AST 552 Spring 2026 **Homework #5: Shocks & Reconnection**

Due Monday, April 20, 2026

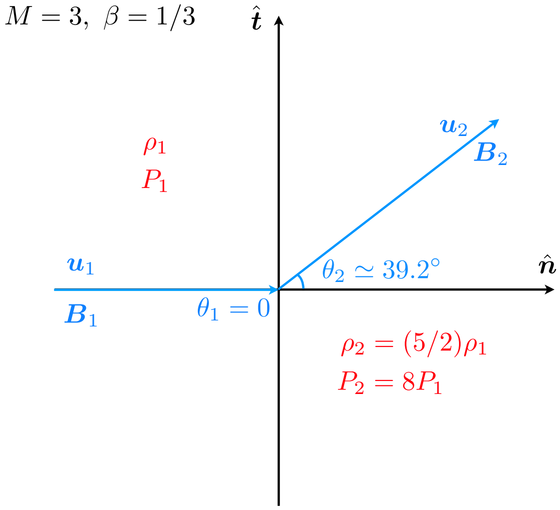
Generals prep. Make sure you can provide brief definitions of the following terms: Rankine–Hugoniot jump conditions, tearing instability, Lundquist number, Δ', Sweet–Parker reconnection, Petschek reconnection, plasmoid instability

1. **Shockspotting.** The following short-answer questions highlight some basic facts of MHD shocks with which you should familiarize yourselves. Consult §VII.2 of the lecture notes.

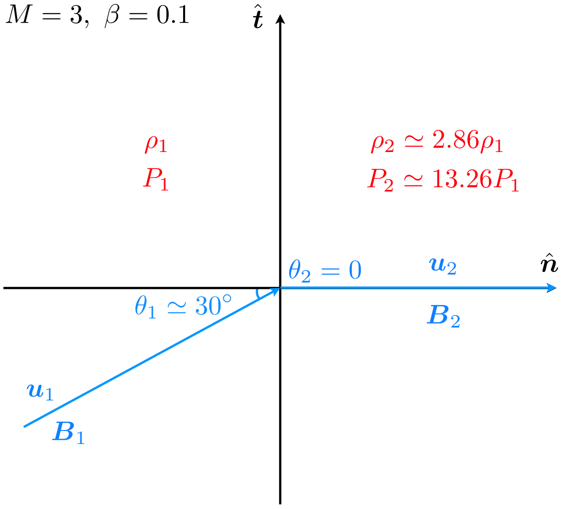
(a) Identify the following features as being associated with an adiabatic shock that is either "parallel", "perpendicular", "switch-off", or "switch-on":

(i) The upstream and downstream magnetic fields are aligned with the shock front.

(ii)



(iii)



(iv) The compression ratio $R \doteq \rho_2/\rho_1 = u_{n1}/u_{n2}$ satisfies

$$R = \frac{\gamma + 1}{\gamma - 1 + 2/M^2},$$

where $M \doteq u_{n1}/c_{\rm s1}$ is the Mach number of the shock and γ is the adiabatic index.

(v) The upstream normal velocity u_{n1} *must* exceed the fast magnetosonic speed ahead of the shock, *viz.* $u_{n1}^2 > c_{\rm s1}^2 + v_{\rm A1}^2$.

(vi) The upstream normal velocity u_{n1} *must* satisfy $u_{n1}^2 = B_{n1}^2/4\pi\rho_1 < B_1^2/4\pi\rho_1$.

(b) The pressure ratio for a strong ($M^2 \gg 1$) hydrodynamic shock in a monatomic gas with $\gamma = 5/3$ satisfies $P_2/P_1 \approx (5/4)M^2$. Explain in a sentence or two why this result implies there must be a maximum value for R.

(c) In an oblique fast-mode shock, the angles $\theta_1 \doteq \cos^{-1}(B_{n1}/B_1)$ and $\theta_2 \doteq \cos^{-1}(B_{n2}/B_2)$ satisfy $|\theta_2| > |\theta_1|$, indicating that the downstream magnetic field is canted towards the shock interface, i.e., $|B_{t2}| > |B_{t1}|$. Explain in a sentence or two why this occurs.

(d) In an oblique slow-mode shock, $|\theta_2| < |\theta_1|$ with B_{t2} and B_{t1} having the same sign. In this case, the downstream magnetic field is oriented closer to the shock normal than the upstream magnetic field. Explain in a sentence or two why this occurs. Hint: slow modes involve an anti-correlation between thermal pressure and magnetic pressure.

2. **Lundquist numbers in nature and the laboratory.** The following table shows (very approximate) parameters for four laboratory experiments used to investigate magnetic reconnection and one type of natural reconnection event. You can click on the experiment's names below to be taken to their websites, where you can find more information about them.

system	n_e (cm^{-3})	T_e (eV)	B (kG)	L (cm)
MRX	10^{13}–10^{14}	3–15	0.01–0.04	20–50
FLARE	10^{13}–10^{14}	7–20	0.2–1	50–150
TREX	10^{12}–10^{13}	5–20	0.01–0.5	$\lesssim$50
Puffin Z-pinch	$\sim$$10^{18}$	10–1000	1–10	$\sim$1
solar flare	10^{9}–10^{10}	100–1000	0.01–0.1	10^{9}–10^{10}

The resistivity in a collisional plasma is given by

$$\eta \doteq \frac{c^2}{4\pi}\frac{\alpha_e m_e}{e^2 n_e \tau_{ei}},$$

where τ_{ei} is the electron–ion collision time and α_e is an order-unity constant that depends on the ion charge number Z. For $Z = 1$, $\alpha_e \simeq 0.51$; for $Z = 2$, $\alpha_e \simeq 0.44$; for $Z \to \infty$, $\alpha_e \simeq 0.29$. The corresponding Lundquist number, evaluated for $Z = 1$ and suitably normalized, is

$$S_L \doteq \frac{v_{\rm A} L}{\eta} \approx 5200 \left(\frac{B}{1~{\rm kG}}\right)\left(\frac{L}{10~{\rm cm}}\right)\left(\frac{T_e}{10~{\rm eV}}\right)^{3/2}\left(\frac{n_e}{10^{13}~{\rm cm}^{-3}}\right)^{-1/2}\left(\frac{\lambda_{ei}}{10}\right)^{-1},$$

where λ_{ei} the Coulomb logarithm. Calculate the possible range of Lundquist numbers in these systems. Which of the above laboratory experiments is most likely to produce conditions for the plasmoid instability to occur?

3. **Simple solutions for shrinking sheets (and their tearing).** In class, we studied the evolution of small-amplitude perturbations to an equilibrium current sheet (CS) described by the flux function $\Psi(x)$. For example, $\Psi = av_{\rm A}\ln[\cosh(x/a)]$ corresponds to a Harris-sheet profile with characteristic thickness a. Under certain restrictions, one can use the Reduced MHD (RMHD) equations to describe a *time-dependent* magnetic configuration and investigate its linear stability to tearing. In this problem, you'll use the RMHD momentum and induction equations (see §V.5 of the lecture notes),

$$\frac{\partial}{\partial t}\nabla_\perp^2\Phi + \{\Phi,\nabla_\perp^2\Phi\} = v_{\rm A}\frac{\partial}{\partial z}\nabla_\perp^2\Psi + \{\Psi,\nabla_\perp^2\Psi\}, \tag{1}$$

$$\frac{\partial\Psi}{\partial t} + \{\Phi,\Psi\} = v_{\rm A}\frac{\partial\Phi}{\partial z}, \tag{2}$$

to obtain simple solutions describing a thinning CS whose width $a(t)$ shrinks in time, causing the tearing-mode stability parameter $\Delta'(t,k)$ to increase in time. These time-dependent solutions are based on S. Chapman & P.C. Kendall, *Proc. Roy. Soc. London Ser. A*, **271**, 435 (1963), and were used by N.F. Loureiro and D.A. Uzdensky, *Phys. Rev. Lett.* **116**, 105003 (2016) and E.A. Tolman *et al.*, *J. Plasma Phys.* **84**, 905840115 (2018) in their studies of the onset of reconnection in a thinning CS. Here we'll sketch out those authors' basic idea.

(a) Consider the following time-dependent stream and flux functions:

$$\Phi(t,x,y) = \Lambda(t)xy \quad \text{and} \quad \Psi(t,x,y) = \frac{B_0}{2}\left[\frac{x^2}{a(t)} - \frac{y^2}{L(t)}\right]. \tag{3}$$

These describe a local incompressible flow that is thinning and lengthening a CS about an X-point. For these potentials to be solutions of the RMHD equations, what must the CS width $a(t)$ and length $L(t)$ satisfy? Plot iso-contours of $\Phi/(\Lambda a^2)$ and $\Psi/(B_0 a)$ in the (x/a)-(y/a) plane for $L/a = 10$ and describe what you see. (You might find it helpful to calculate the flow velocity $\boldsymbol{u} = \hat{\boldsymbol{z}}\times\boldsymbol{\nabla}_\perp\Phi$ and the magnetic field $\boldsymbol{B} = \hat{\boldsymbol{z}}\times\boldsymbol{\nabla}_\perp\Psi$ corresponding to these functions and plot their vector fields.)

(b) Set $\Lambda(t) = \tau^{-1}$ with $\tau = \text{const}$ and solve your equations for $a(t)$ and $L(t)$. (Name the initial values of the CS thickness and length a_0 and L_0, respectively.) Briefly describe in words the evolution of this CS.

(c) Suppose $L(t) = L_0(1 + t/\tau)$ with $\tau = \text{const}$. Obtain the corresponding $\Lambda(t)$ and $a(t)$. Briefly describe in words the evolution of this CS.

(d) Let's adopt the CS model from part (c) and set $\tau \doteq (L_0/v_{\rm A})M_{\rm A}^{-1}$, where $M_{\rm A}$ is the Alfvén Mach number of the incompressible flow. We now ask how linear tearing modes grow on top of this time-dependent background and determine which of these linear modes grows the fastest at any given time in the CS evolution. For that, give the CS some resistivity η, and assume that the outer solution for the CS provides $\Delta'(k) \sim 1/ka^2$ with $ka \ll 1$. (The "$\sim$" here means that we are dropping factors of order unity.) The number of tearing-induced magnetic islands with wavenumber k that can fit inside the length of this CS at any given time is $\sim kL \doteq N$. Because each tearing-mode wavelength k^{-1} is stretched by the flow in the same way as is L, each tearing mode can be labeled by its own unique value of N. With this borne in mind, answer the following:

(i) Take the long-time limit $t \gg \tau$, such that $L(t) \sim L_0(t/\tau)$. Write down how $\Delta'(N)$ evolves in time for this CS. Your answer should involve N, L_0, a_0, and t/τ only.

(ii) Show that, in the FKR regime, the time-dependent growth rate $\gamma_{\rm FKR}$ satisfies (see VIII.3.24 in the lecture notes)

$$\gamma_{\rm FKR}(t)\tau_0 \sim N^{-2/5} M_{\rm A}^{12/5} S_0^{-3/5} \left(\frac{t}{\tau_0}\right)^{12/5}, \tag{4}$$

where $\tau_0 \doteq (a_0 L_0)^{1/2}/v_{\rm A}$ and $S_0 \doteq v_{\rm A}(a_0 L_0)^{1/2}/\eta$. Thus, the fastest-growing FKR mode is the $N = 1$ mode.[1]

(iii) Use (4) to determine the approximate time at which the $N = 1$ FKR mode grows faster than the rate at which the CS thickness is shrinking. (Don't be too fancy here – I'm only looking for a scaling argument.) Name this time $t_{\rm cr}$ and express it in terms of τ_0, $M_{\rm A}$, and S_0.

(iv) Determine the approximate time at which this $N = 1$ mode transitions into the Coppi regime (see VIII.3.29 in the lecture notes). Name this time $t_{\rm tr}$ and express it in terms of τ_0, $M_{\rm A}$, and S_0.

(v) For what combination of $M_{\rm A}$ and S_0 is $t_{\rm tr} \sim t_{\rm cr}$? In this situation, the maximally growing FKR mode enters the Coppi regime just as it starts to grow fast enough to disrupt the evolving CS. Loureiro & Uzdensky argued that, under these conditions, this time marks the onset of reconnection and the disruption of the CS.[2]

(vi) Consider a solar flare powered by a reconnecting CS whose $L_0 \sim a_0 \sim 10^4$ km and which evolves according to our crude model here. Typical photospheric values are $v_{\rm A} \sim 2000$ km s^{-1}, $M_{\rm A} \sim 10^{-3}$, and $S_0 \sim 10^{13}$. If you plug these numbers in to your answer from part (v), you should find that $t_{\rm tr} \sim t_{\rm cr}$. Use this to estimate the time at which reconnection onsets, as well as the aspect ratio of the CS at this time. The former turns out to be reasonably consistent with the observed pre-flare energy-buildup times in the solar photosphere. Nice.

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[1] In writing (4), we are implicitly assuming that the secular evolution of the CS does not greatly affect the instantaneous exponential growth of the tearing modes – only that it changes the instantaneous values of $\tau_{\rm A}$, τ_η, and $\Delta' a$ that figure into the usual FKR growth rate. This is a good approximation when $\gamma_{\rm FKR}(t) \gg |\dot{a}/a|$ – see Tolman *et al.* (2018) if you're interested in the more rigorous details.

[2] They also considered the cases $t_{\rm tr} > t_{\rm cr}$ and $t_{\rm cr} < t_{\rm tr}$; I picked $t_{\rm tr} \sim t_{\rm cr}$ just to keep this problem short(ish).

4. **Tearing mode with rigid conducting walls**. The solution for the tearing instability of a current sheet whose equilibrium magnetic profile is described by the flux function $\Psi = \Psi(x)$ satisfies

$$0 = \left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} - k^2 - \frac{\Psi'''}{\Psi'} \right) \psi_{\mathrm{out}}, \tag{5}$$

where $\psi_{\mathrm{out}} = \psi_{\mathrm{out}}(k, x)$ is the flux function of a magnetic perturbation having wavenumber k evaluated in the "outer region" of the perturbed sheet where the growth rate $\gamma = \gamma(k)$ of the tearing mode satisfies the double inequality $\tau_\eta^{-1} \doteq \eta/a^2 \ll \gamma \ll \tau_{\mathrm{A}}^{-1} \doteq ka\Psi''(0)$ (see §VIII.3.2 of the lecture notes). The linear tearing of an equilibrium Harris sheet having $\Psi'(x) = B_{\mathrm{r}} \tanh(x/a)$ has the following outer solution and tearing instability parameter:

$$\psi_{\mathrm{out}}(k, x) = C\, \mathrm{e}^{-kx} \left[1 + \frac{1}{ka} \tanh\left(\frac{x}{a} \right) \right] \quad \text{and} \quad \Delta'(k) = \frac{2}{a} \left(\frac{1}{ka} - ka \right), \tag{6}$$

where C is a constant. Now place rigid conducting walls at $x = \pm b$, with $b \gg a$. Do you expect the plasma to be more or less stable to tearing than it was without the walls? Construct a physical argument, and then calculate the change in $\Delta'(k)$ because of the walls.

Hint: write

$$\psi(x) = C_-\, \mathrm{e}^{-kx} \left[1 + \frac{1}{ka} \tanh\left(\frac{x}{a} \right) \right] + C_+\, \mathrm{e}^{+kx} \left[1 - \frac{1}{ka} \tanh\left(\frac{x}{a} \right) \right]$$

for $x \geq 0$ and impose $\psi(b) = 0$ with $b \gg a$ and $kb \sim \mathcal{O}(1)$. You should find that the correction to Δ' is small when $\exp(-2kb) \ll 1$.