - p_\parallel = \frac{3p}{\nu}\left(\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} - \frac{\mathsf{\boldsymbol{I}}}{3}\right) \boldsymbol{:} \boldsymbol{\nabla}\boldsymbol{u}} \tag{IX.6.17}$$

Do you see it yet? Substitute (IX.6.17) into (IX.6.5) to find

$$\mathsf{\boldsymbol{P}} = p\mathsf{\boldsymbol{I}} - \frac{3p}{\nu}\left(\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} - \frac{\mathsf{\boldsymbol{I}}}{3}\right)\left(\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} - \frac{\mathsf{\boldsymbol{I}}}{3}\right) \boldsymbol{:} \boldsymbol{\nabla}\boldsymbol{u} = p\mathsf{\boldsymbol{I}} + \boldsymbol{\mathit{\Pi}}. \tag{IX.6.18}$$

If we identify ν with $(0.96\tau)^{-1}$, then the expression for $\boldsymbol{\mathit{\Pi}}$ in (IX.6.18) is precisely the same as was obtained via the Chapman–Enskog procedure for the parallel viscous stress (cf. (IX.5.59))! Pressure anisotropy in a weakly collisional, magnetized plasma is parallel viscosity ("Braginskii viscosity"). Note further that the entropy equation (IX.6.15) becomes

$$\frac{3}{2}p\frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{p}{n^{5/3}} = \frac{3p}{\nu}\left[\left(\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} - \frac{\mathsf{\boldsymbol{I}}}{3}\right) \boldsymbol{:} \boldsymbol{\nabla}\boldsymbol{u}\right]^2; \tag{IX.6.19}$$

i.e., viscous heating. Thus, only motions that change the magnetic-field strength and density (provided that $B \not\propto n^{2/3}$) are viscously damped. Linear shear Alfvén waves are not damped. Nonlinear circularly polarized Alfvén waves are not damped. Neither produce any change in the form of the distribution function; rather, they simply define the frame in which the distribution function is to be measured. Magnetosonic waves, on the other hand, do produce pressure anisotropy, and thus are viscously damped.

IX.6.2. *Discussion of perpendicular transport*

Collecting together all the terms related to collisional transport occurring across the magnetic field, we have

$$\boldsymbol{q}_{\times e} = \frac{5}{2}\frac{p_e}{m_e\mathit{\Omega}_e}\hat{\boldsymbol{b}} \times \boldsymbol{\nabla}T_e, \tag{IX.6.20a}$$

$$\boldsymbol{q}_{\times i} = \frac{5}{2}\frac{p_i}{m_i\mathit{\Omega}_i}\hat{\boldsymbol{b}} \times \boldsymbol{\nabla}T_i, \tag{IX.6.20b}$$

$$\boldsymbol{q}_{\perp e} = -4.66\frac{p_e\tau_{ee}^{-1}}{m_e\mathit{\Omega}_e^2}\boldsymbol{\nabla}_\perp T_e + \frac{3}{2}\frac{p_e\tau_{ei}^{-1}}{\mathit{\Omega}_e}\hat{\boldsymbol{b}} \times (\boldsymbol{u}_i - \boldsymbol{u}_e), \tag{IX.6.20c}$$

$$\boldsymbol{q}_{\perp i} = -2\frac{p_i\tau_i^{-1}}{m_i\mathit{\Omega}_i^2}\boldsymbol{\nabla}_\perp T_i, \tag{IX.6.20d}$$

$$\boldsymbol{\mathit{\Pi}}_{\times i} = \frac{p_i}{4\mathit{\Omega}_i}\left[\hat{\boldsymbol{b}} \times \mathsf{\boldsymbol{W}}_i \boldsymbol{\cdot} (\mathsf{\boldsymbol{I}} + 3\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}) - (\mathsf{\boldsymbol{I}} + 3\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}) \boldsymbol{\cdot} \mathsf{\boldsymbol{W}}_i \times \hat{\boldsymbol{b}}\right], \tag{IX.6.20e}$$

$$\boldsymbol{\mathit{\Pi}}_{\perp i} = -\frac{3}{10}\frac{p_i\tau_i^{-1}}{\mathit{\Omega}_i^2}\left[(\mathsf{\boldsymbol{I}} - \hat{\boldsymbol{b}}\hat{\boldsymbol{b}}) \boldsymbol{\cdot} \mathsf{\boldsymbol{W}}_i \boldsymbol{\cdot} (\mathsf{\boldsymbol{I}} + 3\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}) + (\mathsf{\boldsymbol{I}} + 3\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}) \boldsymbol{\cdot} \mathsf{\boldsymbol{W}}_i \boldsymbol{\cdot} (\mathsf{\boldsymbol{I}} - \hat{\boldsymbol{b}}\hat{\boldsymbol{b}})\right], \tag{IX.6.20f}$$

$$\boldsymbol{R}_{\perp ei} = \frac{3}{2}\frac{m_e}{\mathit{\Omega}_e\tau_{ei}}\hat{\boldsymbol{b}} \times \boldsymbol{\nabla}T_e + \frac{m_e n_e}{\tau_{ei}}(\boldsymbol{u}_{\perp i} - \boldsymbol{u}_{\perp e}). \tag{IX.6.20g}$$

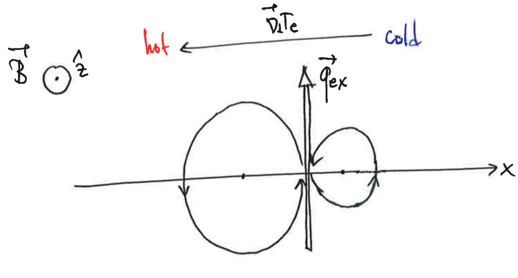
Let us first focus on the terms with subscript "$\perp$" and set aside those with a $\hat{\boldsymbol{b}}\times$ operator in them.

The first thing to notice is that $|q_{\perp i}/q_{\perp e}| \sim \sqrt{m_i/m_e} \gg 1$, and so the ions dominate the perpendicular transport of heat along $\boldsymbol{\nabla}_\perp T_i$. This dominance is the opposite of $|q_{\| i}/q_{\| e}| \sim \sqrt{m_e/m_i} \ll 1$ for the parallel transport. The reason is that $|q_\perp/q_\||\sim(\Omega\tau)^{-2}\sim(\rho/\lambda_{\rm mfp})^2$, and so the species with the larger Larmor radius has more perpendicular transport. Physically, having a larger Larmor radius means that the particle orbits sample more widely differing temperatures for a given $\boldsymbol{\nabla}_\perp T$. Put differently, particles travel a distance $\Delta\ell\sim\rho$ in a collision time, and so diffusive transport $\sim(\Delta\ell)^2/\Delta t\sim\rho^2\nu$ is larger for larger ρ. By contrast, parallel diffusive transport $\sim(\Delta\ell)^2/\Delta t\sim\lambda_{\rm mfp}^2\nu\sim\lambda_{\rm mfp}v_{\rm th}$, and so the species with the larger thermal speed (viz., the electrons) has the faster parallel diffusion of heat.

Next, the $(m_e n_e/\tau_{ei})(\boldsymbol{u}_{\perp i}-\boldsymbol{u}_{\perp e})$ term in $\boldsymbol{R}_{ei}$. This, again, is just the standard friction force due to collisional drag, but is oriented across the field lines. Since cross-field drifts are, at least in Braginskii's "high-flow" ordering, species independent (think $\boldsymbol{E}\times\boldsymbol{B}$), this terms is small.

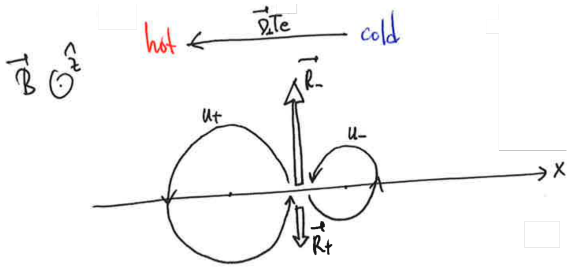
Finally, $\boldsymbol{\Pi}_{\perp i}$. This is $\sim(\Omega_i\tau_i)^{-2}$ times smaller than $\boldsymbol{\Pi}_{\| i}$. Gradients in velocity are sampled across the magnetic field by gyromotion, and so diffusive transport of momentum by collisions is constrained by how large ρ_i is relative to $\lambda_{\rm mfp}$. Same idea as with the cross-field heat flow.

Next, the weird $\hat{\boldsymbol{b}}\times$ terms: these arise because the act of particle gyration about the magnetic field rotates temperature and velocity differences by 90°. Let us look first at the heat fluxes $\boldsymbol{q}_{\times e}$ and $\boldsymbol{q}_{\times i}$. Note that $|q_{\times e}|\sim|q_{\times i}|$; their magnitude is independent of mass! Also note that they are in different directions, since $\Omega_e<0$ and $\Omega_i>0$. This will make sense, since the sense of rotation in the Larmor orbits of ions and electrons is different, and so the species might herald from different thermal environments. These are fundamentally *finite-Larmor-radius* (FLR) terms; they rely on ρ/L being finite. Note that the collision time is *not* in $\boldsymbol{q}_\times$. Below is a diagram explaining their physical origin:

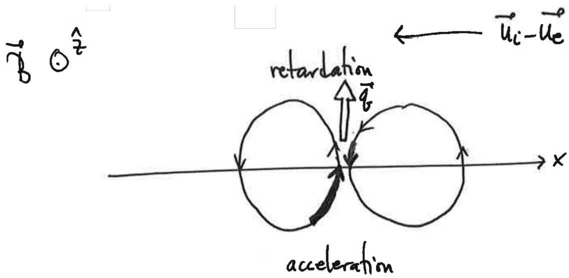


Bigger Larmor orbit on the left, since hotter plasma is over there. This orbit is taken by a faster particle than the one on the right, and so there is a net energy flow in the direction $-\hat{\boldsymbol{b}}\times\boldsymbol{\nabla}T_e$ (upwards in the picture). NB: heat is transported along isotherms!

The friction force $\propto\hat{\boldsymbol{b}}\times\boldsymbol{\nabla}T_e$ and the corresponding (Onsager-symmetric) heat flow $\propto\hat{\boldsymbol{b}}\times(\boldsymbol{u}_i-\boldsymbol{u}_e)$ also have their origin in FLR effects, but for them collisions are involved. The former is due to the fact that, in the presence of a temperature gradient, the energy of a particle is correlated with its direction. This is just as in the parallel friction force, but now its direction is not along the field ($\hat{\boldsymbol{b}}\boldsymbol{\cdot}\boldsymbol{\nabla}T\neq0$) but rather across the radius because of Larmor motion. The particles arrive from a gyroradius away rather than a mean free path away: particles coming from low-temperature regions will collide more often and experience a larger friction force ($R_->R_+$), giving an imbalance along $\hat{\boldsymbol{b}}\times\boldsymbol{\nabla}T/\Omega$:



The latter is caused by the fact that, for one-half of a Larmor orbit, the electrons are traveling against the ion flow and, for the other half, the electrons are traveling with the ion flow. That means that the friction force is different along the Larmor orbit. Picture this:



Electrons arriving from below are gaining energy from ion collisions, whereas those arriving from above are losing energy from ion collisions. This gives a net energy transport at the intersection of these orbits.

Finally, $\boldsymbol{\varPi}_{\times i}$. This is called "gyroviscosity"; it arises from Larmor-scale spatial variations in the guiding-center $\boldsymbol{E} \times \boldsymbol{B}$ drifts. It is dissipationless and transports momentum across the magnetic field. Note that it is independent of collision frequency. Momentum transport is deflected in the cross-field direction due to Larmor motion. Braginskii passes the explanation of this term to a "lucid discussion" by Kaufman (1960). I do not find Kaufman's discussion particularly lucid, so I'll offer up my own.

Gyroviscosity is a peculiar feature of working with particle coordinates rather than guiding-center coordinates. In the latter, with $\boldsymbol{R} \doteq \boldsymbol{r} - \boldsymbol{\rho}$, the distribution function evaluated at the guiding-center position is $f(\boldsymbol{R}) = f(\boldsymbol{r}) - \rho \boldsymbol{\cdot} \boldsymbol{\nabla} f(\boldsymbol{r}) + \mathcal{O}(\rho/L)^2$. The $-\boldsymbol{\rho} \boldsymbol{\cdot} \boldsymbol{\nabla} f(\boldsymbol{r})$ term here is what is responsible for the "diamagnetic" fluxes of momentum and heat that appear in the Braginskii equations as $\hat{\boldsymbol{b}} \times \boldsymbol{\nabla} T$ and $\hat{\boldsymbol{b}} \times \boldsymbol{W}$ terms. "Gyroviscosity" is a bit of a misnomer, though – no momentum is being dissipated! – and it even appears in collisionless plasmas with finite Larmor radii. What is does is *reorient* momentum through rotation.

Consider an oscillatory displacement $\boldsymbol{\xi}_\perp$ perpendicular to the magnetic field in a plasma (collisional or collisionless, doesn't matter). It is straightforward to show using flux freezing and the momentum equation that $\boldsymbol{\xi}_\perp$, in the limit $m_e/m_i \to 0$, satisfies

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + k_\parallel^2 v_\mathrm{A}^2\right)\boldsymbol{\xi}_\perp = -\frac{k_\parallel^2 v_{\mathrm{th}i}^2}{2\varOmega_i}\frac{\mathrm{d}\boldsymbol{\xi}_\perp}{\mathrm{d}t} \times \hat{\boldsymbol{b}}$$

if $\boldsymbol{\xi}_\perp \propto \exp(\mathrm{i}k_\parallel z)$. The right-hand side of this equation is due to off-diagonal components of the pressure tensor – the "gyroviscosity". The left-hand side is just an Alfvén wave.

Note that the right-hand side rotates the displacement about $\hat{\boldsymbol{b}}$:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + k_\parallel^2 v_\mathrm{A}^2\right)\xi_x = -\frac{k_\parallel^2 v_{\mathrm{th}i}^2}{2\Omega_i}\frac{\mathrm{d}\xi_y}{\mathrm{d}t},$$
$$\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + k_\parallel^2 v_\mathrm{A}^2\right)\xi_y = +\frac{k_\parallel^2 v_{\mathrm{th}i}^2}{2\Omega_i}\frac{\mathrm{d}\xi_x}{\mathrm{d}t},$$

which lends the Alfvén wave some circular polarization on scales $k_\parallel \rho_i \sim \beta_i^{-1/2}$. Let $\boldsymbol{\xi}_\perp \propto \exp(-\mathrm{i}\omega t)$ to obtain

$$(-\omega^2 + k_\parallel^2 v_\mathrm{A}^2)\xi_x = +\mathrm{i}\omega\frac{k_\parallel^2 v_{\mathrm{th}i}^2}{2\Omega_i}\xi_y,$$
$$(-\omega^2 + k_\parallel^2 v_\mathrm{A}^2)\xi_y = -\mathrm{i}\omega\frac{k_\parallel^2 v_{\mathrm{th}i}^2}{2\Omega_i}\xi_x,$$

$$\Longrightarrow \quad \omega^2 = k_\parallel^2 v_\mathrm{A}^2 + \frac{1}{2}\left(\frac{k_\parallel^2 v_{\mathrm{th}i}^2}{2\Omega_i}\right)^2 \pm \frac{k_\parallel^2 v_{\mathrm{th}i}^2}{2\Omega_i}\sqrt{k_\parallel^2 v_\mathrm{A}^2 + \frac{1}{4}\left(\frac{k_\parallel^2 v_{\mathrm{th}i}^2}{2\Omega_i}\right)^2}. \qquad \text{(IX.6.21)}$$

Note that gyroviscosity gives dispersion, not dissipation! At long wavelengths, one recovers Alfvén waves. At small wavelengths such that $k_\parallel \rho_i \gg \beta_i^{-1/2}$, one gets either $\omega \approx k_\parallel^2 v_{\mathrm{th}i}^2/2\Omega_i$ or $\omega \approx 2\Omega_i/\beta_i$, depending on whether the wave is right-handed or left-handed.

One way of thinking about this is that, as a particle streams along a field line with speed $\sim v_{\mathrm{th}i}$, it samples different $\boldsymbol{E}\times\boldsymbol{B}$ drifts. If the structure along the field is such that the particle encounters different $\boldsymbol{E}\times\boldsymbol{B}$ drifts in one Larmor orbit, then gyroviscosity is important. This occurs when $k_\parallel \rho_i \sim v_\mathrm{wave}/v_\mathrm{particle}$. For waves/instabilities whose v_wave is proportional to some inverse power of $k_\parallel$, gyroviscosity can be stabilizing (e.g., see §5.2 of Xu & Kunz (2016) or Ferraro (2007) for astrophysical examples).

IX.7. Curvilinear considerations

There are a lot of vectors, tensors, and gradients in the Braginskii transport model, and so it's useful to catalog here a few vector identities and expressions of these objects in curvilinear coordinates. Given a scalar f, vector $\boldsymbol{A}$ with components A_i, and tensor $\boldsymbol{T}$ with components T_{ij}...

Cylindrical polar coordinates (R, ϕ, z):

$$\boldsymbol{\nabla} f = \frac{\partial f}{\partial R}\,\hat{\boldsymbol{R}} + \frac{1}{R}\frac{\partial f}{\partial \phi}\,\hat{\boldsymbol{\phi}} + \frac{\partial f}{\partial z}\,\hat{\boldsymbol{z}}, \tag{IX.7.1}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{A} = \frac{1}{R}\frac{\partial(RA_R)}{\partial R} + \frac{1}{R}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}, \tag{IX.7.2}$$

$$\begin{aligned}\boldsymbol{\nabla}\boldsymbol{A} = {}& \frac{\partial A_R}{\partial R}\,\hat{\boldsymbol{R}}\hat{\boldsymbol{R}} + \left(\frac{1}{R}\frac{\partial A_R}{\partial \phi} - \frac{A_\phi}{R}\right)\hat{\boldsymbol{R}}\hat{\boldsymbol{\phi}} + \frac{\partial A_R}{\partial z}\,\hat{\boldsymbol{R}}\hat{\boldsymbol{z}} \\ &+ \frac{\partial A_\phi}{\partial R}\,\hat{\boldsymbol{\phi}}\hat{\boldsymbol{R}} + \left(\frac{1}{R}\frac{\partial A_\phi}{\partial \phi} + \frac{A_R}{R}\right)\hat{\boldsymbol{\phi}}\hat{\boldsymbol{\phi}} + \frac{\partial A_\phi}{\partial z}\,\hat{\boldsymbol{\phi}}\hat{\boldsymbol{z}} \\ &+ \frac{\partial A_z}{\partial R}\,\hat{\boldsymbol{z}}\hat{\boldsymbol{R}} + \frac{1}{R}\frac{\partial A_z}{\partial \phi}\,\hat{\boldsymbol{z}}\hat{\boldsymbol{\phi}} + \frac{\partial A_z}{\partial z}\,\hat{\boldsymbol{z}}\hat{\boldsymbol{z}},\end{aligned} \tag{IX.7.3}$$

$$\begin{aligned}(\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{A} = {}& \left(B_R\frac{\partial A_R}{\partial R} + \frac{B_\phi}{R}\frac{\partial A_R}{\partial \phi} + B_z\frac{\partial A_R}{\partial z} - \frac{B_\phi A_\phi}{R}\right)\hat{\boldsymbol{R}} \\ &+ \left(B_R\frac{\partial A_\phi}{\partial R} + \frac{B_\phi}{R}\frac{\partial A_\phi}{\partial \phi} + B_z\frac{\partial A_\phi}{\partial z} + \frac{B_\phi A_R}{R}\right)\hat{\boldsymbol{\phi}} \\ &+ \left(B_R\frac{\partial A_z}{\partial R} + \frac{B_\phi}{R}\frac{\partial A_z}{\partial \phi} + B_z\frac{\partial A_z}{\partial z}\right)\hat{\boldsymbol{z}},\end{aligned} \tag{IX.7.4}$$

$$\begin{aligned}\boldsymbol{\nabla}\cdot\boldsymbol{\mathsf{T}} = {}& \left(\frac{\partial \mathsf{T}_{RR}}{\partial R} + \frac{1}{R}\frac{\partial \mathsf{T}_{\phi R}}{\partial \phi} + \frac{\partial \mathsf{T}_{zR}}{\partial z} + \frac{\mathsf{T}_{RR} - \mathsf{T}_{\phi\phi}}{R}\right)\hat{\boldsymbol{R}} \\ &+ \left(\frac{\partial \mathsf{T}_{R\phi}}{\partial R} + \frac{1}{R}\frac{\partial \mathsf{T}_{\phi\phi}}{\partial \phi} + \frac{\partial \mathsf{T}_{z\phi}}{\partial z} + \frac{\mathsf{T}_{R\phi} + \mathsf{T}_{\phi R}}{R}\right)\hat{\boldsymbol{\phi}} \\ &+ \left(\frac{\partial \mathsf{T}_{Rz}}{\partial R} + \frac{1}{R}\frac{\partial \mathsf{T}_{\phi z}}{\partial \phi} + \frac{\partial \mathsf{T}_{zz}}{\partial z} + \frac{\mathsf{T}_{Rz}}{R}\right)\hat{\boldsymbol{z}}.\end{aligned} \tag{IX.7.5}$$

For example, the rate-of-strain tensor for a differentially rotating plasma with velocity $\boldsymbol{u} = R\varpi(R,z)\hat{\boldsymbol{\phi}}$ is given by

$$\boldsymbol{\mathsf{W}} = \frac{\partial\varpi}{\partial\ln R}\,(\hat{\boldsymbol{R}}\hat{\boldsymbol{\phi}} + \hat{\boldsymbol{\phi}}\hat{\boldsymbol{R}}) + R\frac{\partial\varpi}{\partial z}\,(\hat{\boldsymbol{\phi}}\hat{\boldsymbol{z}} + \hat{\boldsymbol{z}}\hat{\boldsymbol{\phi}}). \tag{IX.7.6}$$

Spherical polar coordinates (r, θ, ϕ):

$$\nabla f = \frac{\partial f}{\partial r}\,\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\,\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\,\hat{\boldsymbol{\phi}}, \tag{IX.7.7}$$

$$\nabla \cdot \boldsymbol{A} = \frac{1}{r^2}\frac{\partial (r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta A_\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}, \tag{IX.7.8}$$

$$\begin{aligned}\nabla \boldsymbol{A} = {} & \frac{\partial A_r}{\partial r}\,\hat{\boldsymbol{r}}\hat{\boldsymbol{r}} + \left(\frac{1}{r}\frac{\partial A_r}{\partial \theta} - \frac{A_\theta}{r}\right)\hat{\boldsymbol{r}}\hat{\boldsymbol{\theta}} + \left(\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{A_\phi}{r}\right)\hat{\boldsymbol{r}}\hat{\boldsymbol{\phi}} \\ & + \frac{\partial A_\theta}{\partial r}\,\hat{\boldsymbol{\theta}}\hat{\boldsymbol{r}} + \left(\frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{A_r}{r}\right)\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}} + \left(\frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial \phi} - \cot\theta\frac{A_\phi}{r}\right)\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\phi}} \\ & + \frac{\partial A_\phi}{\partial r}\,\hat{\boldsymbol{\phi}}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial A_\phi}{\partial \theta}\,\hat{\boldsymbol{\phi}}\hat{\boldsymbol{\theta}} + \left(\frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi} + \frac{A_r}{r} + \cot\theta\frac{A_\theta}{r}\right)\hat{\boldsymbol{\phi}}\hat{\boldsymbol{\phi}},\end{aligned} \tag{IX.7.9}$$

$$\begin{aligned}(\boldsymbol{B}\cdot\nabla)\boldsymbol{A} = {} & \left(B_r\frac{\partial A_r}{\partial r} + \frac{B_\theta}{r}\frac{\partial A_r}{\partial \theta} + \frac{B_\phi}{r\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{B_\theta A_\theta + B_\phi A_\phi}{r}\right)\hat{\boldsymbol{r}} \\ & + \left(B_r\frac{\partial A_\theta}{\partial r} + \frac{B_\theta}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{B_\phi}{r\sin\theta}\frac{\partial A_\theta}{\partial \phi} + \frac{B_\theta A_r}{r} - \cot\theta\frac{B_\phi A_\phi}{r}\right)\hat{\boldsymbol{\phi}} \\ & + \left(B_r\frac{\partial A_\phi}{\partial r} + \frac{B_\theta}{r}\frac{\partial A_\phi}{\partial \theta} + \frac{B_\phi}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi} + \frac{B_\phi A_r}{r} + \cot\theta\frac{B_\phi A_\theta}{r}\right)\hat{\boldsymbol{\phi}},\end{aligned} \tag{IX.7.10}$$

$$\begin{aligned}\nabla\cdot\boldsymbol{\mathsf{T}} = {} & \left(\frac{\partial \mathsf{T}_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \mathsf{T}_{\theta r}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \mathsf{T}_{\phi r}}{\partial \phi} + \frac{2\mathsf{T}_{rr} - \mathsf{T}_{\theta\theta} - \mathsf{T}_{\phi\phi}}{r} + \cot\theta\frac{\mathsf{T}_{\theta r}}{r}\right)\hat{\boldsymbol{r}} \\ & + \left(\frac{\partial \mathsf{T}_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \mathsf{T}_{\theta\theta}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \mathsf{T}_{\phi\theta}}{\partial \phi} + \frac{2\mathsf{T}_{r\theta} + \mathsf{T}_{\theta r}}{r} + \cot\theta\frac{\mathsf{T}_{\theta\theta} - \mathsf{T}_{\phi\phi}}{r}\right)\hat{\boldsymbol{\theta}} \\ & + \left(\frac{\partial \mathsf{T}_{r\phi}}{\partial r} + \frac{1}{r}\frac{\partial \mathsf{T}_{\theta\phi}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \mathsf{T}_{\phi\phi}}{\partial \phi} + \frac{2\mathsf{T}_{r\phi} + \mathsf{T}_{\phi r}}{r} + \cot\theta\frac{\mathsf{T}_{\theta\phi} + \mathsf{T}_{\phi\theta}}{r}\right)\hat{\boldsymbol{\phi}}.\end{aligned} \tag{IX.7.11}$$

For example, the rate-of-strain tensor for a differentially rotating plasma with velocity $\boldsymbol{u} = R\varpi(r,\theta)\hat{\boldsymbol{\phi}}$ with $R = r\sin\theta$ is given by

$$\boldsymbol{\mathsf{W}} = \sin\theta\frac{\partial\varpi}{\partial\ln r}\,(\hat{\boldsymbol{r}}\hat{\boldsymbol{\phi}} + \hat{\boldsymbol{\phi}}\hat{\boldsymbol{r}}) + \sin\theta\frac{\partial\varpi}{\partial\theta}\,(\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\phi}} + \hat{\boldsymbol{\phi}}\hat{\boldsymbol{\theta}}). \tag{IX.7.12}$$

IX.8. Resistivity of a poorly ionized plasma

This is **optional material** presenting a derivation of the electrical resistivity of a poorly ionized, magnetized plasma. Its content is important in the study of astrophysical plasmas – namely, molecular clouds, protostellar cores, and protoplanetary disks – but is somewhat peripheral to what else is covered in this course. The closest chapter to which it's related is Chapter IX.1 on the Spitzer–Härm problem, but here the current-bearing species are taken to be much less abundant than a population of neutral particles and a magnetic field is present that effectively promotes the electrical conductivity from a scalar to a tensor (à la Braginskii). Let's get started.

Consider a collisional plasma composed of neutrals, ions, and electrons. To give some physical context here, the cold ($T \sim 10$ K) plasma out of which stars form is comprised primarily of neutral molecular hydrogen H_2 ($n_{H_2} \gtrsim 10^3$ cm^{-3}) with 20% He by number, along with trace ($\lesssim 10^{-7}$) amounts of electrons, molecular ions (primarily HCO^+), and atomic ions (primarily Na^+, Mg^+, K^+). There are also neutral, negatively charged, and positively charged dust grains, conglomerates of silicate and carbonaceous materials that are between a few molecules to 0.1 μm in size. While of critical importance to interstellar chemistry and thermodynamics, and magnetic-field diffusion, we ignore dust grains in what follows. Molecular clouds are poorly ionized because their densities are large enough to screen the most potent sources of ionization (e.g., UV radiation) and their temperatures are low enough to render thermal ionization completely negligible. This leaves only infrequent cosmic rays of energy $\gtrsim$100 MeV (and extremely weak radioactive nuclides like ^{26}Al and ^{40}K) to ionize the plasma. So sad.

Assuming these species are collisional enough for their local distribution functions to described by isotropic Maxwellians, the momentum equations for the neutrals, ions, and electrons are

$$m_\mathrm{n} n_\mathrm{n} \left(\frac{\partial}{\partial t} + \boldsymbol{u}_\mathrm{n} \cdot \boldsymbol{\nabla} \right) \boldsymbol{u}_\mathrm{n} = -\boldsymbol{\nabla} p_\mathrm{n} + \boldsymbol{R}_\mathrm{ni} + \boldsymbol{R}_\mathrm{ne} + m_\mathrm{n} n_\mathrm{n} \boldsymbol{g}, \tag{IX.8.1a}$$

$$m_\mathrm{i} n_\mathrm{i} \left(\frac{\partial}{\partial t} + \boldsymbol{u}_\mathrm{i} \cdot \boldsymbol{\nabla} \right) \boldsymbol{u}_\mathrm{i} = -\boldsymbol{\nabla} p_\mathrm{i} + \boldsymbol{R}_\mathrm{in} + \boldsymbol{R}_\mathrm{ie} + Zen_\mathrm{i} \left(\boldsymbol{E} + \frac{\boldsymbol{u}_\mathrm{i} \times \boldsymbol{B}}{c} \right) + m_\mathrm{i} n_\mathrm{i} \boldsymbol{g}, \tag{IX.8.1b}$$

$$m_\mathrm{e} n_\mathrm{e} \left(\frac{\partial}{\partial t} + \boldsymbol{u}_\mathrm{e} \cdot \boldsymbol{\nabla} \right) \boldsymbol{u}_\mathrm{e} = -\boldsymbol{\nabla} p_\mathrm{e} + \boldsymbol{R}_\mathrm{en} + \boldsymbol{R}_\mathrm{ei} - en_\mathrm{e} \left(\boldsymbol{E} + \frac{\boldsymbol{u}_\mathrm{e} \times \boldsymbol{B}}{c} \right) + m_\mathrm{e} n_\mathrm{e} \boldsymbol{g}, \tag{IX.8.1c}$$

respectively, where $\boldsymbol{R}_{\alpha\beta}$ is the friction force on α due to collisions with β and $\boldsymbol{g}$ is the gravitational acceleration. The other symbols have their usual meanings: m_α is the mass, n_α is the number density, $\boldsymbol{u}_\alpha$ is the fluid velocity, $p_\alpha = n_\alpha T_\alpha$ is the gas pressure, and T_α is the temperature, all of which refer to species α; $\boldsymbol{E}$ is the electric field and $\boldsymbol{B}$ is the magnetic field. The ion charge $q_\mathrm{i} = Ze$. The goal is to use these equations to derive the electrical conductivity tensor of a poorly ionized, quasi-neutral, collisional plasma with $Zn_\mathrm{i} = n_\mathrm{e} \ll n_\mathrm{n}$.

Under molecular-cloud and protostellar conditions, interspecies collisions are strong enough to guarantee that $T_\mathrm{n} = T_\mathrm{i} = T_\mathrm{e}$. The friction forces are primarily due to elastic collisions and are accurately modeled by

$$\boldsymbol{R}_\mathrm{in} = -\boldsymbol{R}_\mathrm{ni} = \frac{m_\mathrm{n} n_\mathrm{n}}{\tau_\mathrm{ni}} (\boldsymbol{u}_\mathrm{n} - \boldsymbol{u}_\mathrm{i}) \tag{IX.8.2a}$$

$$\text{with } \tau_\mathrm{ni} = \frac{\varrho_\mathrm{n}}{\varrho_\mathrm{i}} \tau_\mathrm{in} = 1.23 \frac{m_\mathrm{i} + m_{\mathrm{H}_2}}{\varrho_\mathrm{i} \langle \sigma w \rangle_{\mathrm{iH}_2}} \simeq 0.23 \text{ Myr} \left(1 + \frac{m_{\mathrm{H}_2}}{m_\mathrm{i}} \right) \left(\frac{10^{-7}}{x_\mathrm{i}} \right) \left(\frac{10^3 \text{ cm}^{-3}}{n_\mathrm{n}} \right),$$

$$\boldsymbol{R}_\mathrm{en} = -\boldsymbol{R}_\mathrm{ne} = \frac{m_\mathrm{n} n_\mathrm{n}}{\tau_\mathrm{ne}} (\boldsymbol{u}_\mathrm{n} - \boldsymbol{u}_\mathrm{e}) \tag{IX.8.2b}$$

$$\text{with } \tau_\mathrm{ne} = \frac{\varrho_\mathrm{n}}{\varrho_\mathrm{e}} \tau_\mathrm{en} = 1.21 \frac{m_\mathrm{e} + m_{\mathrm{H}_2}}{\varrho_\mathrm{e} \langle \sigma w \rangle_{\mathrm{eH}_2}} \simeq 0.29 \text{ Myr} \, \frac{m_{\mathrm{H}_2}}{m_\mathrm{e}} \left(\frac{10^{-7}}{x_\mathrm{i}} \right) \left(\frac{10^3 \text{ cm}^{-3}}{n_\mathrm{n}} \right) \left(\frac{10 \text{ K}}{T} \right)^{1/2},$$

$$\boldsymbol{R}_\mathrm{ie} = -\boldsymbol{R}_\mathrm{ei} = \frac{m_\mathrm{i} n_\mathrm{i}}{\tau_\mathrm{ie}} (\boldsymbol{u}_\mathrm{e} - \boldsymbol{u}_\mathrm{i}) \tag{IX.8.2c}$$

$$\text{with } \tau_\mathrm{ie} = \frac{\varrho_\mathrm{i}}{\varrho_\mathrm{e}} \tau_\mathrm{ei} \simeq 1.2 \text{ hr} \, \frac{m_\mathrm{i}}{m_\mathrm{e}} \left(\frac{10^{-7}}{x_\mathrm{i}} \right) \left(\frac{10^3 \text{ cm}^{-3}}{n_\mathrm{n}} \right) \left(\frac{T}{10 \text{ K}} \right)^{3/2},$$

where $x_\mathrm{i} \doteq n_\mathrm{i}/n_\mathrm{n}$ is the degree of ionization, $\langle \sigma w \rangle_{\mathrm{iH}_2} \simeq 1.69 \times 10^{-9}$ cm^3 s^{-1} for HCO^+–H_2 collisions, and $\langle \sigma w \rangle_{\mathrm{eH}_2} \simeq 1.3 \times 10^{-9}$ cm^3 s^{-1} for e–H_2 collisions. The collision timescales are calculated using fiducial molecular-cloud parameters; the numerical pre-factors of 1.23 and 1.21 are the factors by which the presence of He lengthens the slowing-down time relative to the value it would have if only H_2–s collisions were considered. The following mass ratios are useful: $m_\mathrm{i}/m_\mathrm{p} = 29$ for HCO^+, $m_\mathrm{i}/m_\mathrm{p} = 23$ for Na^+, $m_\mathrm{i}/m_\mathrm{p} = 24$ for

Mg^+, and $m_p/m_e = 1836$. The mean mass per neutral particle in molecular clouds is $m_n = 2.33m_p$.

To give the above timescales some context, and to educate you a bit on a non-laboratory plasma, dynamical timescales in star-forming molecular clouds are $\sim$0.1–10 Myr. Magnetic-field strengths are $\sim$10–100 μG, giving an ion cyclotron frequency $\Omega_i \sim 0.1$ Hz and an Alfvén speed $\sim$1 km s^{-1}. Every plasma astrophysicist should know that 1 km s$^{-1} \simeq$ 1 pc Myr^{-1}, and so an Alfvén wave crosses a typical molecular cloud of size $\sim$10 pc in $\sim$10 Myr and a typical pre-stellar core of size $\sim$0.1 pc in $\sim$0.1 Myr. Sound travels slower at $\simeq$0.2 km s^{-1}, and so the plasma $\beta \sim 0.01$ or so. The gravitational free-fall time is roughly $\sim$1 Myr at the mean density of a molecular cloud, although support against gravitational collapse provided by magnetic tension renders this timescale almost meaningless.

Okay, enough of this astrophysics propaganda...

Add (IX.8.1b) and (IX.8.1c) together:

$$m_i n_i \frac{\mathrm{D}\boldsymbol{u}_i}{\mathrm{D}t_i} + m_e n_e \frac{\mathrm{D}\boldsymbol{u}_e}{\mathrm{D}t_e} = -\boldsymbol{\nabla}(p_i + p_e) + \boldsymbol{R}_{in} + \cancel{\boldsymbol{R}_{ie}} + \boldsymbol{R}_{en} + \cancel{\boldsymbol{R}_{ei}} + (m_i n_i + m_e n_e)\boldsymbol{g}$$
$$+ \underbrace{(q_i n_i - e n_e)}_{\text{= 0 by quasi-neutrality}} \boldsymbol{E} + \frac{1}{c} \underbrace{(q_i n_i \boldsymbol{u}_i - e n_e \boldsymbol{u}_e)}_{= \boldsymbol{j} \text{ by def'n}} \times \boldsymbol{B}. \tag{IX.8.3}$$

Now add (IX.8.1a) and (IX.8.3):

$$m_n n_n \frac{\mathrm{D}\boldsymbol{u}_n}{\mathrm{D}t_n} + m_i n_i \frac{\mathrm{D}\boldsymbol{u}_i}{\mathrm{D}t_i} + m_e n_e \frac{\mathrm{D}\boldsymbol{u}_e}{\mathrm{D}t_e} = -\boldsymbol{\nabla}(p_n + p_i + p_e) + (m_n n_n + m_i n_i + m_e n_e)\boldsymbol{g}$$
$$+ \cancel{\boldsymbol{R}_{ni}} + \cancel{\boldsymbol{R}_{ne}} + \cancel{\boldsymbol{R}_{in}} + \cancel{\boldsymbol{R}_{en}} + \frac{\boldsymbol{j}}{c} \times \boldsymbol{B}. \tag{IX.8.4}$$

All the friction forces cancel by Newton's third law. Recalling (IX.2.11)–(IX.2.13), the left-hand side of (IX.8.4) may be written as

$$\varrho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \boldsymbol{\nabla} \cdot \left(\sum_\alpha m_\alpha n_\alpha \Delta\boldsymbol{u}_\alpha \Delta\boldsymbol{u}_\alpha \right),$$

where $\varrho \doteq \sum_\alpha m_\alpha n_\alpha$ and $\Delta\boldsymbol{u}_\alpha \doteq \boldsymbol{u}_\alpha - \boldsymbol{u}$ are the species drifts relative to the center-of-mass velocity $\boldsymbol{u}$. Further using Ampère's law to write

$$\frac{\boldsymbol{j}}{c} \times \boldsymbol{B} = \boldsymbol{\nabla} \cdot \left(\frac{\boldsymbol{BB}}{4\pi} - \boldsymbol{I}\frac{B^2}{8\pi} \right), \tag{IX.8.5}$$

equation (IX.8.4) becomes

$$\varrho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\boldsymbol{\nabla} \cdot \left[\boldsymbol{I} \left(\sum_\alpha p_\alpha + \frac{B^2}{8\pi} \right) + \sum_\alpha m_\alpha n_\alpha \Delta\boldsymbol{u}_\alpha \Delta\boldsymbol{u}_\alpha - \frac{\boldsymbol{BB}}{4\pi} + \varrho\boldsymbol{g} \right]. \tag{IX.8.6}$$

With $n_i, n_e \ll n_n$, equation (IX.8.6) is, to a very good approximation,

$$m_n n_n \left(\frac{\partial}{\partial t} + \boldsymbol{u}_n \cdot \boldsymbol{\nabla} \right) \boldsymbol{u}_n = -\boldsymbol{\nabla} \left(p_n + \frac{B^2}{8\pi} \right) + \frac{\boldsymbol{B} \cdot \boldsymbol{\nabla}\boldsymbol{B}}{4\pi} + m_n n_n \, \boldsymbol{g}. \tag{IX.8.7}$$

So, collisions between the charged species and the neutrals are responsible for transmitting the Lorentz force to the bulk neutral plasma. By virtue of the relatively large mass of the neutrals and the low degree of ionization in many system, $\boldsymbol{u} \simeq \boldsymbol{u}_n$, and so it *looks* like the neutrals are magnetized. Not true. They just need to collide often enough with the magnetized particles.

With that borne in mind, let us again return to the MHD induction equation,

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}).$$

Now, that $\boldsymbol{u}$ *cannot* be the neutral velocity; it would make no sense for the magnetic flux to be frozen into a neutral fluid! Let us instead write

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{u}_\mathrm{f} \times \boldsymbol{B}), \tag{IX.8.8}$$

where $\boldsymbol{u}_\mathrm{f}$ is the velocity of the field lines. This must be true: field lines are frozen into themselves (i.e., there exists a frame where the electric field vanishes). Now add zero:

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times \Bigl[\underbrace{(\boldsymbol{u}_\mathrm{f} - \boldsymbol{u}_\mathrm{e})}_{\substack{\text{electron-}\boldsymbol{B}\\ \text{drift}\\ \text{O}}} \times \boldsymbol{B} + \underbrace{(\boldsymbol{u}_\mathrm{e} - \boldsymbol{u}_\mathrm{i})}_{\substack{\text{ion-electron}\\ \text{drift}\\ \text{H}}} \times \boldsymbol{B} + \underbrace{(\boldsymbol{u}_\mathrm{i} - \boldsymbol{u}_\mathrm{n})}_{\substack{\text{ion-neutral}\\ \text{drift}\\ \text{A}}} \times \boldsymbol{B} + \underbrace{\boldsymbol{u}_\mathrm{n} \times \boldsymbol{B}}_{\substack{\text{advection}\\ \text{by neutrals}\\ \text{I}}} \Bigr]. \tag{IX.8.9}$$

The terms in (IX.8.9) labelled O (Ohmic), H (Hall), and A (ambipolar) are formally zero in ideal MHD.[20] Let us estimate their relative sizes:

$$\frac{\text{O}}{\text{I}} \sim \frac{1}{\mathrm{Rm}} \doteq \frac{\eta}{v_\mathrm{A} \ell_B} \sim \underbrace{\left(\frac{d_\mathrm{e}}{\ell_B}\right)}_{\text{small}} \underbrace{\left(\frac{d_\mathrm{e}}{v_\mathrm{A} \tau_\mathrm{en}}\right)}_{\substack{\text{could be}\\ \text{large}}} \tag{IX.8.10}$$

$$\frac{\text{H}}{\text{I}} \sim \left| \frac{\boldsymbol{j}/en_\mathrm{e}}{\boldsymbol{u}_\mathrm{n}} \right| \sim \underbrace{\left(\frac{d_\mathrm{i}}{\ell_B}\right)}_{\text{small}} \underbrace{\left(\frac{\varrho}{\varrho_\mathrm{i}}\right)^{1/2}}_{\substack{\sim 1\text{, but}\\ \text{could be}\\ \text{large}}} \underbrace{\left|\frac{v_\mathrm{A}}{u_\mathrm{n}}\right|}_{\sim 1} \tag{IX.8.11}$$

$$\frac{\text{A}}{\text{I}} \sim \left| \frac{\boldsymbol{R}_\mathrm{ni} \tau_\mathrm{ni}}{\varrho_\mathrm{n} \boldsymbol{u}_\mathrm{n}} \right| \sim \left| \frac{\boldsymbol{j} \times \boldsymbol{B}}{c} \right| \frac{\tau_\mathrm{ni}}{\varrho_\mathrm{n} v_\mathrm{A}} \left| \frac{v_\mathrm{A}}{u_\mathrm{n}} \right| \sim \underbrace{\frac{v_\mathrm{A} \tau_\mathrm{ni}}{\ell_B}}_{\substack{\text{could be}\\ \sim 1}} \underbrace{\left(\frac{\varrho}{\varrho_\mathrm{n}}\right)^{1/2}}_{\gtrsim 1} \underbrace{\left|\frac{v_\mathrm{A}}{u_\mathrm{n}}\right|}_{\sim 1} \tag{IX.8.12}$$

Note that H/I is the only ratio not involving collisions... we'll come back to this.

Our task now is to compute these Ohmic, Hall, and ambipolar terms more rigorously. To do that, start with the momentum equation for the charged species s $(= \mathrm{i}, \mathrm{e})$, which under poorly ionized, collisional, and strongly magnetized conditions may be simplified

[20] Plasma physicists and plasma astrophysicists have different definitions of "ambipolar diffusion". The former use the term to describe the diffusion of oppositely charged species as they interact via an electric field that is trying to enforce quasi-neutrality. The idea is that an electric field is set up to ensure electrons and ions diffuse at the same rate, preserving quasi-neutrality ("ambi" means "both".) The latter community uses the term to describe ion-neutral drifts, by which the magnetic flux diffuses along with flux-frozen ions (and electrons) through a predominantly neutral fluid (Mestel & Spitzer 1956; Mouschovias 1979).

without great consequence to obtain

$$0 = q_s n_s \left(\boldsymbol{E} + \frac{\boldsymbol{u}_s \times \boldsymbol{B}}{c} \right) + \boldsymbol{R}_{s\mathrm{n}}. \tag{IX.8.13}$$

The assumptions here are that collisions between charged species, the thermal pressure of charged species, and the inertia of charged species are all negligible compared to collisions with the neutrals and electromagnetic forces. Introduce the velocity of species s relative to the neutrals, $\boldsymbol{w}_s \doteq \boldsymbol{u}_s - \boldsymbol{u}_\mathrm{n}$, and the electric field in the frame of the neutrals, $\boldsymbol{E}_\mathrm{n} \doteq \boldsymbol{E} + \boldsymbol{u}_\mathrm{n} \times \boldsymbol{B}/c$. With these definitions, equation (IX.8.13) may be written as

$$0 = q_s n_s \left(\boldsymbol{E}_\mathrm{n} + \frac{\boldsymbol{w}_s \times \boldsymbol{B}}{c} \right) - \frac{m_s n_s}{\tau_{s\mathrm{n}}} \boldsymbol{w}_s. \tag{IX.8.14}$$

(Note that $\tau_{s\mathrm{n}} \neq \tau_{\mathrm{n}s}$, or else Newton would be very unhappy!) Using quasi-neutrality, the current density

$$\boldsymbol{j} = \sum_s q_s n_s \boldsymbol{u}_s = \sum_s q_s n_s \boldsymbol{w}_s. \tag{IX.8.15}$$

Equation (IX.8.14) may be solved to relate the components of $\boldsymbol{j}$ in the directions parallel ($\|$) and perpendicular ($\perp$) to the magnetic field to the corresponding components of $\boldsymbol{E}_\mathrm{n}$.

Start by taking the cross product of (IX.8.14) with $\boldsymbol{B}$ and multiplying the result by $q_s \tau_{s\mathrm{n}}/m_s c$ to find

$$0 = \frac{q_s^2 n_s \tau_{s\mathrm{n}}}{m_s c} \left(\boldsymbol{E}_\mathrm{n} \times \boldsymbol{B} - \frac{\boldsymbol{w}_{s\perp}}{c} B^2 \right) - q_s n_s \frac{\boldsymbol{w}_s}{c} \times \boldsymbol{B}. \tag{IX.8.16}$$

Adding (IX.8.16) to (IX.8.14) and multiplying by $\tau_{s\mathrm{n}}/\varrho_s$,

$$0 = \frac{q_s \tau_{s\mathrm{n}}}{m_s} \boldsymbol{E}_\mathrm{n} + (\Omega_s \tau_{s\mathrm{n}})^2 \left(\frac{c}{B} \boldsymbol{E}_\mathrm{n} \times \hat{\boldsymbol{b}} - \boldsymbol{w}_{s\perp} \right) - \boldsymbol{w}_s. \tag{IX.8.17}$$

Note that if the entire charged plasma is well magnetized, *viz.* $(\Omega_s \tau_{s\mathrm{n}})^2 \gg 1$ for each s, then the leading-order motion of all species consists of the same $\boldsymbol{E} \times \boldsymbol{B}$ drift.

We solve (IX.8.17) by examining its parallel and perpendicular components separately. The former gives

$$\boldsymbol{w}_{s\|} = \frac{q_s \tau_{s\mathrm{n}}}{m_s} \boldsymbol{E}_{\mathrm{n}\|} \quad \Longrightarrow \quad \boldsymbol{j}_\| = \left(\sum_s \frac{q_s^2 n_s \tau_{s\mathrm{n}}}{m_s} \right) \boldsymbol{E}_{\mathrm{n}\|} \doteq \left(\sum_s \sigma_s \right) \boldsymbol{E}_{\mathrm{n}\|} \doteq \sigma_\| \boldsymbol{E}_{\mathrm{n}\|}, \tag{IX.8.18}$$

where the parallel conductivity $\sigma_\|$ has been defined *in situ*. The perpendicular component of (IX.8.17) may be rearranged to obtain

$$\boldsymbol{w}_{s\perp} = \frac{q_s \tau_{s\mathrm{n}}}{m_s} \left[\frac{1}{1 + (\Omega_s \tau_{s\mathrm{n}})^2} \boldsymbol{E}_{\mathrm{n}\perp} + \frac{\Omega_s \tau_{s\mathrm{n}}}{1 + (\Omega_s \tau_{s\mathrm{n}})^2} \boldsymbol{E}_\mathrm{n} \times \hat{\boldsymbol{b}} \right]$$

$$\Longrightarrow \quad \boldsymbol{j}_\perp = \left[\sum_s \frac{\sigma_s}{1 + (\Omega_s \tau_{s\mathrm{n}})^2} \right] \boldsymbol{E}_{\mathrm{n}\perp} + \left[\sum_s \frac{\sigma_s \Omega_s \tau_{s\mathrm{n}}}{1 + (\Omega_s \tau_{s\mathrm{n}})^2} \right] \boldsymbol{E}_\mathrm{n} \times \hat{\boldsymbol{b}}$$

$$\doteq \sigma_\perp \boldsymbol{E}_{\mathrm{n}\perp} - \sigma_\mathrm{H} \boldsymbol{E}_\mathrm{n} \times \hat{\boldsymbol{b}}, \tag{IX.8.19}$$

where the perpendicular conductivity $\sigma_\perp$ and Hall conductivity σ_H have been defined *in situ*. (*Question:* What if $\Omega_s \tau_{s\mathrm{n}} \gg 1$ for all charged species? What if $\Omega_s \tau_{s\mathrm{n}} \ll 1$ for all charged species? Do the asymptotic values of $\sigma_\|$, $\sigma_\perp$, and σ_H make sense to you?)

Combining (IX.8.18) and (IX.8.19), the total current density

$$\boldsymbol{j} = \sigma_\| \boldsymbol{E}_{\mathrm{n}\|} + \sigma_\perp \boldsymbol{E}_{\mathrm{n}\perp} - \sigma_\mathrm{H} \boldsymbol{E}_\mathrm{n} \times \hat{\boldsymbol{b}}, \tag{IX.8.20}$$

which may be inverted to find

$$\boldsymbol{E}_{\rm n} = \eta_\parallel \boldsymbol{j}_\parallel + \eta_\perp \boldsymbol{j}_\perp + \eta_{\rm H} \boldsymbol{j} \times \hat{\boldsymbol{b}}, \tag{IX.8.21}$$

where the parallel, perpendicular, and Hall resistivities are

$$\eta_\parallel \doteq \frac{1}{\sigma_\parallel}, \quad \eta_\perp \doteq \frac{\sigma_\perp}{\sigma_\perp^2 + \sigma_{\rm H}^2}, \quad \eta_{\rm H} \doteq \frac{\sigma_{\rm H}}{\sigma_\perp^2 + \sigma_{\rm H}^2}, \tag{IX.8.22}$$

respectively. Knowing that Ohmic dissipation affects the total current while ambipolar diffusion affects only the perpendicular component, the Ohmic (O) and ambipolar (A) resistivities are

$$\eta_{\rm O} \doteq \eta_\parallel \quad \text{and} \quad \eta_{\rm A} \doteq \eta_\perp - \eta_\parallel, \tag{IX.8.23}$$

respectively. Thus,

$$\boldsymbol{E} = -\frac{\boldsymbol{u}_{\rm n}}{c} \times \boldsymbol{B} + \eta_{\rm O} \boldsymbol{j} + \eta_{\rm A} \boldsymbol{j}_\perp + \eta_{\rm H} \boldsymbol{j} \times \hat{\boldsymbol{b}} \tag{IX.8.24}$$

is the generalized Ohm's law. Note that an arbitrary number of species may be included in this expression by simply adding their contributions to the conductivity tensor $\boldsymbol{\sigma}$, so long as their abundances are small enough that they may be considered inertia- and pressure-less and so long as the dominant collisional processes affecting their dynamics involve only the neutrals. For example, the collision time between a neutral and a charged grain is

$$\tau_{\rm ng} = \frac{\varrho_{\rm n}}{\varrho_{\rm g}} \tau_{\rm gn} = 1.09\, \frac{m_{\rm g} + m_{\rm H_2}}{\varrho_{\rm g} \langle \sigma w \rangle_{\rm gH_2}}, \tag{IX.8.25}$$

where the mean collisional rate between the grain species and H_2 is

$$\langle \sigma w \rangle_{\rm gH_2} = \pi a_{\rm gr}^2 \left(\frac{8 k_{\rm B} T}{\pi m_{\rm H_2}} \right)^{1/2} \tag{IX.8.26}$$

for sub-sonic drift speeds and grains of radius $a_{\rm gr}$. Inelastic collisions between charged grains, neutral grains, ions, and electrons can also be included (with some effort; see Kunz & Mouschovias 2009.)

Equation (IX.8.24 can be inserted into Faraday's law to obtain

$$\frac{\partial \boldsymbol{B}}{\partial t} = -c \boldsymbol{\nabla} \times \boldsymbol{E} = \boldsymbol{\nabla} \times \left[\boldsymbol{u}_{\rm n} \times \boldsymbol{B} - c\eta_{\rm O} \boldsymbol{J} - c\eta_{\rm A} \boldsymbol{J}_\perp - c\eta_{\rm H} \boldsymbol{J} \times \hat{\boldsymbol{b}} \right]. \tag{IX.8.27}$$

In the ideal-MHD limit $\boldsymbol{\eta} \rightarrow \boldsymbol{0}$, the flux is effectively frozen into the neutrals. Note that the Hall effect depends on the sign of the magnetic field, which can make for interesting dynamics (e.g., Wardle 1999; Balbus & Terquem 2001; Kunz 2008; Kunz & Lesur 2013).

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