Due Wednesday, March 18, 2026

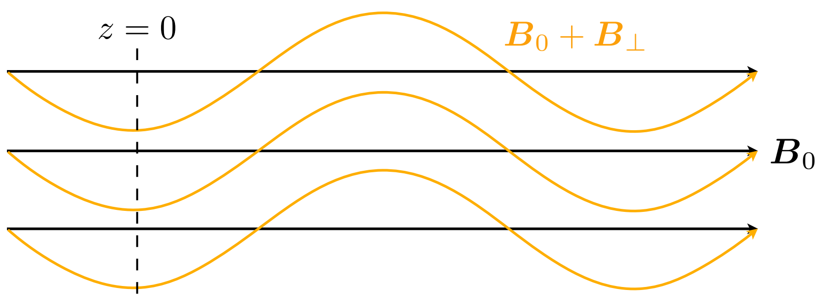
Generals prep. Make sure you can provide brief definitions of the following terms: Alfvén wave, magnetosonic wave, entropy mode, toroidal Alfvén eigenmode, Elsasser fields.

1. **Drifting in the waves.**

(a) A small-amplitude, linearly polarized Alfvén wave of amplitude $B_\perp$ and wavenumber $k_\parallel > 0$ propagates along a uniform magnetic field $B_0\hat{\boldsymbol{z}}$ through an otherwise stationary, uniform, ideal-MHD plasma. The magnetic field and fluid velocity are given by

$$\boldsymbol{B} = B_0\hat{\boldsymbol{z}} + B_\perp \sin\big[k_\parallel(z - v_\mathrm{A}t)\big]\hat{\boldsymbol{x}} \quad \text{and} \quad \boldsymbol{u} = -v_\mathrm{A}\frac{B_\perp}{B_0}\sin\big[k_\parallel(z - v_\mathrm{A}t)\big]\hat{\boldsymbol{x}},$$

respectively, where $v_\mathrm{A} \doteq B_0/\sqrt{4\pi\rho_0}$ is the Alfvén speed of the background plasma (see figure below). Neglecting terms of order $B_\perp^2$ and higher, compute the current density $\boldsymbol{j}_\mathrm{pol}$ associated with the particles' polarization drift in this wave and show that it is equal to the total current density from Ampère's law, $\boldsymbol{j} = (c/4\pi)\boldsymbol{\nabla}\times\boldsymbol{B}$.[1]



(b) Following on from part (a), there is also a current associated with the curvature drift experienced by the particles in the bent field lines, which is equal to

$$\boldsymbol{j}_\mathrm{curv} = \frac{ck_\parallel B_\perp}{4\pi}\frac{4\pi P_0}{B_0^2}\cos\big[k_\parallel(z - v_\mathrm{A}t)\big]\,\hat{\boldsymbol{y}},$$

where P_0 is the thermal pressure in the background plasma. If $\boldsymbol{j}_\mathrm{pol} = \boldsymbol{j}$, then what balances $\boldsymbol{j}_\mathrm{curv}$? Prove it.[2]

(c) **(Optional)** A small-amplitude fast mode of amplitude $B_\parallel$ and wavenumber $k_\perp > 0$ propagates across a uniform magnetic field $B_0\hat{\boldsymbol{z}}$ through an otherwise stationary, uniform, ideal-MHD plasma. The magnetic field and fluid velocity are given by

$$\boldsymbol{B} = B_0\hat{\boldsymbol{z}} + B_\parallel \sin\big[k_\perp(x - v_\mathrm{f}t)\big]\hat{\boldsymbol{z}} \quad \text{and} \quad \boldsymbol{u} = v_\mathrm{f}\frac{B_\parallel}{B_0}\sin\big[k_\perp(x - v_\mathrm{f}t)\big]\hat{\boldsymbol{x}},$$

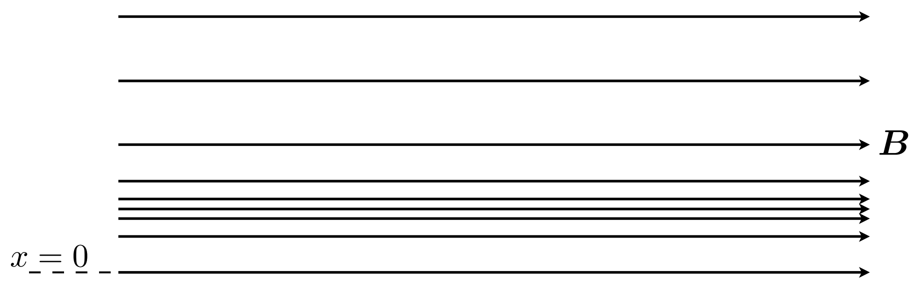
[1] Here the analogy to standard electrodynamics in a dielectric medium is illuminating: $\boldsymbol{j}_\mathrm{pol} = (\varepsilon/4\pi)\partial\boldsymbol{E}/\partial t$, where ε is the permittivity. Noting that $\boldsymbol{E} = -(\boldsymbol{u}/c)\times\boldsymbol{B}$ in ideal MHD, you should find that $\varepsilon = (c/v_\mathrm{A})^2$, making a direct connection between the speed at which Alfvén waves travel and the polarizability of the plasma. Since we often have $c/v_\mathrm{A} \gg 1$, most plasmas have $\varepsilon \gg 1$, i.e., they behave as strongly polarizable media.

[2] The balance you'll derive here does not hold in a pressure-anisotropic plasma with $P_\perp \neq P_\parallel$. Stay tuned.

respectively, where $v_{\rm f} \doteq \sqrt{v_{\rm A}^2 + c_{\rm s}^2}$ is the fast magnetosonic speed and $c_{\rm s} \doteq (\gamma P_0/\rho_0)^{1/2}$ is the adiabatic sound speed in the background plasma (see figure below). Neglecting terms of order $B_\parallel^2$ and higher, compute the current density $\boldsymbol{j}_{\nabla B}$ associated with the particles' grad-B drift in this wave. Then calculate the current density $\boldsymbol{j}_{\rm pol}$ associated with the particles' polarization drift in this wave. Show that their sum is equal to

$$\boldsymbol{j}_{\nabla B} + \boldsymbol{j}_{\rm pol} = \boldsymbol{j} - (\gamma - 1)\frac{ck_\perp B_\parallel}{4\pi}\,\frac{4\pi P_0}{B_0^2}\cos\bigl[k_\perp(x - v_{\rm f}t)\bigr]\,\hat{\boldsymbol{y}},$$

where $\boldsymbol{j} = (c/4\pi)\boldsymbol{\nabla}\times\boldsymbol{B}$ is the current density. What accounts for the final term on the right-hand side? Prove it.



2. **Magnetic braking.** A classic problem in astrophysical fluid dynamics, formulated by Leon Mestel in 1965 and refined by Telemachos Mouchovias and his students in the late '70s and '80s, concerns the means by which angular momentum is transported away from objects contracting under their own self-gravity. An illustrative, back-of-the-envelope calculation goes as follows. Take a spherical clump of one solar mass (1 $\mathrm{M}_\odot$) out of the interstellar medium (ISM) of our Galaxy and try to form the Sun. In ballpark numbers, the mean density of the ISM is $n_{\rm ISM} \sim 1~{\rm cm}^{-3}$ and the mean density of the Sun is $n_\odot \sim 10^{24}~{\rm cm}^{-3}$ ($\sim$1 g cm^{-3}, the density of water). That's a long way to go. By virtue of residing in a rotating galaxy, that 1 $\mathrm{M}_\odot$ chunk of the ISM you have in your eager hands will inherit angular momentum corresponding to a rotation rate of $\Omega_{\rm ISM} \sim 10^{-15}~{\rm s}^{-1}$. Supposing that the mass and angular momentum of the sphere were to be conserved during its contraction, we have

$$\frac{\Omega_{\rm final}}{\Omega_{\rm init}} = \frac{R_{\rm init}^2}{R_{\rm final}^2} = \left(\frac{n_\odot}{n_{\rm ISM}}\right)^{2/3} \sim 10^{16} \quad\Longrightarrow\quad \Omega_{\rm final} \sim 10~{\rm s}^{-1}.$$

Our Sun rotates every $\approx$24 days (at its equator), not every $2\pi/\Omega_{\rm final} \sim 1$ s. If angular momentum were conserved from the initial galactic rotation, centrifugal forces would not allow even the formation of its natal molecular cloud ($n \sim 300~{\rm cm}^{-3}$), let alone the star that formed within it. And yet observations of embedded fragments in such molecular clouds (the prestellar condensations that ultimately collapse to form stars) rarely exhibit rotation significantly greater than that of the background medium (e.g., Goldsmith & Arquila 1985). Where does the more than six orders of magnitude in angular momentum go? Let's find out.

To keep things relatively simple, imagine a rigid, cylindrical disk of (fixed) half-thickness Z comprised of perfectly conducting material. Thread this disk along its symmetry (z) axis by a uniform magnetic field $\boldsymbol{B} = B_0\hat{\boldsymbol{z}}$ that is anchored into the surrounding "external" medium.

Let us not concern ourselves with equilibrium states or the possible self-gravitational contraction of the disk, but rather focus on what happens to this rigidly rotating disk if its rotation frequency were to exceed that of the external medium. Denote the magnetic-field strength by B_0, the uniform density of the external medium as $\rho_{\rm ext}$, and the uniform density of the disk as $\rho_{\rm d}$. Suppose at $t = 0$ that the external medium is at rest and that the disk is suddenly given an initial angular velocity Ω_0:

$$\Omega(t \leq 0, |z| \geq Z) = 0, \quad \Omega_{\rm d}(t < 0) = 0, \quad \Omega_{\rm d}(t = 0) = \Omega_0.$$

Require that the angular velocity vanish at infinity, be continuous at the disk surface, and be an even function of z:

$$\Omega(t, |z| = \infty) = 0, \quad \Omega(t > 0, Z) = \Omega_{\rm d}(t > 0), \quad \Omega(t, z) = \Omega(t, -z).$$

Assume axial symmetry and negligibly small velocities v_z in the z-direction (compared with the Alfvén velocity), in which case velocities in the radial direction direction also vanish if no sources or sinks of matter exist on the z-axis. It then follows from flux freezing that B_R and B_z are constants of the motion:

$$B_R(t, R, z) = 0 \quad \text{and} \quad B_z(t, R, z) = B_0.$$

Hence, the solenoidality constraint $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$ is satisfied identically at all times. The figure below ought to help you visualize what's going on.

(a) Use the ideal-MHD induction and force equations to obtain the following equations describing torsional Alfvén waves propagating in the external medium ($|z| > Z$):

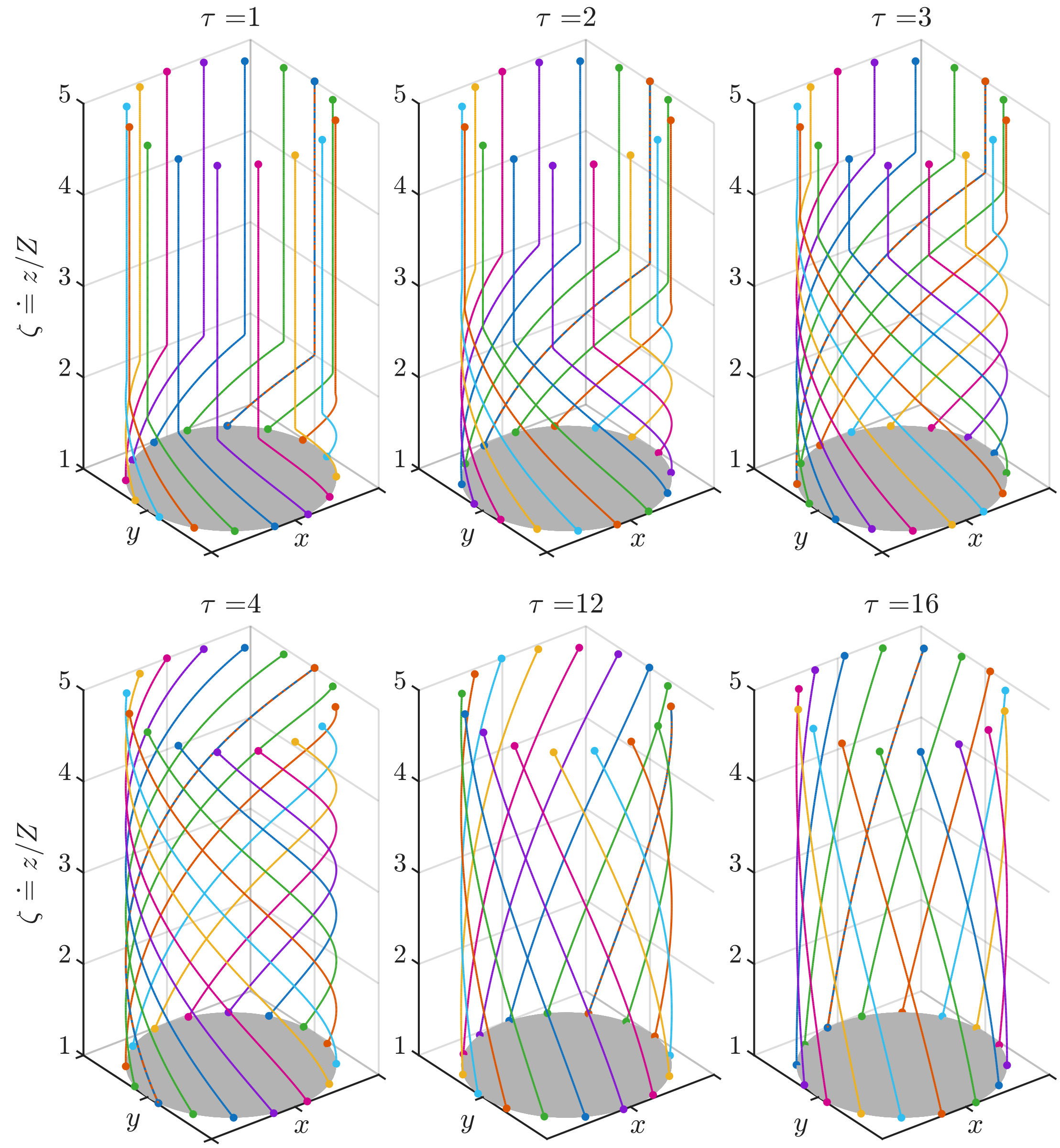
$$\frac{\partial B_\varphi(t, R, z)}{\partial t} = R B_0 \frac{\partial \Omega(t, z)}{\partial z} \quad \text{and} \quad \frac{\partial \Omega(t, z)}{\partial t} = \frac{B_0}{4\pi R \rho_{\rm ext}} \frac{\partial B_\varphi(t, R, z)}{\partial z}$$
$$\Longrightarrow \quad \frac{\partial^2 \Omega(t, z)}{\partial t^2} = v_{\rm A,ext}^2 \frac{\partial^2 \Omega(t, z)}{\partial z^2}, \tag{1}$$

where $v_{\rm A,ext} \doteq B_0/(4\pi\rho_{\rm ext})^{1/2}$.

(b) To obtain an equation of motion of the disk, set the rate of change of its angular momentum equal to the instantaneous magnetic torque exerted on its surfaces. After a few additional steps (which you should reason out), you should find that

$$\frac{\partial^2 \Omega_{\rm d}(t)}{\partial t^2} = \frac{1}{Z} \frac{\rho_{\rm ext}}{\rho_{\rm d}} v_{\rm A,ext}^2 \frac{\partial \Omega(t, z)}{\partial z}\bigg|_{z=Z}. \tag{2}$$

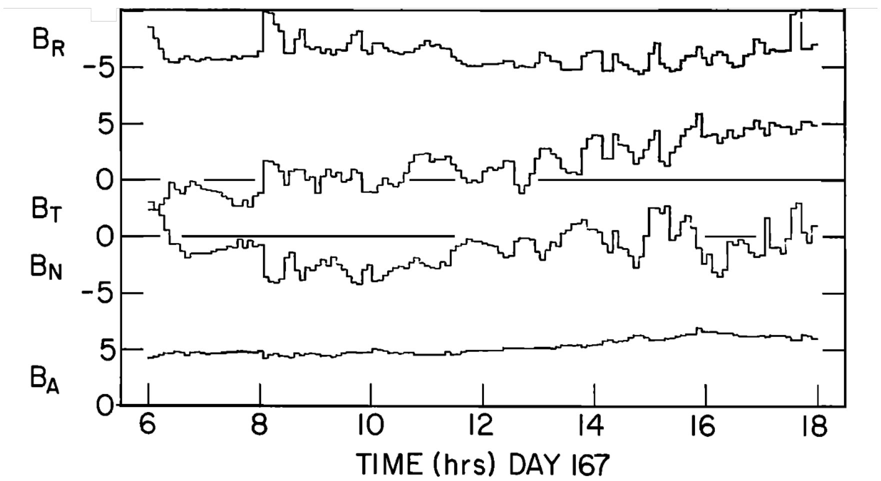
(c) Introduce the dimensionless coordinates $\tau \doteq t v_{\rm A,ext}/Z$ and $\zeta \doteq z/Z$, and the (only) dimensionless free parameter $\rho \doteq \rho_{\rm d}/\rho_{\rm ext}$; put (1) and (2) into dimensionless form; and solve your equations. The combination $(\tau - \zeta + 1)$ that establishes a causal connection between the disk and the external medium should feature in your solution. Use your solution to obtain $\Omega_{\rm d}(\tau)$ and $B_\varphi(\tau, \zeta)$, and identify the characteristic timescale $\tau_\parallel$ for the *magnetic braking* of an aligned rotator; be sure to transform it back into dimensionful units. (Note that B_φ vanishes inside the disk at all times by the assumption of rigid rotation.) When plotted (take $\rho = 10$), your solution for $\Omega_{\rm d}(\tau)$ and $B_\varphi(\tau, \zeta)$ should be consistent with the following figure, which shows a selection of magnetic-field lines at different times above the upper surface of the disk, $\zeta > 1$:



(d) Set aside your solution. Use simple arguments to calculate the amount of time it takes for a torsional Alfvén wave launched from the surface of the torqued disk to traverse an amount of the external medium having a moment of inertia equal to that of the disk. Compare this timescale with the timescale you obtained in part (c) and describe what is going on physically in this system.

The calculation from here can go in a wide variety of ways: one could relax the assumption that the disk is a rigid rotator; one could relax the assumption that the initial rotation is discontinuous at the disk surface; one could orient the background magnetic field at an angle with respect to the disk's rotation axis; one could consider a system of magnetically linked, aligned rotators, with the generated torsional Alfvén waves bouncing back and forth among the disks and redistributing rotational kinetic energy; one could introduce non-ideal (e.g., two-fluid) effects. It's a fun dynamical problem, but this is where we'll leave it…

3. **Spherically polarized Alfvén waves.** The Sun loses $\sim 10^{-14}$ $\mathrm{M}_\odot$ worth of plasma from its surface each year, amounting to $\sim 10^6$ tons per second. This *solar wind* fills interplanetary space and is threaded by a magnetic field in a geometry known as the *Parker spiral*. Follow §IV.3 of the lecture notes to make sure you understand the basic properties of this wind and its magnetic field, namely, that the wind's radial speed $u \gtrsim 400$ km s^{-1} surpasses the Alfvén speed at a distance ~ 20 $\mathrm{R}_\odot$, and that $B_\varphi \sim B_0(\Omega \mathrm{R}_\odot/u)(\mathrm{R}_\odot/R)$. This problem, however, is not about the mean properties of the solar wind, but rather concerns the turbulent Alfvénic fluctuations that have been measured within the wind since the seminal papers of Coleman (1968; using *Mariner 2* data) and Belcher & Davis (1971; using *Mariner 5* data). A peculiar property of many of these Alfvénic fluctuations is that they are "spherically polarized" – meaning that, while the individual components of the turbulent magnetic field fluctuate three-dimensionally, the total magnetic-field strength obtained by summing those components in quadrature is nearly constant:



This figure, taken from Belcher, Davis & Smith (*J. Geophys. Res.* **74**, 2302; 1969), shows the three components of the magnetic field (B_R, B_T, B_N) and the total field strength (B_A) measured by *Mariner 5*. Note that B_A is approximately constant! In this sense, spherical polarization refers to the fact that the magnetic-field vector moves on the surface of a sphere. More recent spacecraft (e.g., *Parker Solar Probe*) obtain similar findings, with such spherically polarized states comprising as much as 90% of the fluctuations close to the Sun.[3]

Producing and maintaining a spherically polarized Alfvénic state in nature seems like quite a geometrical trick, especially in an expanding solar wind in which $\delta B/B$ grows with expansion. Indeed, the constraints of $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$ and $B = |\boldsymbol{B}| = \text{const}$ leave only one degree of freedom from which to construct the 3D vector field $\boldsymbol{B}$. This is easy when the perturbation amplitude $\delta\boldsymbol{B} \doteq \boldsymbol{B} - \langle\boldsymbol{B}\rangle$ is much smaller than the mean field $\langle\boldsymbol{B}\rangle$, but becomes more difficult as $\delta\boldsymbol{B}$ approaches $\langle\boldsymbol{B}\rangle$. This problem explores a method, recently devised by Jono Squire and Alfy Mallet, for generating nonlinear, spherically polarized, Alfvénic states. The basic idea is to construct a small-amplitude spherically polarized wave and then "grow" it to nonlinear amplitudes via an induction equation that forces $B^2 = |\langle\boldsymbol{B}\rangle + \delta\boldsymbol{B}|^2 = \text{const}$.

[3] One of your classmates, Corina Dunn, has used *Parker Solar Probe* (PSP) data to investigate such Alfvénic fluctuations and their impact on the magnetic energy spectrum in the solar wind: click here if you're interested in learning more. For a figure similar to that shown above but updated to 2019 with PSP measurements, see figure 3 of Kasper, Bale, Belcher *et al.*, *Nature* **576**, 228 (2019).

(a) Suppose $\delta\boldsymbol{B}$ were to evolve according to

$$\frac{\partial \delta\boldsymbol{B}}{\partial t} = \delta\boldsymbol{B} + \boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{B}), \tag{3}$$

where $\boldsymbol{v} = \boldsymbol{\nabla}\phi$ is some prescribed potential flow that changes the shape of $\delta\boldsymbol{B}$ as it grows in time in order to maintain constant B. (Note: $\boldsymbol{v}$ is *not* the flow velocity $\boldsymbol{u} = -\delta\boldsymbol{B}/\sqrt{4\pi\rho}$ that would be consistent with the Alfvénic state.) By construction, equation (3) maintains $\langle\delta\boldsymbol{B}\rangle = 0$ and $\boldsymbol{\nabla}\cdot\delta\boldsymbol{B} = 0$. By demanding that $\boldsymbol{\nabla}B^2 = 0$, show that ϕ must satisfy the equation

$$B^2\boldsymbol{\nabla}_\perp^2\phi \doteq B^2\boldsymbol{\nabla}^2\phi - \boldsymbol{B}\boldsymbol{B}\,\boldsymbol{:}\,\boldsymbol{\nabla}\boldsymbol{\nabla}\phi = -\langle\boldsymbol{B}\rangle\cdot\delta\boldsymbol{B}. \tag{4}$$

(b) To keep this problem relatively simple, let us focus on one-dimensional solutions, in which $\delta\boldsymbol{B}$ varies only in the $\hat{\boldsymbol{k}}$ direction with $\hat{\boldsymbol{k}}\cdot\langle\boldsymbol{B}\rangle$ allowed to be arbitrary. Denote the coordinate in the $\hat{\boldsymbol{k}}$ direction by λ, such that $\boldsymbol{\nabla}\phi = \hat{\boldsymbol{k}}\,(\mathrm{d}\phi/\mathrm{d}\lambda)$. Show that (3) and (4) simplify to

$$\frac{\partial \delta\boldsymbol{B}}{\partial t} = \delta\boldsymbol{B} - \frac{\partial}{\partial\lambda}\left[\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\big(\delta\boldsymbol{B} + \langle\boldsymbol{B}_\mathrm{t}\rangle\big)\right], \qquad \big|\delta\boldsymbol{B} + \langle\boldsymbol{B}_\mathrm{t}\rangle\big|^2\frac{\mathrm{d}^2\phi}{\mathrm{d}\lambda^2} = -\langle\boldsymbol{B}\rangle\cdot\delta\boldsymbol{B}, \tag{5}$$

where $\langle\boldsymbol{B}_\mathrm{t}\rangle \doteq \langle\boldsymbol{B}\rangle - \hat{\boldsymbol{k}}(\hat{\boldsymbol{k}}\cdot\langle\boldsymbol{B}\rangle)$ is the part of $\langle\boldsymbol{B}\rangle$ that is transverse to the wavevector.

(c) Go here and download `largeAmplitudeB_1D.m` (or find it on the canvas page for HW03). This matlab script, written by Jono Squire, uses sixth-order finite differencing to solve (5) starting from an initial small-amplitude state with constant B^2 and random amplitudes and phases placed at wavenumbers $k\lambda = (\pm2\pi, \pm4\pi, \pm6\pi)$. The wavevector is oriented such that the angle between the wavevector and the mean field is $\theta_{(k,B)} = \cos^{-1}(\hat{\boldsymbol{k}}\cdot\langle\boldsymbol{B}\rangle/|\langle\boldsymbol{B}\rangle|) = 85°$. Run the code a few times and see how constant-B states grow (the result will be different each time because of a new random-number seed; you can pick different initial conditions by changing `ifield`). At large amplitudes, the solutions exhibit magnetic-field reversals ($|\delta B_\parallel| \doteq \langle\boldsymbol{B}\rangle\cdot\delta\boldsymbol{B} > |\langle\boldsymbol{B}\rangle|$), a commonly used definition of "switchbacks" that are routinely seen in the solar wind.

Good. Now, vary $\theta_{(k,B)} \in [5°, 85°]$ and make a plot of $\max(|\delta B_\parallel|/|\boldsymbol{B}|)$ versus $\theta_{(k,B)}$. (This max value is printed for you into the console.) Write something intelligent about what you've found. It might help to note that $\boldsymbol{\nabla}\cdot\delta\boldsymbol{B} = \boldsymbol{\nabla}_\perp\cdot\delta\boldsymbol{B}_\perp + \nabla_\parallel\delta B_\parallel = 0$, and that a cylindrically polarized Alfvén wave achieves $B = \text{const}$ by having $\delta B_\parallel = 0$.

While equation (3) may look contrived, it is (not coincidentally) reminiscent of the induction equation written in the co-moving frame of an expanding plasma. Indeed, in the expanding solar wind a one-dimensional spherically polarized Alfvénic state should evolve such that the amplitudes of the fluctuating field and (radial) mean field decrease proportionally to $a^{-1/2}$ and a^{-1}, respectively, where a is the plasma's expansion factor. This evolution causes $\delta\boldsymbol{B}$ to grow compared to $\langle\boldsymbol{B}\rangle$. Mallet, Squire *et al.* (2021) explain the connection further: "Expansion of a nearly-constant-B^2 Alfvén wave constantly drives a small fluctuation in the magnetic pressure; this drives a compressive flow which nonlinearly distorts the main Alfvénic fluctuation, keeping the fluctuations in the magnetic pressure small and thus largely preserving the Alfvénic character of the wave."

4. **Shear-Alfvén continuum and toroidal Alfvén eigenmodes (TAEs)**. This problem also starts with some education. Consider a straight screw-pinch equilibrium in cylindrical (R, φ, z) coordinates with magnetic field $\boldsymbol{B} = B_z(R)\hat{\boldsymbol{z}} + B_\varphi(R)\hat{\boldsymbol{\varphi}}$. Small-amplitude perturbations to this equilibrium state having the form

$$\boldsymbol{\xi}(t, R, \varphi, z) = \sum_{m,k_z} \boldsymbol{\xi}_{m,k_z}(R) \exp(-\mathrm{i}\omega t - \mathrm{i}m\varphi + k_z z) \quad \text{with} \quad \boldsymbol{\nabla\cdot\xi} = 0 \tag{6}$$

satisfy the incompressible Hain–Lüst equation for the evolution of the radial displacement, $\xi_R \doteq \hat{\boldsymbol{R}}\boldsymbol{\cdot}\boldsymbol{\xi}_{m,k_z}$ (see §VI.9 of the lecture notes):

$$\frac{k^2R^2}{\rho}\frac{\mathrm{d}}{\mathrm{d}R}\left[\frac{\rho}{k^2R}\left(\omega^2 - k_\parallel^2 v_\mathrm{A}^2\right)\frac{\mathrm{d}(R\xi_R)}{\mathrm{d}R}\right] + \left[\left(\omega^2 - k_\parallel^2 v_\mathrm{A}^2\right)\left(1 - k^2R^2\right) + 2k^2 v_{\mathrm{A}\varphi}^2\left(1 - \frac{\mathrm{d}\ln B_\varphi}{\mathrm{d}\ln R}\right) + \frac{4k_\parallel^2 v_\mathrm{A}^2 k_z^2 v_{\mathrm{A}\varphi}^2}{\omega^2 - k_\parallel^2 v_\mathrm{A}^2} - \frac{k^2R^3}{\rho}\frac{\mathrm{d}}{\mathrm{d}R}\left(\frac{2mk_\parallel \rho v_\mathrm{A} v_{\mathrm{A}\varphi}}{k^2R^3}\right)\right]\xi_R = 0, \tag{7}$$

where $k^2 \doteq (m/R)^2 + k_z^2$ and $k_\parallel \doteq \boldsymbol{k\cdot}\hat{\boldsymbol{b}}$. What a mess. But the two main things I'd like you to notice in (7) for this problem are that: (i) equation (7) is a Sturm–Liouville equation with a singularity that occurs at $\omega^2 = k_\parallel^2 v_\mathrm{A}^2$; and (ii) the linear modes with (m, k_z) are independent. With both $v_\mathrm{A}^2(R) = (B_\varphi^2(R) + B_z^2(R))/4\pi\rho(R)$ and $k_\parallel(R) = -(m/R)b_\varphi(R) + k_z b_z(R)$ being functions of R, this dispersion relation for ω^2 defines a singular radius in the screw pinch where shear-Alfvén waves can be resonantly excited. Namely, for any frequency ω_0, the radial eigenfunction exhibits a logarithmic singularity at the Alfvén-resonant surface R_A where $\omega_0^2 = k_\parallel^2(R_\mathrm{A})v_\mathrm{A}^2(R_\mathrm{A})$. This feature corresponds to what is called the *shear-Alfvén continuum* – a continuous range of resonant frequencies at which shear-Alfvén waves can be locally excited in the screw pinch. It is because of this shear-Alfvén continuum that global (i.e., discrete) shear-Alfvén modes in the screw pinch do not exist.

This shear-Alfvén-wave branch may be isolated rigorously by applying the following ordering to the Hain–Lüst equation:

$$\frac{B_\varphi}{B_z} \sim \frac{k_\parallel}{k_\perp} \sim \frac{k_\perp}{k_R} \sim \epsilon \ll 1 \quad \text{with} \quad k_\parallel R \sim 1,$$

where $k_R \sim \mathrm{d}\ln\xi_R/\mathrm{d}R$ is a WKB-like wavenumber and $k_\perp \doteq \hat{\boldsymbol{R}}\boldsymbol{\cdot}(\boldsymbol{k}\times\hat{\boldsymbol{b}}) = -(m/R)b_z - k_z b_\varphi$. In words, the field is predominantly axial, the fluctuations vary slower along the magnetic field than they do across the field (similar to the reduced-MHD ordering), and the fluctuations are radially localized to a flux surface ($k_R R \gg 1$). Under this ordering, the incompressibility constraint implies that the displacement along the magnetic field is small, *viz.* $\xi_\parallel \sim \epsilon\,\xi_\perp$, where $\xi_\perp = \hat{\boldsymbol{R}}\boldsymbol{\cdot}(\boldsymbol{\xi}\times\hat{\boldsymbol{b}}) = \xi_\varphi b_z - \xi_z b_\varphi$ is the component of the displacement simultaneously tangential to the flux surface and perpendicular to the magnetic field. Incompressibility also implies $\mathrm{d}\xi_\mathrm{R}/\mathrm{d}R \sim -\mathrm{i}k_\perp\xi_\perp$, while the momentum equation gives $\xi_R \sim (\omega^2/k_\parallel^2 v_\mathrm{A}^2 - 1)(\xi_\perp/\epsilon)$. As a result,

$$\omega^2 = k_\parallel^2 v_\mathrm{A}^2 + \mathcal{O}(\epsilon^2), \tag{8}$$

as desired – this is the shear-Alfvén continuum. Now for a little bit of work. . .

(a) Show that the Alfvén frequency in the screw pinch satisfies

$$\omega_{\mathrm{A}}(R) = \pm k_{\parallel}(R) v_{\mathrm{A}}(R) = \pm\left(k_z - \frac{m}{q(R)}\right) v_{\mathrm{A}z}, \quad \text{where} \quad q(R) \doteq \frac{R B_z(R)}{B_\varphi(R)} \tag{9}$$

is the screw-pinch safety factor (see (III.4.5) in the lecture notes). Use this expression to argue that, at rational surfaces in the screw pinch, there is always a mode with $k_\parallel = 0$ and therefore zero Alfvén frequency. Furthermore, show that, in the neighborhood of a rational surface R_{s} where $k_\parallel(R_{\mathrm{s}}) = 0$, the local Alfvén frequency of a mode with azimuthal mode number m behaves as

$$\omega_{\mathrm{A},m}(R) \approx v_{\mathrm{A}z} |R - R_{\mathrm{s}}| \left| \frac{m}{q} \frac{\mathrm{d} \ln q}{\mathrm{d} R} \right|.$$

Provide a qualitative sketch of the local structure of the shear-Alfvén continuum near R_{s} for a few neighboring values of m.

(b) If your sketch is correct, you should find that the frequencies of different m modes cross and produce degenerate frequencies. Take a neighboring pair of modes with (m, k_z) and $(m+1, k_z)$ and show that the safety factor at the radial location of their crossing, R_{cross}, satisfies

$$q(R_{\mathrm{cross}}) = \frac{m + 1/2}{k_z}.$$

This implicitly defines the crossing radius for these two modes.

(c) According to (7), modes with $k_\parallel = 0$ satisfy

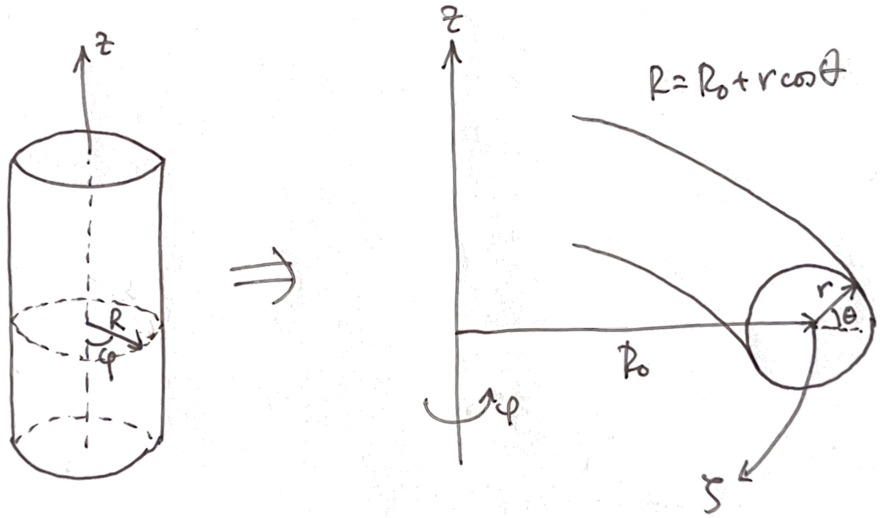
$$\frac{1}{\rho} \frac{\mathrm{d}}{\mathrm{d} R} \left[\frac{\rho}{k_\perp^2 R} \frac{\mathrm{d}(R \xi_R)}{\mathrm{d} R} \right] + \frac{\xi_R}{k_\perp^2 R^2} = \left[1 - \frac{2 v_{\mathrm{A}\varphi}^2}{\omega^2 R^2} \left(1 - \frac{\mathrm{d} \ln B_\varphi}{\mathrm{d} \ln R} \right) \right] \xi_R.$$

At $k_\perp^2 R^2 \gg 1$ (here corresponding to $m^2 \gg 1$), the left-hand side of this equation is small and we obtain global *flute modes* ($k_\parallel = 0$) with frequency

$$\omega^2 \approx \frac{2 v_{\mathrm{A}\varphi}^2}{R^2} \left(1 - \frac{\mathrm{d} \ln B_\varphi}{\mathrm{d} \ln R} \right).$$

This expression corresponds to the $\mathcal{O}(\epsilon^2)$ part of (8). When the term in parentheses is positive, these modes oscillate. What is their restoring force? Knowing that $k_\parallel = 0$ and $m^2 \gg 1$, what do you think these modes look like? Provide a qualitative sketch.

The stage is set. We have the shear-Alfvén continuum, we have mode crossings, and we have flute modes with frequencies $\sim v_{\mathrm{A}\varphi}/R$ taking the place of Alfvén waves at the rational surfaces. Now take our screw pinch and wrap it into a periodic, large-aspect-ratio tokamak with circular, concentric flux surfaces. "Large aspect ratio" means that the minor radius r is much smaller than the major radius R_0, *viz.* $r \sim \epsilon R_0$. Changing our coordinate system from (R, φ, z) to (r, θ, ζ) where $\zeta = -\varphi$, this looks like the following:



The equilibrium magnetic field and the associated safety factor are then

$$\boldsymbol{B} = B_\theta(r)\hat{\boldsymbol{\theta}} + B_\zeta(R)\hat{\boldsymbol{\zeta}} \qquad \text{and} \qquad q(r) \doteq \frac{r}{R_0}\frac{B_\zeta(R_0)}{B_\theta(r)} \sim \mathcal{O}(1),$$

where $R = R_0 + r\cos\theta$. Note that the (larger) toroidal component of the magnetic field, $B_\zeta \sim B_\theta/\epsilon$, is a function of the cylindrical radius R rather than the minor radius r. Indeed, recall that the Grad–Shafranov equation in cylindrical geometry uses $F(\psi) \doteq RB_\zeta$ as a flux function (in this problem, because of the assumption of concentric, circular flux surfaces, the minor radius r serves as a good flux label). This difference between R and r is absolutely essential in what follows.

(d) Equation (9) comes from the local, shear-Alfvén limit of the incompressible Hain–Lüst equation, which may be written as

$$-4\pi\rho\,\omega^2\xi_\perp = (\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{\nabla})^2\xi_\perp.$$

In our new toroidal geometry, this becomes

$$-4\pi\rho\,\omega^2\xi_\perp = \left[\frac{B_\theta(r)}{r}\frac{\partial}{\partial\theta} + \frac{B_\zeta(R)}{R}\frac{\partial}{\partial\zeta}\right]\left[\frac{B_\theta(r)}{r}\frac{\partial\xi_\perp}{\partial\theta} + \frac{B_\zeta(R)}{R}\frac{\partial\xi_\perp}{\partial\zeta}\right]. \tag{10}$$

Following (6), write

$$\xi_\perp(t,r,\theta,\zeta) = \sum_{m,n}\xi_{m,n}(r)\exp(-\mathrm{i}\omega t - \mathrm{i}m\theta + \mathrm{i}n\zeta),$$

with m and n being the poloidal and toroidal mode numbers, respectively. Use (10) to show that, neglecting terms $\sim\mathcal{O}(\epsilon^2)$, the amplitude of the mth harmonic is coupled to those of the $(m+1)$th and $(m-1)$th harmonics as follows:

$$\big(\omega^2 - k_{\parallel,m}^2 v_{\mathrm{A}\zeta}^2\big)\xi_{m,n} = \frac{r}{R_0}\frac{2v_{\mathrm{A}\zeta}^2}{R_0^2}\left[(m+1)^2\,\xi_{m+1,n} + (m-1)^2\,\xi_{m-1,n}\right]. \tag{11}$$

In doing so, provide an explicit expression for $k_{\parallel,m}^2$ in terms of m, n, R_0, and $q(r)$.

The $\mathcal{O}(\epsilon)$ coupling indicated by the right-hand side of (11) was absent in the pinch geometry examined at the start of this problem, and is a direct result of the strength of the toroidal field varying with poloidal angle θ at fixed minor radius r. Now modes with $k_\parallel = 0$ on a rational surface have an $\mathcal{O}(\epsilon)$ contribution to their frequencies! Because shear-Alfvén modes on a given flux surface have their frequencies vary with θ, they can match mode frequencies on a neighboring flux surface and couple...

(e) For simplicity, focus on two such modes: one at m and one at $m+1$. Use (11) to write down a 2×2 coupled system of equations and solve it to obtain a dispersion relation of the form

$$\left(\omega^2 - k_{\parallel,m}^2 v_{\mathrm{A}\zeta}^2\right)\left(\omega^2 - k_{\parallel,m+1}^2 v_{\mathrm{A}\zeta}^2\right) = \Delta_m^4.$$

Provide an expression for Δ_m^2. Show that the minimum separation between the two branches of this dispersion relation occurs where $k_{\parallel,m} = -k_{\parallel,m+1}$ and the dispersion relation takes on the form

$$\left(\omega^2 - \omega_0^2\right)^2 = \Delta_m^2 \quad \Longrightarrow \quad \omega_\pm^2 = \omega_0^2 \pm \Delta_m^2.$$

Thus, toroidicity lifts the degeneracy seen in the screw pinch by splitting the crossing into an avoided crossing. Using your result from part (b), verify that this occurs at the same radius as before, r_{cross}. Evaluate the mode frequency at this crossing radius and show that it satisfies

$$\omega_0 \approx \frac{v_{\mathrm{A}\zeta}}{2q(r_{\text{cross}})R_0}.$$

Sketch the resulting avoided crossing (thereby amending your sketch from part (a)) and calculate the frequency gap, $\Delta\omega = \omega_+ - \omega_-$.

(f) Recall that in the screw pinch, global shear-Alfvén modes do not exist because the Hain–Lüst equation becomes singular at radii where $\omega^2 = k_\parallel^2 v_\mathrm{A}^2$. Using your result from part (e), show that for frequencies lying inside the gap,

$$\omega^2 \neq k_\parallel^2(r) v_\mathrm{A}^2(r) \quad \text{for any } r.$$

What does this imply about the regularity of the radial eigenfunction $\xi_\perp$? Explain why global discrete modes can now exist.

These discrete gap modes are the *toroidal Alfvén eigenmodes* (TAEs).