AST 552 Spring 2026 **Homework #6: Turbulence & Dynamo**

Due Monday, May 4, 2026

Generals prep. Make sure you can provide brief definitions of the following terms: Reynolds number, magnetic Prandtl number, Kolmogorov turbulence, Goldreich–Sridhar turbulence, Iroshnikov–Kraichnan turbulence, Cowling's anti-dynamo theorem, Zel'dovich's anti-dynamo theorem, fluctuation dynamo, α-Ω dynamo

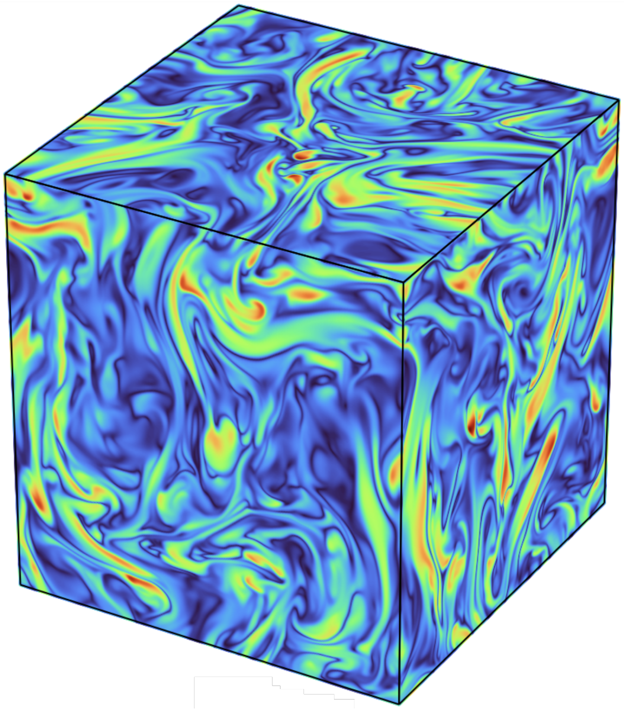
1. **Critical balance in rotating, hydrodynamic turbulence.** In a rigidly rotating, hydrodynamic, incompressible fluid, the characteristic linear frequency of waves is $\omega = \pm(k_\parallel/k)\Omega$, where $\boldsymbol{\Omega} = \Omega\hat{\boldsymbol{z}}$ is the angular velocity of the flow and $k_\parallel = k_z$ is component the wavenumber oriented parallel to the rotation axis. Suppose that such a fluid is turbulent, with velocity fluctuations satisfying $k_\parallel/k_\perp \ll 1$, i.e., the fluctuations are anisotropic with respect to the rotation axis and elongated in that direction. Assume the turbulence to be strong and critically balanced. Obtain the resulting perpendicular and parallel power spectra of the turbulent velocities and the scaling relation linking $k_\parallel$ and $k_\perp$. (*This should take no more than a few lines!*) Does the anisotropy of the fluctuations increase or decrease as the cascade goes to smaller scales? In other words, does the assumption of anisotropy get better or worse at progressively smaller scales? Is this similar to or different than the situation in Goldreich–Sridhar turbulence? Physically, why?

2. **Weak-to-strong transition in balanced Alfvén-wave turbulence.** Suppose that structures in Alfvén-wave turbulence can be described by a field-parallel scale ℓ, an isotropic field-perpendicular scale λ, and a fluctuation amplitude $\delta Z^\pm_\lambda$. In weak Alfvén-wave turbulence, the linear timescale $\tau_{\rm A} \sim \ell/v_{\rm A}$ is much shorter than the non-linear timescale $\tau_\lambda \sim \lambda/\delta Z^\pm_\lambda$; in strong turbulence, $\tau_{\rm A} \sim \tau_\lambda$. Given balanced fluctuation amplitudes $\delta Z^+_0 \sim \delta Z^-_0 \sim \delta Z_0$ at the outer scales ℓ_0 and λ_0, determine the characteristic amplitude and spatial anisotropy of the fluctuations at the weak-to-strong transition scale in the cascade. Then describe qualitatively and quantitatively (using "$\sim$"-style scaling arguments) what happens to the amplitudes, spatial anisotropies, and energy spectrum of the fluctuations at scales smaller than this transition scale. You may neglect dynamical alignment.

To get you started. . . here's a brief recap of weak-turbulence theory. Counter-propagating Alfvén waves interact and deform one another by an amount $\Delta(\delta Z_\lambda) \sim (\delta Z^2_\lambda/\lambda)(\ell/v_{\rm A})$ per interaction, an expression that approximates the time integral of the nonlinearity $\delta\boldsymbol{Z}^\pm \boldsymbol{\cdot} \boldsymbol{\nabla}\delta\boldsymbol{Z}^\mp$ over the Alfvén-crossing time $\ell/v_{\rm A}$. In weak turbulence, these deformations are small, and so for a structure to decorrelate fully it must undergo many such interactions. Assuming that these interactions are uncorrelated with one another and therefore accumulate as a random walk leads to the scaling $N^{1/2}\Delta(\delta Z_\lambda) \sim \delta Z_\lambda$, where N is the number of interactions required to distort a fluctuation by an amount equal to itself. Solving for N leads to $N \sim (v_{\rm A}/\ell)^2(\lambda/\delta Z_\lambda)^2$. The decorrelation time is then $\tau_\lambda \sim N\tau_{\rm A} \sim (v_{\rm A}/\ell)(\lambda/\delta Z_\lambda)^2$. A constant-flux cascade has $(\delta Z_\lambda)^2/\tau_\lambda = {\rm const}$, which after some rearrangement implies that $\delta Z_\lambda/\delta Z_0 \sim (\ell_0/\ell)^{1/4}(\lambda/\lambda_0)^{1/2}$. Weak-turbulence theory assumes that the wave–wave interactions keep the parallel scale $\ell = {\rm const}$, in which case $\delta Z_\lambda/\delta Z_0 \sim (\lambda/\lambda_0)^{1/2}$, corresponding to an energy spectrum $E(k_\perp) \propto k_\perp^{-2}$.

3. **(Optional) Study.** Consult the distributed list of accumulated "generals prep" vocabulary words and their definitions, and make sure you understand them and can describe them in your own words if asked (because you will be asked on the final exam and, if you're in the Program in Plasma Physics, likely during the oral portion of generals).

4. **(Optional guided problem) Tearing in the fluctuation dynamo.** As discussed in class, the fluctuation dynamo is particularly good at producing folded magnetic fields. Below is a snapshot of the magnetic-field strength in the saturated state of the fluctuation dynamo, as realized in a 560^3 MHD simulation with $\mathrm{Pm} \doteq \nu/\eta = 10$ and $\mathrm{Rm} \doteq u_{\mathrm{rms}}/k_0\eta \approx 1900$, where $k_0 \doteq 2\pi/L$ is the box wavenumber:

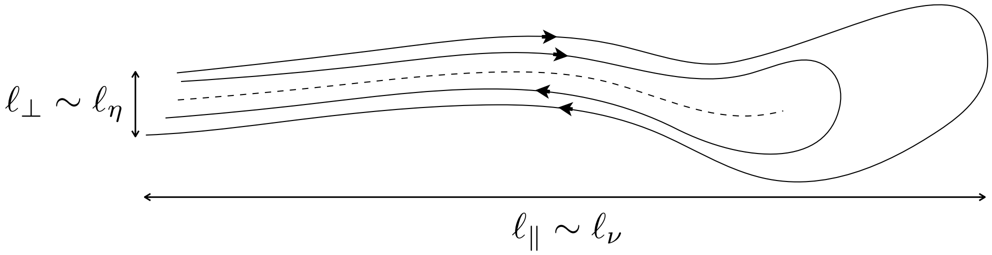


Note the elongated folds exhibiting sharp reversals. This problem follows a calculation by Galishnikova, Kunz & Schekochihin (2022), who proposed that such magnetic folds would be susceptible to tearing, thereby changing the geometry of the amplified magnetic field. The authors accompanied that calculation with several MHD simulations, reaching a resolution of 2240^3 with $\mathrm{Rm} \approx 5 \times 10^4$. Fortunately, many of their analytical predictions were borne out by the simulations (or at least not contradicted by them)... so perhaps this problem won't be a waste of your time.

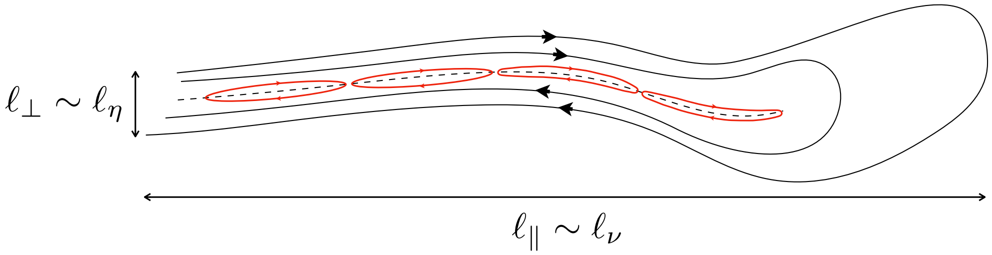
(a) Consider a statistically homogeneous MHD plasma with constant magnetic resistivity η and kinematic viscosity ν ($\gg\eta$), in which an initially weak, zero-net-flux magnetic field is amplified via random stretching by three-dimensional, incompressible turbulence. Take this turbulence to consist of fluid motions that are injected with root-mean-square (rms) velocity U at the outer (forcing) scale L and cascaded conservatively through an inertial range down to a viscous scale ℓ_ν, at and below which dissipation occurs. This assumes that the magnetic field is weak enough that the Lorentz force is negligible throughout this inertial range – the *kinematic stage*. Assuming Kolmogorov scalings, obtain an expression for the maximal stretching rate of the eddies in the turbulent cascade in terms of U, L, and the Reynolds number $\mathrm{Re} \doteq UL/\nu \geq 1$. Show that it occurs at the viscous scale $\ell_\nu \sim L\,\mathrm{Re}^{-3/4}$. Using the ideal induction equation in the form $\mathrm{D}\ln B/\mathrm{D}t = (\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} - \mathsf{I})\,\boldsymbol{:}\,\boldsymbol{\nabla}\boldsymbol{u}$, obtain a scaling relation for the exponential growth of the magnetic-field strength during this kinematic stage.

(b) As the magnetic-field strength is exponentially amplified, the field itself is stretched, folded, and ultimately organized into a highly intermittent patchwork of long, thin structures ("folds") whose energy spectrum $M(k) \propto k^{3/2}$ – the Kazantsev spectrum – peaks at the smallest available scale on which the magnetic field can reverse its direction, *viz.*, the resistive scale ℓ_η. Balance the maximal stretching rate from part (a) with the rate of resistive decay of the folds to obtain an expression for ℓ_η, written in terms of the magnetic Reynolds number $\mathrm{Rm} \doteq UL/\eta$ and the magnetic Prandtl number $\mathrm{Pm} \doteq \nu/\eta$. Then provide an expression for the ratio ℓ_ν/ℓ_η in terms of Pm.

(c) Schekochihin *et al.* (2002) showed that the characteristic parallel length of the magnetic folds, $\ell_\parallel$, is inherited from the velocity fluctuations with the fastest rate of strain. This means that, during the kinematic stage, $\ell_\parallel \sim \ell_\nu$. Thus, Pm controls the aspect ratio of the folds, with $\mathrm{Pm} \gg 1$ implying large-aspect-ratio current sheets:



Now here's the key idea. Take the view of these elongated, folded fields as being current sheets that might be susceptible to the tearing instability:



In this illustration, the red islands spanning the null line represent possible tearing modes. But for those tearing modes to be effective at disrupting a fold, they must grow faster than both the rate at which the fold decorrelates and the rate at which the fold resistively diffuses – otherwise the fold would be replaced by a brand new fold before any tearing can occur. Use this idea to determine whether or not a typical magnetic fold generated during the kinematic stage of the fluctuation dynamo lives long enough to experience appreciable tearing (this is tantamount to asking whether the maximum current-sheet thickness at which tearing can onset, which we'll call λ_*, is larger than the resistive scale ℓ_η).

For your calculation, you'll need to know the growth rate γ_t and wavenumber k_t of the fastest-growing tearing mode for a current sheet having thickness λ_*. In a $\mathrm{Pm} \gg 1$

MHD plasma, these satisfy

$$\gamma_{\rm t}\,\frac{\lambda_*}{v_{{\rm A},\lambda_*}} \sim S_{\lambda_*}^{-(n+1)/(3n+1)}\,{\rm Pm}^{-n/(3n+1)}, \tag{1a}$$

$$k_{\rm t}\,\lambda_* \sim S_{\lambda_*}^{-1/(3n+1)}\,{\rm Pm}^{1/2(3n+1)}, \tag{1b}$$

where $v_{{\rm A},\lambda_*}$ is the Alfvén speed associated with the local dynamo-generated field whose reversal scale is λ, $S_{\lambda_*} \doteq v_{{\rm A},\lambda_*}\lambda_*/\eta$ is the relevant Lundquist number, and we have taken the tearing stability parameter Δ' to satisfy $\Delta'\lambda \sim (k_{\rm t}\lambda_*)^{-n}$ with $n > 1/3$. If the current sheets were to resemble the Harris (tanh) profile, then $n = 1$; for a sinusoidal profile that is more reminiscent of the folds produced by the fluctuation dynamo, $n = 2$.[1]

There are two important things to note about (1). First, these scalings reproduce the scalings for the fastest-growing "Coppi mode" that were discussed in class and derived in §VIII.3.4 of the lecture notes, modified by multiplication by powers of Pm. As Pm increases, the tearing growth rate decreases because of viscous damping of the fluid motions. Clearly there is a sweet spot in the value of Pm: it should be large enough to produce the elongated, spatially anisotropic folds that favor tearing modes, but not so large that viscous stresses suppress tearing in the first place. Second, when applying (1), one must verify *a posteriori* that $k_{\rm t}\ell_\parallel \gtrsim 1$, i.e., that the fastest-growing tearing mode actually fits into the current sheet associated with the folded magnetic fold.

What you should find is that a typical current sheet produced during the kinematic stage of the dynamo should resistively diffuse before tearing has a chance to onset, a conclusion that is independent of n. This is because the onset of tearing in a current sheet of length $\ell_\parallel \sim \ell_\nu$ requires that $B^2_{\rm rms} \gtrsim U^2\,{\rm Re}^{-1/2}$ (which you might recall marks the end of the kinematic stage).

(d) By the end of the kinematic stage, magnetic folds having their reversal scale ℓ_η set by resistive diffusion also have large enough $v_{{\rm A},\ell_\eta}$ that $\lambda_* \sim \ell_\eta$. For tearing to onset in the subsequent nonlinear stage of the dynamo, during which $B^2_{\rm rms}$ grows secularly in time, the maximal current-sheet thickness λ_* must grow faster than ℓ_η does. In part (b), you should have found that $\ell_\eta \sim L\,(B_{\rm rms}/U)\,{\rm Rm}^{-1/2}$; with $B^2_{\rm rms} \propto t$ during this nonlinear stage of the dynamo, this implies that $\ell_\eta \propto t^{1/2}$. Galishnikova *et al.* showed that, indeed, λ_* grows faster than ℓ_η during this stage of the fluctuation dynamo, i.e., tearing can onset during this stage of evolution. They then conjectured that a typical current sheet with reversal scale

$$\lambda_* \sim L\left[\frac{(B^2_{\rm rms}/U^2)^{4n+1}}{{\rm Rm}^{n+1}{\rm Pm}^n}\right]^{1/2(2n+1)}$$

would decorrelate at a rate slower than the growth rate of the tearing instability. With $k_{\rm t}$ given by (1b), such tearing should spawn

$$N_{\rm t} \sim k_{\rm t}\ell_\parallel \sim \left[\frac{B^2_{\rm rms}}{U^2}{\rm Re}^{1/2}\right]^{n/(2n+1)}{\rm Pm}^{1/2} \gtrsim 1$$

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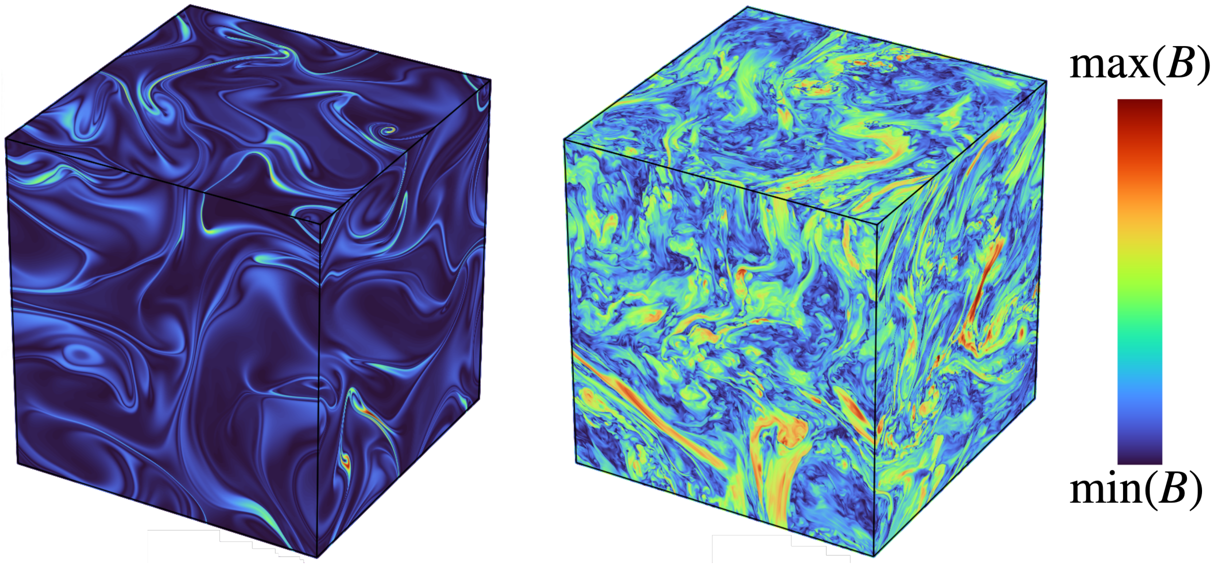
[1] There was a possible homework problem for HW05 asking you to calculate $\Delta'(k)$ for a sinusoidally varying reconnecting magnetic field, but I dropped it because HW05 was already long enough for this late in the semester. The answer to that problem is $\Delta'\lambda \simeq (8/\pi)(k_{\rm t}\lambda)^{-2}$ when $k_{\rm t}\lambda \ll 1$, *viz.* $n = 2$.

magnetic islands. Use these expressions to obtain predictions for λ_* and N_t in the *saturated state* of the dynamo, when $B_\mathrm{rms} \sim U$. Evaluate these for $n = 2$ (sinusoidal sheets) and estimate how large Re must be for there to be a range of scales on which tearing acts much faster than resistive decay and nonlinear decorrelation (i.e., $\lambda_* \gg \ell_\eta$). With $\mathrm{Pm} \sim 10$, how large does Rm have to be? This estimate shows why high numerical resolution is required to capture the tearing disruption of dynamo-generated folds.

The theory goes on from there to conjecture that the role of the tearing instability is to break up the folds into a succession of smaller structures that exhibit a $k^{-19/9}$ energy spectrum. Some qualitative evidence that the authors were on to something is shown in the images below, which are of the magnetic-field strength in the kinematic stage (left) and saturated state (right) of the fluctuation dynamo in a 2240^3 simulation with $\mathrm{Pm} = 10$ and $\mathrm{Rm} \approx 5{\times}10^4$:



Note the laminar, coherent folds on the left and the large number of small-scale plasmoids on the right. For *quantitative* evidence that the authors were on to something – from how the fold thickness depends on Rm and Pm, to the aspect ratios of the disrupted folds, to the magnetic energy spectrum, to the ratio of viscous and resistive dissipation – read the paper. If you're interested, there is much more to be done on this topic, both in MHD and in collisionless kinetic plasmas.