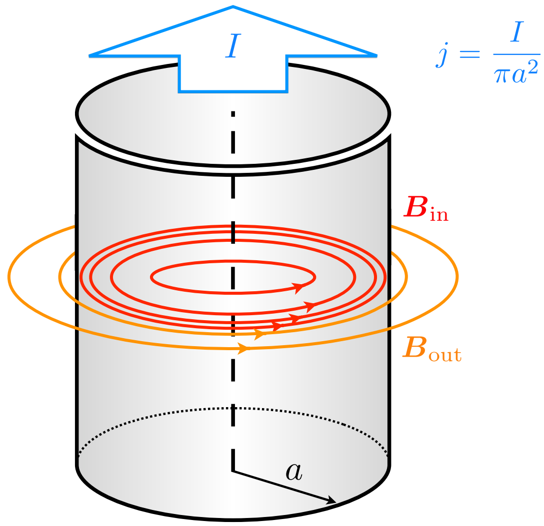
Due Monday, February 23, 2026

Generals prep. Make sure you can provide brief definitions of the following terms: flux freezing, Z-pinch, θ-pinch, screw pinch, flux function, Clebsch potentials, rotational transform, safety factor, magnetic axis, Grad–Shafranov equation, magnetic helicity, J.B. Taylor relaxation

1. **Bennett relation and ignition**. A cylindrical plasma of radius a and length $L \gg a$ is confined by an azimuthal magnetic field $\boldsymbol{B} = B(R)\hat{\boldsymbol{\varphi}}$ and conducting end plates. The magnetic field is produced by a uniform axial current density $j = I/(\pi a^2)$, associated with a current I that is driven through the plasma by electrically biasing the end plates. This is a classic Z-pinch, as shown in the figure to the right. Answer the following; you may use CGS or SI units, but please be consistent.



(a) Calculate $B(R)$ inside and outside of the cylinder and write it in terms of I, a, and R. Neglect end effects.

(b) Use radial MHD force balance to calculate the pressure profile $P(R)$ of the confined plasma. Set $P(a) = 0$.

(c) Obtain a relationship between the current I and the volume-averaged pressure $\langle P \rangle$. This relationship is called the *Bennett relation*, after W.H. Bennett (*Phys. Rev.* **45**, 890; 1934). Take the temperatures of the ions ($T_{\rm i}$) and electrons ($T_{\rm e}$) to be uniform, adopt quasi-neutrality, and rewrite the Bennett relation in terms of the ion line density $N_{\rm i} = \langle n_{\rm i} \rangle \pi a^2$. Given $N_{\rm i}$ and I, this relation provides the equilibrium temperature.

(d) Define the energy confinement time by

$$\tau_E = \frac{3}{2} \frac{\langle P \rangle \, \pi a^2 L}{IV},$$

where V is the voltage applied to the Z-pinch. This is the stored energy divided by the power input; note that this definition corresponds to a steady-state energy balance. Relate the voltage to the (assumed uniform) current via the following Ohm's law for a quasi-static collisional plasma,

$$V = \frac{I}{\sigma} \frac{L}{\pi a^2}, \quad \text{where} \quad \sigma = \frac{e^2 n_{\rm i} \tau_{\rm ei}}{m_{\rm e}} \propto T_{\rm e}^{3/2}$$

is the plasma conductivity and $\tau_{\rm ei}$ is the electron–ion collision time. Use the Bennett relation to show that the product

$$n_{\rm i} \tau_E = \frac{3\mu_0}{16\pi} N_i \sigma \ \text{(SI)} \quad \text{or} \quad \frac{3}{4} N_i \sigma \ \text{(CGS)}.$$

(e) Suppose that the Z-pinch is in steady state for long enough to achieve fusion conditions given by the Lawson criteria for a deuterium–tritium plasma:

$$n_{\rm i} T_{\rm i} \tau_E \gtrsim 3 \times 10^{21} \text{ keV s m}^{-3} \quad \text{at} \quad T_{\rm i} \approx 14 \text{ keV}.$$

Take $T_{\rm e} \approx T_{\rm i}$ for simplicity, and estimate the minimum values of $N_{\rm i}$, I, and the effective electric field $V\pi a^2/L$ required for ignition. (Set the Coulomb logarithm λ that appears in $\tau_{\rm ei}$ to 10.)

(f) On Sandia National Laboratory's Z facility, hundreds of 10μm-diameter wires are arranged in a cylinder that is $\approx$1 cm in radius and $\approx$1 cm in length. Once formed, the Z-pinch plasma is compressed down to a radius $a \approx 0.3$ mm, at which point the compression stagnates. Use $L = 1$ cm and $a = 0.3$ mm in your expressions to estimate the voltage and therefore the power required for ignition. Use these to calculate an approximate value for τ_E. Explain why your estimates dramatically underestimate the difficulty of achieving ignition in a Z-pinch; indeed, peak currents of $\sim$20 MA are delivered on Sandia's Z machine over $\sim$100 ns, but ignition hasn't yet been achieved and stagnation persists for only a few ns. Your answer should include at least one instability, one loss mechanism, one transport effect, and one modeling assumption. Feel free to read about Sandia's Z machine online.

2. ***B* from variational principles.** The magnetic energy in a volume $\mathcal{V}$ bounded by a surface $\mathcal{S}$ is

$$\mathscr{E}_{\rm mag} \doteq \int_{\mathcal{V}} {\rm d}V \, \frac{B^2}{8\pi}.$$

Answer the following; you might want to consult §II.2.1 for some useful vector identities. Note that you do not need to employ Lagrange multipliers to solve these problems.

(a) Write $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$. Show that the magnetic field that minimizes $\mathscr{E}_{\rm mag}$, subject to the tangential components of $\boldsymbol{A}$ being fixed (i.e., $\hat{\boldsymbol{n}} \times \delta \boldsymbol{A} = 0$) on $\mathcal{S}$, is a potential field. Argue that this constraint corresponds to fixing the normal component of $\boldsymbol{B}$ on $\mathcal{S}$.

(b) Write $\boldsymbol{B} = \boldsymbol{\nabla}\alpha \times \boldsymbol{\nabla}\beta$. Show that the magnetic field that minimizes $\mathscr{E}_{\rm mag}$, subject to α and β being fixed (i.e., $\delta\alpha = \delta\beta = 0$) on $\mathcal{S}$, is a force-free field. Argue that this constraint corresponds to fixing both the normal component of $\boldsymbol{B}$ on $\mathcal{S}$ and the magnetic-field-line connectivity between boundary points on $\mathcal{S}$.

3. **Grad–Shafranov, Solov'ev, & shaping.** The Grad–Shafranov (GS) equation describes an MHD equilibrium state in which thermal and magnetic forces balance and for which there is a continuous spatial symmetry. In this problem, you'll explore a useful analytic solution to the GS equation due to Solov'ev (1968), and then append to that solution a flux function that respects a particular set of boundary conditions. Aided by a python code written by Rishin Madan, you'll then investigate the impact of the plasma beta and various shaping factors on the equilibrium and the so-called Shafranov shift.

The GS equation describes the equilibrium relationship between the poloidal flux function $\psi = \psi(\boldsymbol{r})$, the plasma pressure profile $P = P(\psi)$, and the poloidal current profile $F = F(\psi)$.

In axisymmetric ($\partial/\partial\varphi = 0$) geometry and expressed in cylindrical (R, φ, Z) coordinates, it reads

$$\Delta^*\psi \doteq R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right) + \frac{\partial^2\psi}{\partial Z^2} = -\frac{4\pi}{c}Rj_\varphi = -4\pi R^2\frac{\mathrm{d}P}{\mathrm{d}\psi} - \frac{1}{2}\frac{\mathrm{d}F^2}{\mathrm{d}\psi}, \tag{1}$$

where j_φ is the toroidal current density. The flux function $\psi = \psi(R, Z)$ and the poloidal current $F \doteq -2I_\mathrm{p}/c$ provide the poloidal and toroidal components of the magnetic field

$$\boldsymbol{B} = \boldsymbol{B}_\mathrm{p} + B_\varphi\hat{\boldsymbol{\varphi}} = \frac{1}{R}\boldsymbol{\nabla}\psi \times \hat{\boldsymbol{\varphi}} + \frac{F(\psi)}{R}\hat{\boldsymbol{\varphi}}.$$

Note that $\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{\nabla}P = \boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{\nabla}F = 0$. We wish to solve (1) for a confined plasma within a separatrix on which $P(\psi) = 0$. To do so, we adopt the following *Solov'ev profiles*:

$$\frac{1}{2}\frac{\mathrm{d}F^2}{\mathrm{d}\psi} = A \qquad \text{and} \qquad 4\pi\frac{\mathrm{d}P}{\mathrm{d}\psi} = -\frac{C}{R_0^2}, \tag{2}$$

where A, C, and R_0 are constants. Then (1) becomes

$$\Delta^*\psi = R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right) + \frac{\partial^2\psi}{\partial Z^2} = -A + C\frac{R^2}{R_0^2}. \tag{3}$$

In what follows, two model solutions will be considered. The first is a fully analytic toy model designed to highlight some features that occur in a more general Solov'ev solution. The second is such a more general solution, taken from Cerfon & Freidberg (2010), which exhibits enough freedom to describe a variety of magnetic configurations: the standard tokamak, the spherical tokamak, the spheromak, and the field-reversed configuration (FRC).

(a) Prove by substitution that the following flux function is a solution to (3):

$$\psi(R, Z) = \frac{A}{2\gamma R_0^2}\left[\left(R^2 - \gamma R_0^2\right)Z^2 + \frac{\kappa_0^2}{4}\left(R^2 - R_0^2\right)\left(R^2 - R_0^2 - 4DR_0^2\right)\right], \tag{4}$$

where $\gamma < 1$ is a constant, $D > 0$ is a constant, and $\kappa_0^2 \doteq \gamma C/A - 1$.

(b) Compute $\boldsymbol{\nabla}\psi$. Use it to show that the magnetic axis for equation (4) is located at

$$R_\mathrm{ax} = R_0\sqrt{1 + 2D} > R_0 \quad \text{and} \quad Z_\mathrm{ax} = 0. \tag{5}$$

The difference between R_ax and R_0 is called the *Shafranov shift*. Without the term proportional to D in (4), the magnetic axis is located at $(R, Z) = (R_0, 0)$.

In this simple model, the Shafranov shift is baked into the chosen form of ψ through the (arbitrary) parameter D. In reality, the shift is a consequence of the plasma's diamagnetism in a device constrained by boundary conditions: the plasma pressure pushes towards the outboard side of the torus to displace the magnetic axis by an amount that is approximately linear in the poloidal plasma beta β_p. Stay tuned.

(c) Taylor-expand (4) about the magnetic axis at $(R, Z) = (R_\mathrm{ax}, 0)$ by writing

$$\psi \simeq \psi_\mathrm{ax} + \frac{1}{2}\left[\left(\frac{\partial^2\psi}{\partial R^2}\right)_{R=R_\mathrm{ax}} x^2 + \left(\frac{\partial^2\psi}{\partial Z^2}\right)_{Z=0} z^2\right],$$

where $x \doteq R - R_{\rm ax}$, $z \doteq Z$, and $\psi_{\rm ax} = \psi(R_{\rm ax}, 0)$. Use the resulting equation to show that, near the magnetic axis, the flux surfaces are ellipses:

$$x^2 + \frac{z^2}{\kappa^2} = \text{const.}$$

Determine the local *elongation* κ, and then simplify it for the case of zero Shafranov shift, $D = 0$. Does the local elongation near the magnetic axis differ from the boundary elongation parameter κ_0? Why might they differ?

One can show (though you need not) that the *safety factor*

$$q = \frac{1}{2\pi} \oint \mathrm{d}\ell \, \frac{B_\varphi}{RB_p}$$

evaluated at the magnetic axis satisfies

$$q \simeq q_0 = \frac{\gamma\kappa F_0}{A\kappa_0^2} \frac{R_0^2}{R_{\rm ax}^2},$$

where $F_0 \doteq R_{\rm ax} B_\varphi(R_{\rm ax})$. Increasing the elongation increases q_0, increases the so-called beta limit, and improves stability to ballooning modes. Those are three reasons why spherical tokamaks are so attractive.

(d) Use $\boldsymbol{\nabla}\psi$ to show further that there are two X-points located at

$$R_X = \sqrt{\gamma} R_0 < R_0 \quad \text{and} \quad Z_X = \pm \kappa_0 R_0 \sqrt{\frac{1-\gamma}{2} + D}. \tag{6}$$

These points mark the *separatrix*, which is the last closed flux surface separating the confined plasma from the open field lines of the edge region (the "scrape-off layer"). Obtain an expression for $\psi(R_X, Z_X)$.

(e) Make an isocontour plot of $\psi(R, Z)$ in the (R, Z) plane. Set $R_0 = 1$, $\gamma = 0.6$, $D = 0.1$, and $\kappa_0^2 = 3\gamma - 1$ (in this case, A is an inconsequential normalization constant). I recommend using $R \in [0.5, 1.5]$ and $Z \in [-0.6, 0.6]$; make sure you plot enough contours to see the structure of the flux surfaces. Use (5) and (6) to plot the separatrix and the location of the magnetic axis on your figure. Describe what you see, highlighting the elements featured in parts (b), (c), and (d).

Now for something a bit more realistic. In Freidberg's textbook (§6.6), he writes the Solov'ev solution in the following form:[1]

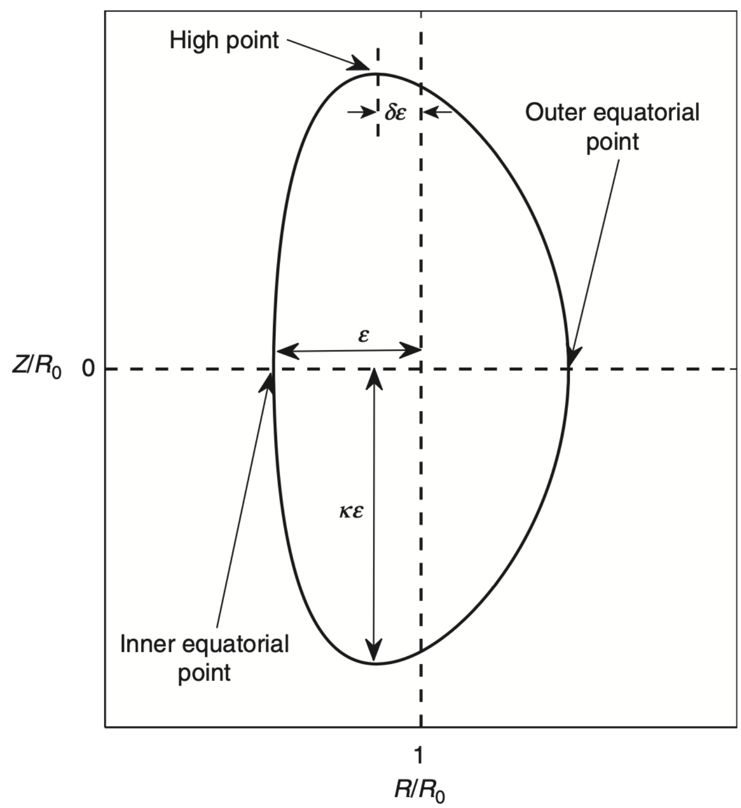
$$\psi(R, Z) = -\frac{\alpha}{2} R^2 \ln R + \frac{1+\alpha}{8} R^4 + \sum_{j=0}^{6} c_j \psi_j(R, Z), \tag{7}$$

[1] Rishin found a typo in Freidberg's equation (6.156): N1 should have a negative sign and N2 shouldn't have a negative sign. I've also switched the signs of Friedberg's A and α coefficients to make them manifestly positive, corresponding to a diamagnetic plasma.

where $\alpha \doteq A/(C-A) > 0$, the flux function ψ has been normalized using $(C-A)R_0^2$, and the cylindrical coordinates R and Z have been normalized using R_0. The summation includes various terms proportional to combinations and powers of R^2, Z^2, and $R^2 \ln R$. Those terms are part of the homogeneous solution to the GS equation (i.e., $\Delta^* \psi_{\rm h} = 0$), specially designed to account for the boundary conditions imposed on ψ by the coils. I won't write out the ψ_j functions here, nor will I specify the c_j coefficients (which are determined numerically from the boundary conditions), but suffice it to say that $\psi(R, Z)$ is taken to be up-down symmetric (i.e., $\psi(R, Z) = \psi(R, -Z)$) and to equal zero on a boundary parametrized by

$$R(\tau) = 1 + \varepsilon \cos(\tau + \delta_0 \sin \tau), \quad Z(\tau) = \varepsilon\kappa \sin(\tau), \tag{8}$$

where $\varepsilon \doteq a/R_0$ is the inverse aspect ratio, $\delta \doteq \sin \delta_0$ is the triangularity, and κ is the elongation. These parameters are diagrammed in the figure below of a poloidal cross section of a flux surface, taken from Freidberg's figure 6.19:



Lucky for you (and me), your excellent TA Rishin has written a python code that provides the solution to the GS equation given the Solov'ev profiles (2) and the boundary (8). Download it by clicking here or by going to the course canvas page; you'll need sympy to run it (type "`pip install sympy`" in a terminal window if you don't have it). The input parameters that you must specify are at the top of the code; they are: the inverse aspect ratio $\varepsilon \doteq a/R_0$, the elongation κ, the triangularity δ, and the Solov'ev coefficients A and C that control the plasma current and pressure profiles, respectively. In units where $R_0 = 1$, $C = 1 + A$ fixes the normalization of the flux function, so one need only specify A. This means four free parameters: ε, κ, δ, A. The additional parameter `external_coil_current` in the file does not affect ψ at all, but it does affect the toroidal beta β_t; it should be larger than the poloidal current in the plasma or else the toroidal field will vanish somewhere.

I've put a set of parameters in the code that approximately match TFTR, NSTX, DIII-D, JET, MAST, and ITER. The default set of parameters in the code ($\varepsilon = 0.34$, $\kappa = 1$, $\delta = 0$, $A = 1$) corresponds to TFTR, which operated at PPPL from 1982 to 1997 and set a world record in 1994 with 10.7 MW of fusion power from a 50-50 deuterium-tritium fuel mix. To run the code, type "`python3 solovev_profile_solver.py`"; it will generate two plots (`boundary_surface_3D.png` and `flux_surface_cross_section.png`) and some information:

```
position of magnetic axis is R=1.106
poloidal beta is 2.246
toroidal beta is 0.02856
beta is 0.0282
safety factor on boundary is 5.046
kink safety factor is 3.058
toroidal beta times kink safety factor squared on boundary is 0.2671
```

Note that the position of the magnetic axis is greater than unity (Shafranov shift!). Check out the generated plots, and proceed...

(f) Keep all the TFTR parameters fixed except for A, which you should gradually increase starting from 0 up to 2.1, beyond which you should see the nature of the solution change. Plot the resulting β_p versus the radial position of the magnetic axis R_{ax}, both of which are reported by the code. Describe what you see and why you think it's happening. In particular, what happens when A, and thus β_p, gets too large? Why?

(g) Now move on to NSTX ($\varepsilon = 0.78$, $\kappa = 2$, $\delta = 0.35$). This spherical tokamak is vertically elongated and exhibits triangularity. Uncomment the NSTX parameters, run the code, and examine the generated plots. Comment on what you see in comparison to TFTR. Then set $A = 0$ and increase it gradually up to 0.8, beyond which you should see the nature of the solution change. As in part (f), plot the resulting β_p versus the radial position of the magnetic axis R_{ax}. Which has a higher β_p limit, TFTR or NSTX? How does β_p at its limit scale with elongation? Make an illustrative plot.

(h) In 2023, the DIII-D tokamak had a negative triangularity campaign, in which it did not experience an edge-localized mode (ELM) in a single discharge so long as sufficiently strong negative triangularity was maintained. That's great. Uncomment the DIII-D parameters in the code ($\varepsilon = 0.37$, $\kappa = 2.1$), set $\delta = -0.5$, and check it out. Weird, huh? Investigate how changes in triangularity from positive to negative values (at fixed κ) affect the maximum attainable β_p and report on your findings.

Feel free to explore the JET, MAST, and ITER parameters! If you want even more fun, try an FRC ($\varepsilon = 0.99$, $\kappa = 10$, $\delta = 0.7$, $A = 0$).