AST 552 Spring 2026 **Homework #1: Preparation MHD**

Due Monday, February 9, 2026

Generals prep. Make sure you can write down the equations of ideal, adiabatic MHD from memory and can state and explain the assumptions involved in their derivation.

1. **Shrinking sink streams.** Go to the bathroom and turn on the sink slowly to get a nice, laminar stream flowing down from the faucet. Go on, I'll wait. If you followed instructions, then you'll see that the stream becomes more narrow as it descends. Knowing that the density of water is very nearly constant, use the continuity equation to show that the cross-sectional area of the stream $A(z)$ as a function of distance from the faucet z is

$$A(z) = \frac{A_0}{\sqrt{1 + 2gz/u_0^2}},$$

where A_0 is the cross-sectional area of the stream upon exiting the faucet with velocity u_0 and g is the gravitational acceleration. If you turn the faucet to make the water flow faster, what happens to the tapering of the stream?

2. **Straining in cylindricals.** Show that the $R\varphi$-component in cylindrical coordinates of the rate-of-strain tensor

$$W_{ij} \doteq \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}$$

is given by

$$W_{R\varphi} = \frac{1}{R}\frac{\partial u_R}{\partial \varphi} + R\frac{\partial}{\partial R}\frac{u_\varphi}{R}.$$

(Hint: $\partial u_i/\partial x_j = [(\hat{\boldsymbol{e}}_j \cdot \boldsymbol{\nabla})\boldsymbol{u}] \cdot \hat{\boldsymbol{e}}_i$ is coordinate invariant.) Such an object shows up in the theory of angular-momentum transport in accretion discs, for which $u_\varphi/R = \Omega(R)$ is the rotation frequency.

3. **Helicity conservation.** Given the fluid vorticity $\boldsymbol{\omega} \doteq \boldsymbol{\nabla} \times \boldsymbol{u}$, the kinetic helicity of a region of plasma is defined to be $\mathcal{H} \doteq \int_{\mathcal{V}} \mathrm{d}\boldsymbol{r}\, \boldsymbol{\omega} \cdot \boldsymbol{u}$, where the integral is taken over the volume $\mathcal{V} = \mathcal{V}(t)$ of that region. Likewise, given the magnetic field $\boldsymbol{B} \doteq \boldsymbol{\nabla} \times \boldsymbol{A}$, the magnetic helicity of a region of plasma is defined to be $\mathcal{H}_{\mathrm{m}} \doteq \int_{\mathcal{V}} \mathrm{d}\boldsymbol{r}\, \boldsymbol{B} \cdot \boldsymbol{A}$.

(a) Assume that the circulation $\Gamma \doteq \int_{\mathcal{S}} \boldsymbol{\omega} \cdot \mathrm{d}\boldsymbol{S} = \text{const}$ and that $\boldsymbol{\omega} \cdot \hat{\boldsymbol{n}}$ vanishes over the surface $\mathcal{S} = \mathcal{S}(t)$ that bounds $\mathcal{V}$, where $\hat{\boldsymbol{n}}$ is the unit normal to that surface. Use the momentum equation to show that $\mathcal{H}$ is conserved in a frame co-moving with the fluid, *viz.* $\mathrm{D}\mathcal{H}/\mathrm{D}t = 0$ with $\mathrm{D}/\mathrm{D}t \doteq \partial/\partial t + \boldsymbol{u} \cdot \boldsymbol{\nabla}$. (Include a gravitational potential in your calculation.) Note that the fluid need not be incompressible for this property to hold. Explain in physical terms how baroclinicity and flux-frozen magnetic fields can each break kinetic helicity conservation.

(b) Assume that the magnetic flux $\Phi_B \doteq \int_{\mathcal{S}} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = \text{const}$ and that $\boldsymbol{B} \cdot \hat{\boldsymbol{n}}$ vanishes over the surface $\mathcal{S} = \mathcal{S}(t)$ that bounds $\mathcal{V}$. Prove that $\mathcal{H}_{\mathrm{m}}$ is conserved in a frame co-moving with the fluid, *viz.* $\mathrm{D}\mathcal{H}_{\mathrm{m}}/\mathrm{D}t = 0$.

4. ***Z*-pinch and diamagnetic flow.** In this problem, you will learn about the role diamagnetic flow plays in plasma equilibria. The equilibrium we consider here is one of magnetohydrostatic balance in a radially stratified, cylindrical plasma – the so-called "pinch" geometry, expressed in cylindrical coordinates (R, φ, z). The plasma is composed of ions (subscript i) and electrons (subscript e) and placed in a magnetic field $\boldsymbol{B} = B(R)\hat{\boldsymbol{\varphi}}$, about which the particles spiral. There is also an equilibrium electric field $\boldsymbol{E} = -(\mathrm{d}\Phi/\mathrm{d}R)\hat{\boldsymbol{R}}$ in the plasma, whose physical origin you will explain in part (a). For simplicity, let us assume that

- the number densities $n_\alpha = n_\alpha(R)$ and temperatures $T_\alpha = T_\alpha(R)$ of each species α are functions only of the cylindrical radial coordinate R;
- the velocity distribution functions of both species are isotropic, so that the thermal pressures $P_\alpha = n_\alpha T_\alpha$ are scalars;
- the gradient scales of both species are the same, *viz.* $L_R \doteq |\mathrm{d}R/\mathrm{d}\ln P_\mathrm{i}| = |\mathrm{d}R/\mathrm{d}\ln P_\mathrm{e}|$;
- the Larmor radii of both species, $\varrho_\alpha \doteq v_{\mathrm{th},\alpha}/\Omega_\alpha$ where $v_{\mathrm{th},\alpha} \doteq (2T_\alpha/m_\alpha)^{1/2}$ and $\Omega_\alpha \doteq q_\alpha B/m_\alpha c$, are much smaller than the gradient scale, *viz.* $\varrho^*_\alpha \doteq \varrho_\alpha/L_R \ll 1$.

Answer the following questions.

(a) In this equilibrium state, the force balance on each species $\alpha = \mathrm{i, e}$ is given by

$$0 = \frac{\mathrm{d}P_\alpha}{\mathrm{d}R} + q_\alpha n_\alpha \frac{\mathrm{d}\Phi}{\mathrm{d}R} + \frac{q_\alpha n_\alpha u_{\alpha,z}}{c} B, \tag{1}$$

where q_α and m_α the charge and mass of species α. Note that summing over species and adopting quasi-neutrality provides MHD force balance:

$$0 = \frac{\mathrm{d}P}{\mathrm{d}R} + \frac{j_z}{c}B = \frac{\mathrm{d}P}{\mathrm{d}R} + \frac{B}{4\pi R}\frac{\mathrm{d}(RB)}{\mathrm{d}R} \quad \text{with} \quad P \doteq P_\mathrm{i} + P_\mathrm{e}. \tag{2}$$

Multiply (1) by m_α/q_α, sum over species, and use the fact that no mass is flowing along the pinch axis to obtain an equation for the equilibrium electrostatic potential Φ in terms of the pressure (it will involve sums over species). What condition must the ion-to-electron temperature ratio $T_\mathrm{i}/T_\mathrm{e}$ satisfy in order for the equilibrium electric field to vanish? If this condition isn't satisfied, why is an equilibrium electric field even necessary? Explain your answer in physical terms.

(b) Compute the species-dependent cross-field particle drift velocity $\boldsymbol{v}_{\mathrm{dr},\alpha}$ due to the electric and magnetic fields. Prove that its magnitude is small, meaning that $v_{\mathrm{dr},\alpha} \sim \varrho^*_\alpha v_{\mathrm{th},\alpha}$.

(c) Given an equilibrium distribution function $f_\alpha = f_\alpha(R, \boldsymbol{v})$ for each species, calculate the current density $\boldsymbol{j}_\mathrm{dr}$ associated with the drifts. Under what condition does it vanish? [Reminder: $\int \mathrm{d}\boldsymbol{v}\,(1, m_\alpha w_\perp^2/2, m_\alpha w_\parallel^2) f_\alpha = (n_\alpha, P_{\perp\alpha}, P_{\parallel\alpha})$, where $\boldsymbol{w} \doteq \boldsymbol{v} - \boldsymbol{u}_\alpha$.] Compare this current density to $\boldsymbol{j} = (c/4\pi)\boldsymbol{\nabla}\times\boldsymbol{B}$.

If you did things right, your expressions for $\boldsymbol{j}_\mathrm{dr}$ and $\boldsymbol{j}$ don't match. Indeed, using (2) you should have found that

$$\boldsymbol{j} - \boldsymbol{j}_\mathrm{dr} = -\frac{c}{R}\frac{\mathrm{d}}{\mathrm{d}R}\left(\frac{RP}{B}\right)\hat{\boldsymbol{z}}.$$

There must be an additional source of current that compensates for this difference. In what follows, you will obtain this *magnetization current* rigorously.

(d) As with the equilibrium electric field, the equilibrium magnetic field may be written in terms of a potential. Define $\boldsymbol{A} = A(R)\hat{\boldsymbol{z}}$, such that

$$B(R) = -\frac{\mathrm{d}A}{\mathrm{d}R}.$$

Note that $\boldsymbol{B}\cdot\boldsymbol{\nabla}A = 0$, so that we may label the flux surfaces by the values of $A(R)$. As the particles move around in the potentials Φ and A, they conserve three quantities. Use the following Lagrangian describing particle motion in this equilibrium state,

$$\mathcal{L}(\boldsymbol{r},\boldsymbol{v}) = \frac{1}{2}m_\alpha v^2 - q_\alpha\left(\Phi - \frac{\boldsymbol{v}\cdot\boldsymbol{A}}{c}\right) = \frac{1}{2}m_\alpha v^2 - q_\alpha\Phi + q_\alpha v_z\frac{A}{c},$$

to find the three invariants conserved by the particles. According to Jeans' theorem, the equilibrium distribution function F_α of each species must be a function of *only* these invariants. Because we have assumed velocity-space isotropy of the particle distribution functions, one of these invariants doesn't matter. Write the other two in terms of the particle coordinates $(\boldsymbol{r},\boldsymbol{v})$ and identify their physical significance. For the rest of this problem, it will help to assign these quantities the labels $\mathcal{E}_\alpha$ (with units of energy) and $\mathcal{R}_\alpha > 0$ (with units of length). Do so. (It should be clear which one to name $\mathcal{E}_\alpha$ and which to name $\mathcal{R}_\alpha$; if you're unsure, please ask!)

(e) So far, we know that the particles spiral around the magnetic-field lines and drift across the field lines, all while conserving $\mathcal{E}_\alpha$ and $\mathcal{R}_\alpha$. As such, it will prove beneficial to separate the gyromotion from this drift by decomposing the particle velocities in terms of the parallel velocity $v_\parallel$, the perpendicular velocity $v_\perp$ relative to $v_{\mathrm{dr},\alpha}$, and the gyrophase angle ϑ:

$$\boldsymbol{v} = v_\parallel\hat{\boldsymbol{\varphi}} + v_\perp(\cos\vartheta\hat{\boldsymbol{R}} + \sin\vartheta\hat{\boldsymbol{z}}) + v_{\mathrm{dr},\alpha}\hat{\boldsymbol{z}}.$$

Substitute this decomposition into $\mathcal{E}_\alpha$ and $\mathcal{R}_\alpha$, and use the smallness of $v_{\mathrm{dr},\alpha}/v_{\mathrm{th},\alpha} \sim \varrho^*_\alpha$ to Taylor-expand the equilibrium distribution function $F_\alpha(\mathcal{E}_\alpha,\mathcal{R}_\alpha)$ about their leading-order counterparts ε_α and R. (Be sure to specify what you think ε_α is!) In other words, write

$$F_\alpha(\mathcal{E}_\alpha,\mathcal{R}_\alpha) \simeq f_\alpha(\varepsilon_\alpha, R) + \mathcal{O}(\varrho^*_\alpha)$$

and calculate the $\mathcal{O}(\varrho^*_\alpha)$ term.

STOP! Seriously, stop!

Do **not** progress to the next page until you've completed parts (a)–(e)!

This is for your own good!!

What you should have found in part (e) is that the difference between the equilibrium distribution function $F_\alpha(\mathcal{E}_\alpha, \mathcal{R}_\alpha)$ and its approximation $f_\alpha(\varepsilon_\alpha, R)$ is dependent upon gyrophase. This is the difference between the distribution function of guiding centers (F_α), and the distribution function of particles (f_α). Guiding centers sit at fixed radial positions $\mathcal{R}_\alpha$, while the particles sinusoidally sample the radial gradients as they gyrate, conserving $\mathcal{E}_\alpha$ as they do so. You'll now show that this difference is what is responsible for the magnetization current.

(f) Use your result from part (e) to compute the mean flow velocity of each species,

$$\boldsymbol{u}_\alpha \doteq \frac{1}{n_\alpha} \int \mathrm{d}^3\boldsymbol{v}\, \boldsymbol{v} F_\alpha(\mathcal{E}_\alpha, \mathcal{R}_\alpha),$$

including contributions from all terms at $\mathcal{O}(\varrho^*_\alpha)$. For concreteness, assume the Maxwellian form

$$F_\alpha(\mathcal{E}_\alpha, \mathcal{R}_\alpha) = \frac{\mathcal{N}_\alpha(\mathcal{R}_\alpha)}{\pi^{3/2} v^3_{\mathrm{th},\alpha}(\mathcal{R}_\alpha)} \exp\left[-\frac{\mathcal{E}_\alpha}{T_\alpha(\mathcal{R}_\alpha)}\right], \tag{3}$$

where $\mathcal{N}_\alpha$ is the number density of guiding centers. In doing so, show that the number density of particle $n_\alpha \doteq \int \mathrm{d}\boldsymbol{v}\, F_\alpha(\mathcal{E}_\alpha, \mathcal{R}_\alpha)$ is related to that of the guiding centers via a Boltzmann factor:

$$\mathcal{N}_\alpha(R) = n_\alpha(R) \exp\left[\frac{q_\alpha \Phi(R)}{T_\alpha(R)}\right] + \mathcal{O}(\varrho^{*2}_\alpha).$$

Hint: when you expand (3) about (ε_α, R), be careful with what is held fixed under partial differentiation.

What you should find is that a piece of $\boldsymbol{u}_\alpha$ cancels the drift velocity $\boldsymbol{v}_{\mathrm{dr}}$, leaving

$$\boldsymbol{u}_\alpha = -\frac{c}{B}\left(\frac{T_\alpha}{q_\alpha}\frac{\mathrm{d}\ln P_\alpha}{\mathrm{d}R} + \frac{\mathrm{d}\Phi}{\mathrm{d}R}\right)\hat{\boldsymbol{z}}. \tag{4}$$

The first term here is referred to as the "diamagnetic flow". The current that it produces,

$$\boldsymbol{j}_{\mathrm{dia}} = \sum_\alpha q_\alpha n_\alpha \left(-\frac{c}{B}\frac{T_\alpha}{q_\alpha}\frac{\mathrm{d}\ln P_\alpha}{\mathrm{d}R}\right)\hat{\boldsymbol{z}} = -\frac{c}{B}\frac{\mathrm{d}P}{\mathrm{d}R}\hat{\boldsymbol{z}} = \frac{c}{B}\hat{\boldsymbol{b}} \times \boldsymbol{\nabla} P,$$

balances the perpendicular current in the Lorentz force when in magnetohydrostatic equilibrium. Use (4) to calculate the total current $\boldsymbol{j} = \sum_\alpha q_\alpha n_\alpha \boldsymbol{u}_\alpha$, infer the magnetization current $\boldsymbol{j}_M = \boldsymbol{j} - \boldsymbol{j}_{\mathrm{dr}}$, and compare it with the missing current in part (c). In words, the total current is a superposition of the particle drifts, which are caused by the equilibrium electric and magnetic fields, and the magnetization current, which is driven by the inhomogeneity in the energies, positions, and orientations of the gyrating particles (which act as magnetic dipoles).