Due Monday, April 6, 2026

Generals prep. Make sure you can provide brief definitions of the following terms: Kelvin–Helmholtz instability, Rayleigh–Taylor instability, Schwarzschild convective instability, Parker instability, interchange instability, magnetorotational instability, kink instability, sausage instability, energy principle.

1. **Interchange instability, three ways.** Consider a stratified atmosphere whose equilibrium state represents force balance between the acceleration of a constant gravitational field $\boldsymbol{g} = -g\hat{\boldsymbol{z}}$ and supporting gradients in the thermal and magnetic pressures:

$$g = -\frac{1}{\rho}\left(\frac{\mathrm{d}P_0}{\mathrm{d}z} + \frac{1}{8\pi}\frac{\mathrm{d}B_0^2}{\mathrm{d}z}\right) = -c_\mathrm{s}^2\frac{\mathrm{d}\ln P_0^{1/\gamma}}{\mathrm{d}z} - v_\mathrm{A}^2\frac{\mathrm{d}\ln B_0}{\mathrm{d}z} = \mathrm{const},$$

where $c_\mathrm{s}^2 \doteq \gamma P_0/\rho_0$ and $v_\mathrm{A}^2 \doteq B_0^2/4\pi\rho_0$ are the squares of the sound and Alfvén speeds. To this equilibrium we add small-amplitude perturbations that depend *only* on z:

$$\rho = \rho_0(z) + \delta\rho(z)\,\mathrm{e}^{-\mathrm{i}\omega t},\ \boldsymbol{u} = \delta\boldsymbol{u}(z)\,\mathrm{e}^{-\mathrm{i}\omega t},\ \boldsymbol{B} = B_0(z)\hat{\boldsymbol{x}} + \delta\boldsymbol{B}(z)\,\mathrm{e}^{-\mathrm{i}\omega t},\ P = P_0(z) + \delta P(z)\,\mathrm{e}^{-\mathrm{i}\omega t}.$$

This situation is notably different from that associated with the Parker instability, in which $k_\parallel \neq 0$ is required to bend the magnetic-field lines and sinusoidally relieve the magnetic field of the weight of its confining mass. This problem guides you through three derivations of the *interchange instability*, which occurs when rigid, mass-supporting flux tubes are re-arranged vertically to liberate gravitational potential energy and take the system to an overall lower energy state.

(a) The linearized MHD equations may be written as

$$\text{continuity}: \quad \frac{\delta\rho}{\rho} = -\boldsymbol{\xi}\boldsymbol{\cdot}\boldsymbol{\nabla}\ln\rho - \boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\xi},$$

$$\text{momentum}: \quad \frac{\mathrm{D}^2\boldsymbol{\xi}}{\mathrm{D}t^2} = -\frac{1}{\rho}\boldsymbol{\nabla}\left(\delta P + \frac{\boldsymbol{B}\boldsymbol{\cdot}\delta\boldsymbol{B}}{4\pi}\right) + \frac{\delta\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{B}}{4\pi\rho} + \frac{\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{\nabla}\delta\boldsymbol{B}}{4\pi\rho} + \boldsymbol{g}\frac{\delta\rho}{\rho},$$

$$\text{induction}: \quad \delta\boldsymbol{B} = \boldsymbol{\nabla}\times(\boldsymbol{\xi}\times\boldsymbol{B}),$$

$$\text{entropy}: \quad \frac{\delta P}{P} = \gamma\frac{\delta\rho}{\rho} - \boldsymbol{\xi}\boldsymbol{\cdot}\boldsymbol{\nabla}\ln P\rho^{-\gamma},$$

where $\boldsymbol{\xi}$ is the displacement of a fluid element, which satisfies $\mathrm{D}\boldsymbol{\xi}/\mathrm{D}t \doteq \Delta\boldsymbol{u} = \delta\boldsymbol{u}$ (the final equality is because $\boldsymbol{u} = \boldsymbol{0}$ in the background state). For notational ease, I've dropped all the "0" subscripts on the background quantities, at no consequence for the linear theory. Apply this set of equations to the problem at hand, and then combine them to derive the following differential equation for the vertical displacement ξ_z:

$$\left(v_\mathrm{A}^2 + c_\mathrm{s}^2\right)\frac{\mathrm{d}^2\xi_z}{\mathrm{d}z^2} + \left(v_\mathrm{A}^2\frac{\mathrm{d}\ln B^2}{\mathrm{d}z} + c_\mathrm{s}^2\frac{\mathrm{d}\ln P}{\mathrm{d}z}\right)\frac{\mathrm{d}\xi_z}{\mathrm{d}z} + \omega^2\xi_z = 0. \qquad (1)$$

(b) Equation (1) combines the physics of both buoyancy and magnetosonic waves, and is therefore more complicated than it most likely needs to be. Namely, if $\mathrm{d}\xi_z/\mathrm{d}z = \mathrm{i}k_z\xi_z$ with $k_zH \gg 1$ (H the scale height of the equilibrium), then the dominant terms in (1) are the first and third terms; balancing them gives

$$\omega^2 \approx \omega_{\mathrm{fast}}^2 \doteq k_z^2\big(v_{\mathrm{A}}^2 + c_{\mathrm{s}}^2\big),$$

representing fast waves propagating vertically in the atmosphere. We can separate this branch from the buoyancy branch by adopting the *Boussinesq approximation*, which amounts to an assumption of total pressure balance between the displaced fluid elements and their surroundings. In this approximation, the relevant (buoyancy) frequency satisfies $\omega_{\mathrm{buoy}} \sim (g/H)^{1/2} \ll \omega_{\mathrm{fast}}$.

Enact the Boussinesq approximation by calculating the total pressure perturbation $\delta P_{\mathrm{tot}} \doteq \delta P + B\,\delta B_x/4\pi$, imposing $\delta P_{\mathrm{tot}} \approx 0$ to eliminate compressive (fast) dynamics, and using the result to eliminate all derivatives of ξ_z that appear in (1). You should find that

$$\omega^2 \approx \omega_{\mathrm{buoy}}^2 \doteq \frac{g}{v_{\mathrm{A}}^2 + c_{\mathrm{s}}^2}\left(v_{\mathrm{A}}^2\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{B}{\rho} + c_{\mathrm{s}}^2\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{P^{1/\gamma}}{\rho}\right). \tag{2}$$

(c) When $v_{\mathrm{A}}^2/c_{\mathrm{s}}^2 \to 0$, equation (2) returns the Brunt–Väisälä frequency that features in the Schwarzschild criterion for convective instability:

$$\omega_{\mathrm{buoy}}^2 \approx g\,\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{P^{1/\gamma}}{\rho}.$$

As discussed in class, if the entropy increases upwards ($\omega_{\mathrm{buoy}}^2 > 0$), the fluid is stable. This was explained by displacing a fluid element from height z_1 to height z_2 while maintaining pressure balance with its surroundings (so that $P_{\mathrm{new}} = P_2$) while conserving its entropy (so that $\sigma_{\mathrm{new}} = \sigma_1$); and then asking whether the fluid element is too heavy or too light to stay at its new position (*viz.*, is $\rho_{\mathrm{new}} > \rho_2$ or $< \rho_2$?)

When $c_{\mathrm{s}}^2/v_{\mathrm{A}}^2 \to 0$, equation (2) instead becomes

$$\omega_{\mathrm{buoy}}^2 \approx g\,\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{B}{\rho}.$$

Construct an analogous physical argument to explain why having B/ρ increase upwards results in a stable MHD configuration. Support your argument with a sketch of what you think is happening. <u>Hint</u>: Mass and magnetic flux are conserved in ideal MHD.

(d) Neglecting surface terms, the energy integral for this problem's equilibrium state reads (see equation (VI.11.13) of the lecture notes)

$$\delta W_2 = \int \mathrm{d}\boldsymbol{r}\,\frac{\rho}{2}\left[c_{\mathrm{s}}^2\left(\frac{\mathrm{d}\xi_z}{\mathrm{d}z} - \frac{g}{c_{\mathrm{s}}^2}\xi_z\right)^2 + v_{\mathrm{A}}^2\left(\frac{\mathrm{d}\xi_z}{\mathrm{d}z}\right)^2 + \frac{g}{c_{\mathrm{s}}^2}\left(v_{\mathrm{A}}^2\frac{\mathrm{d}\ln B}{\mathrm{d}z} + c_{\mathrm{s}}^2\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{P^{1/\gamma}}{\rho}\right)\xi_z^2\right]. \tag{3}$$

First, explain in your own words how the energy principle works. Then use δW_2 to obtain a necessary condition for *instability* and compare it to what you found in parts (b) and (c). In particular, use (3) to argue why pressure-balanced perturbations with $\delta P_{\mathrm{tot}} \approx 0$ are the most unstable perturbations. <u>Hint</u>: When $\delta P_{\mathrm{tot}} \approx 0$, the energy integral is $\delta W_2 \approx \int \mathrm{d}\boldsymbol{r}\,(1/2)\rho\omega_{\mathrm{buoy}}^2\xi_z^2$.

2. **Magnetorotational instability via springs.** The acknowledgement at the end of Balbus & Hawley (1992a) reads, "It is fitting and proper to acknowledge Alar Toomre for this important insight that the Hill equations had something to contribute to the MHD stability problem." This insight is what led Steve Balbus and John Hawley to develop the now-famous spring model of the MRI, which they immediately used to conjecture that the so-called "Oort A-value",

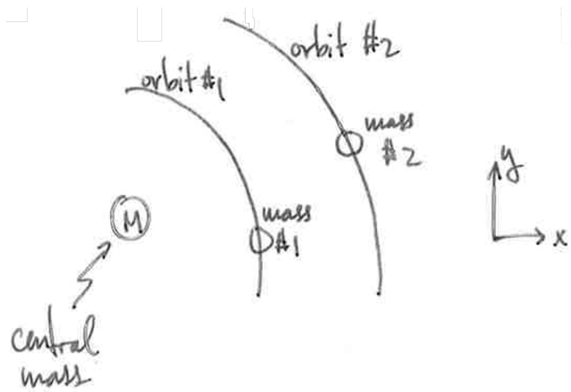
$$A \doteq -\frac{1}{2}\frac{\mathrm{d}\Omega}{\mathrm{d}\ln R} \quad \left(= \frac{3}{4}\Omega \text{ for Keplerian rotation}\right),$$

is a universal limit for the growth rate of accretion-disk shear instabilities (it's not, btw). The Hill equations describe the local dynamics of orbiting point masses as measured in a rotating frame – *local* in that they describe small excursions $x \doteq R - R_0$ and $y \doteq R_0(\phi - \Omega_0 t)$ from a circular orbit described by $R = R_0$, $\phi = \Omega_0 t$. They are

$$\ddot{x} - 2\Omega_0\dot{y} = 4A_0\Omega_0 x + f_x, \tag{4a}$$

$$\ddot{y} + 2\Omega_0\dot{x} = f_y, \tag{4b}$$

where the overdot indicates a time derivative and f_x and f_y represent local forces in the x and y directions. The terms proportional to $2\Omega_0$ represent the Coriolis force; the term proportional to A_0 represents the local "tidal force" – the difference between the local centrifugal force (outward) and the gravitational force from the central object (inward) that is caused by the differential rotation. Visually,

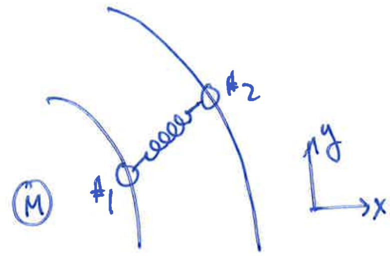


with that inner mass typically orbiting faster than the outer mass (because typically $A > 0$). In the absence of the local forces f_x and f_y, small displacements $\boldsymbol{\xi}(t) \propto \exp(-\mathrm{i}\omega t)$ from the equilibrium position at $(x, y) = (0, 0)$ satisfy the linear equation

$$\begin{bmatrix} -\omega^2 - 4A_0\Omega_0 & 2\Omega\mathrm{i}\omega \\ -2\Omega\mathrm{i}\omega & -\omega^2 \end{bmatrix}\begin{bmatrix} \xi_x \\ \xi_y \end{bmatrix} = 0 \quad \Longrightarrow \quad \omega^2(\omega^2 - \kappa^2) = 0,$$

where $\kappa^2 \doteq 4\Omega_0^2(1 - A_0/\Omega_0)$ is the square of the local epicyclic frequency. (For Keplerian rotation, $\kappa^2 = \Omega_0^2$, which explains why Keplerian orbits are closed.) With $\kappa^2 > 0$ in most all astrophysical disks, this dispersion relation states that perturbations about an equilibrium undergo epicyclic oscillations at the epicyclic frequency, $\omega = \kappa$.

The MRI analogy goes as follows. Consider the forces f_x and f_y to be nondissipative and to act by restoring a displacement back to its equilibrium position. The leading-order contribution to f_x and f_y in a Taylor expansion about $(R_0, \Omega_0 t)$ is linear, i.e., Hooke's law. Balbus and Hawley therefore chose $f_x = -Kx$ and $f_y = -Ky$, where $K > 0$ is the spring constant. Visually,



Then (4) becomes

$$\ddot{x} - 2\Omega_0 \dot{y} = 4A_0\Omega_0 x - Kx, \tag{5a}$$

$$\ddot{y} + 2\Omega_0 \dot{x} = -Ky. \tag{5b}$$

Let's use these equations in what follows.

(a) Consider small perturbations $\boldsymbol{\xi}(t) \propto \exp(-\mathrm{i}\omega t)$ about the equilibrium position $(x, y) = (0, 0)$ and calculate the new dispersion relation using (5). Compare your result with the MRI dispersion relation (consult §VI.8 of the lecture notes) and identify what physical effect provides the restoring force analogous to K, and what parameter replaces K in the actual MRI dispersion relation. Finally, obtain a *necessary condition for instability.*

(b) By setting $\mathrm{d}\gamma/\mathrm{d}K = 0$ where $\gamma = -\mathrm{i}\omega$, calculate the maximum growth rate of this instability, the value of K at which it is obtained, and the eigenvector associated with this maximum growth rate. Use this eigenvector to calculate the linear change in the angular momentum $\ell \doteq \Omega R^2$ of a mass displaced by a small distance from $(R_0, \Omega_0 t)$,

$$\frac{\Delta\ell}{\ell_0} = \frac{\Delta\Omega}{\Omega_0} + 2\frac{\xi_x}{R_0} = \left(\frac{\gamma}{\Omega_0}\frac{\xi_y}{\xi_x} + 2\right)\frac{\xi_x}{R_0}.$$

Note that the difference in angular momentum between two nearby orbits, one at R_0 and one at $R_0 + \xi_x$, satisfies $\Delta\ell/\ell_0 \simeq 2(1 - A_0/\Omega_0)(\xi_x/R_0)$. Leverage all of this information to describe in physical terms how the spring force couples radial displacements to the exchange of angular momentum between neighboring fluid elements.

(c) **(Optional)** Just for fun, consider an anisotropic spring: $f_x = -K_x x$ and $f_y = -K_y y$, with $K_x \neq K_y$ being positive constants. Use (4) to compute the new dispersion relation governing the time-evolution of small displacements. Is the growth rate larger or smaller than the Oort-A value for $K_x > K_y$? for $K_x < K_y$? From this result, find the maximum growth rate and the (hint: asymptotic) values of K_x and K_y at which $\gamma_{\max}$ is achieved. (It may help you to make a contour plot of the growth rate in the K_x–K_y plane using your dispersion relation.) What happens to $\Delta\ell/\ell_0$ in this fastest-growing limit? What do you think is going on physically?

It just so happens that this anisotropic-spring model is a good description of the linear, axisymmetric MRI in a weakly collisional (Braginskii) plasma and in a collisionless (drift-kinetic) plasma; for both cases, K_y includes a contribution from a viscous stress associated with the pressure anisotropy of the perturbed plasma. (I have a problem set from AST521 and some notes on this topic if you're interested, but this goes beyond the content of GPP2. You can also read more in Quataert, Dorland & Hammett (2002).)

3. **Parametric decay instability.** Consider a circularly polarized Alfvén wave of amplitude $B_\perp$ and wavenumber k_0,

$$\boldsymbol{B}_\perp = -\sqrt{4\pi\rho_0}\,\boldsymbol{u}_\perp = B_\perp \cos\bigl[k_0(z - v_{\rm A}t)\bigr]\hat{\boldsymbol{x}} + B_\perp \sin\bigl[k_0(z - v_{\rm A}t)\bigr]\hat{\boldsymbol{y}}, \tag{6}$$

propagating along a uniform magnetic field $\boldsymbol{B}_0 = B_0\hat{\boldsymbol{z}}$ in an otherwise homogeneous, stationary plasma having mass density ρ_0 and Alfvén speed $v_{\rm A} \doteq B_0/\sqrt{4\pi\rho_0}$. Because $B^2 = B_0^2 + B_\perp^2 = \text{const}$, this wave constitutes an exact nonlinear solution of ideal-MHD equations. Because it is a nonlinear solution, you'd be forgiven for thinking this wave can propagate perfectly without any losses. What happens if we introduce just a little bit of noise. . .

Perturb this wave-supporting background state as follows:

$$\begin{array}{rcccccccl} \rho(t,z) & = & \rho_0 & & & & & + & \delta\rho(t,z), \\ \boldsymbol{u}(t,z) & = & & & \boldsymbol{u}_\perp(t,z) & + & \delta\boldsymbol{u}_\perp(t,z) & + & \delta u_\parallel(t,z)\hat{\boldsymbol{z}}, \\ \boldsymbol{B}(t,z) & = & \boldsymbol{B}_0 & + & \boldsymbol{B}_\perp(t,z) & + & \delta\boldsymbol{B}_\perp(t,z) & + & \delta B_\parallel(t,z)\hat{\boldsymbol{z}}. \end{array} \tag{7}$$

Because $\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{B} = 0$, we must have $\delta B_\parallel = 0$. As a result, the fourth column on the right-hand side represents a purely acoustic perturbation to the background; the third column represents a purely Alfvénic perturbation. These perturbations will ultimately represent "daughter" sound and Alfvén waves, which feed off the energy of the "parent" Alfvén wave (6) and grow exponentially. This problem has you derive and analyze this *parametric decay instability*.

(a) Substitute (7) into the ideal MHD equations and linearize in the perturbation amplitudes $\delta\boldsymbol{u}_\perp$, $\delta\boldsymbol{B}_\perp$, $\delta\rho$, $\delta u_\parallel$, and $\delta P \doteq c_{\rm s}^2\delta\rho$. Obtain the following equations describing a small-amplitude sound wave and a small-amplitude Alfvén wave that are coupled by the space-time dependence of the parent Alfvén wave:

$$\left(\frac{\partial^2}{\partial t^2} - c_{\rm s}^2\frac{\partial^2}{\partial z^2}\right)\frac{\delta\rho}{\rho_0} = v_{\rm A}^2\frac{\partial^2}{\partial z^2}\left(\frac{\boldsymbol{B}_\perp}{B_0}\boldsymbol{\cdot}\frac{\delta\boldsymbol{B}_\perp}{B_0}\right), \tag{8a}$$

$$\left(\frac{\partial^2}{\partial t^2} - v_{\rm A}^2\frac{\partial^2}{\partial z^2}\right)\frac{\delta\boldsymbol{B}_\perp}{B_0} = -\frac{\partial^2}{\partial t\partial z}\left(\frac{\boldsymbol{B}_\perp}{B_0}\,\delta u_\parallel\right) - \frac{\partial}{\partial z}\left(\frac{\partial\boldsymbol{u}_\perp}{\partial z}\,\delta u_\parallel\right) - \frac{\partial}{\partial z}\left(\frac{\partial\boldsymbol{u}_\perp}{\partial t}\frac{\delta\rho}{\rho_0}\right). \tag{8b}$$

(b) Introduce the unit vectors $\hat{\boldsymbol{e}}_\pm \doteq (\hat{\boldsymbol{x}} \pm {\rm i}\hat{\boldsymbol{y}})/\sqrt{2}$ and rewrite (6) as

$$\boldsymbol{B}_\perp = -\sqrt{4\pi\rho_0}\,\boldsymbol{u}_\perp = \frac{B_\perp}{\sqrt{2}}\left({\rm e}^{{\rm i}k_0 z - {\rm i}\omega_0 t}\,\hat{\boldsymbol{e}}_- + {\rm e}^{-{\rm i}k_0 z + {\rm i}\omega_0 t}\,\hat{\boldsymbol{e}}_+\right),$$

where $\omega_0 \doteq k_0 v_{\rm A}$ is the frequency of the parent Alfvén wave. It is important in what follows to realize that $\hat{\boldsymbol{e}}_\pm\boldsymbol{\cdot}\hat{\boldsymbol{e}}_\mp = 1$ and $\hat{\boldsymbol{e}}_\pm\boldsymbol{\cdot}\hat{\boldsymbol{e}}_\pm = 0$. Use

$$f(\omega,k) = \iint {\rm d}t\,{\rm d}z\,{\rm e}^{{\rm i}\omega t - {\rm i}kz} f(t,z)$$

to Fourier transform (8) in time and space. In doing so, obtain the following coupled equations for $\delta\rho(\omega,k)$, $\delta B_+(\omega,k) \doteq \hat{\boldsymbol{e}}_+\boldsymbol{\cdot}\delta\boldsymbol{B}(\omega,k)$, and $\delta B_-(\omega,k) \doteq \hat{\boldsymbol{e}}_-\boldsymbol{\cdot}\delta\boldsymbol{B}(\omega,k)$:

$$\bigl(-\omega^2 + k^2c_{\rm s}^2\bigr)\frac{\delta\rho(\omega,k)}{\rho_0} = -\frac{k^2v_{\rm A}^2}{\sqrt{2}}\frac{B_\perp}{B_0}\left[\frac{\delta B_+(\omega_-,k_-) + \delta B_-(\omega_+,k_+)}{B_0}\right], \tag{9a}$$

$$\bigl(-\omega^2 + k^2v_{\rm A}^2\bigr)\frac{\delta B_\pm(\omega,k)}{B_0} = \mp\frac{\omega_0 k v_{\rm A}}{\sqrt{2}}\frac{B_\perp}{B_0}\left(1 \pm \frac{\omega\omega_\mp}{\omega_0^2}\frac{k_0}{k}\right)\frac{\delta\rho(\omega_\pm,k_\pm)}{\rho_0}, \tag{9b}$$

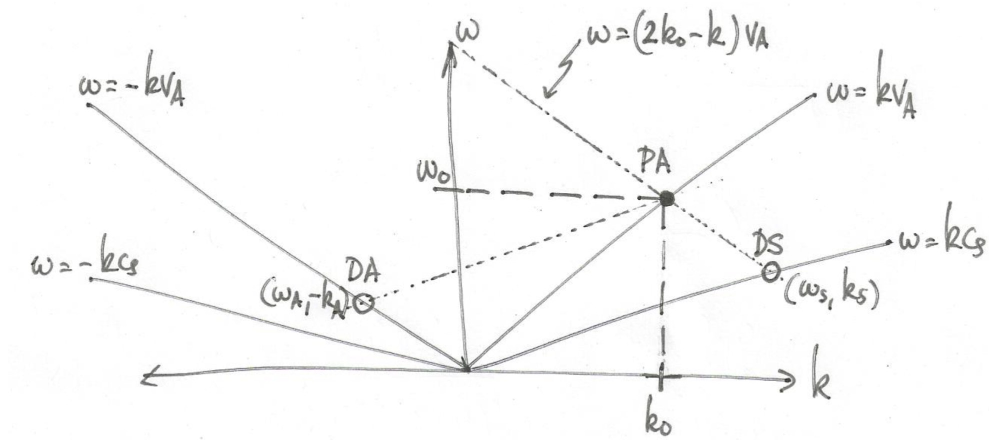
where $\omega_\pm \doteq \omega \pm \omega_0$ and $k_\pm \doteq k \pm k_0$.

Equation (9) indicates that, as the parent Alfvén wave propagates through the noisy plasma, density fluctuations at (ω, k) excite Alfvénic fluctuations at $(\omega+\omega_0, k+k_0)$ and $(\omega-\omega_0, k-k_0)$ (and *vice versa*). To solve this set of equations, we make the replacement $(\omega, k) \to (\omega_\mp, k_\mp)$ in (9b) for $\delta B_\pm(\omega, k)$ and plug the result into (9a). I'll spare you the algebra; in units where $\omega_0 = k_0 = 1$, the resulting dispersion relation is

$$(\omega^2 - \beta k^2)(\omega - k)^2[(\omega + k)^2 - 4] = \alpha^2 k^2 (\omega - k)(\omega^3 + k\omega^2 - 3\omega + k), \tag{10}$$

where $\beta \doteq c_s^2/v_A^2$ and $\alpha \doteq B_\perp/B_0$. In the limit $\alpha = 0$, the right-hand side of (10) vanishes and we obtain two (forward- plus backward-propagating) sound waves at wavenumber k; two (forward-propagating) Alfvén waves at wavenumber k; and two (backward-propagating) Alfvén waves at wavenumbers $2k_0 - k$ and $-(2k_0 + k)$. When $\alpha \neq 0$, all but one of these modes (one of the Alfvén waves at k) are coupled and cease to be normal modes. The instability appears when two of these branches intersect, with the coupling mixing them and producing complex roots.

(c) Analytical progress can be made if the parent Alfvén wave has a small amplitude ($\alpha \ll 1$), in which case the coupled branches are only weakly coupled and nearly retain their normal-mode form. In this limit, and at $\beta < 1$, the diagram looks like this:



The parent Alfvén (PA) wave is at $(\omega_0, k_0) = (1, 1)$. That wave is coupled via the Alfvénic characteristic $\omega = 2 - k$ to a forward-propagating daughter sound (DS) wave at (ω_s, k_s) and a backward-propagating daughter Alfvén (DA) wave at $(\omega_A, -k_A)$. The values of the daughter frequencies and wavenumbers may be found approximately from the *three-wave resonance conditions*

$$\omega_0 = \omega_s + \omega_A \qquad \text{and} \qquad \boldsymbol{k}_0 = \boldsymbol{k}_s + \boldsymbol{k}_A.$$

Use the diagram and these conditions to determine k_s, k_A, ω_s, and ω_A.

(d) When α is finite, the right-hand side of (10) is activated and the mode frequencies depart from those sketched above, acquiring imaginary parts. This departure may be calculated approximately by expanding (10) about the resonance you found where $\omega = \sqrt{\beta}k$ matches $\omega = 2 - k$ by writing $\omega(k) \approx \omega_s(k) + \delta\omega$ with $\delta\omega/\omega_s \sim \alpha \ll 1$. Keep terms to leading order in α, and use your result to obtain an approximate formula for the growth rate γ. (Hint: Near the resonance, the dispersion relation reduces to a quadratic equation for $\delta\omega$.) What happens to γ as $\beta \to 1$? Explain your result physically in terms of the properties of the daughter waves and the three-wave resonance.

(e) Go here and download `PDI.m` (or find it on the canvas page for HW04). This matlab script solves the dispersion relation (10) numerically, with $\alpha = 0.3$ and $\beta = 0.1$ as its default values. Play with the code a bit and explore how the five different roots and the growth rate behave as α and β vary. Write a few sentences on what you find (e.g., how does the maximum growth rate vary with α and β? how does the unstable bandwidth in k change with β? what happens to the instability as $\beta \to 1$? in what ways do the roots depart from the labelled normal modes?). Then name a terrestrial, astrophysical, or space environment in which you think the parametric decay instability might be active, accompanied by a brief discussion of what additional physics (beyond the ideal MHD considered in this problem) might need to be considered to assess whether the decay instability indeed matters there.

(f) The dispersion relation (10) indicates that one of the forward-propagating Alfvén waves is completely decoupled from the instability, with $\omega = k$ being an exact solution for any α. Explain why this mode does not couple to compressive fluctuations and therefore cannot participate in the decay instability. Hint: $\hat{\boldsymbol{e}}^{\pm} \cdot \hat{\boldsymbol{e}}^{\pm} = 0$.

4. **Curvature-driven interchange and ballooning instability.** This problem follows up on some content from HW03 #4. Lucky for you, I've done nearly all of the calculations. The emphasis here is instead on your understanding of those calculations and what they mean in physical terms for the stability of a system. Despite the written length of this problem, the amount of calculation required by you is fairly minimal, but you *will* have to think.

In the absence of gravity, the MHD energy integral for a displacement $\boldsymbol{\xi}$ from magneto-hydrostatic equilibrium is given by

$$\begin{aligned}\delta W_2[\boldsymbol{\xi},\boldsymbol{\xi}] = {} & \frac{1}{2}\oint \mathrm{d}\boldsymbol{S}\cdot\boldsymbol{\xi}\left[-\gamma P(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}) - \boldsymbol{\xi}\cdot\boldsymbol{\nabla}P + \frac{\boldsymbol{B}\cdot\delta\boldsymbol{B}}{4\pi}\right] \\ & + \frac{1}{2}\int \mathrm{d}\boldsymbol{r}\left\{\gamma P(\boldsymbol{\nabla}\cdot\boldsymbol{\xi})^2 + \frac{|\delta\boldsymbol{B}_\perp|^2}{4\pi} + \frac{B^2}{4\pi}\left(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + 2\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa}_\mathrm{c}\right)^2 \right. \\ & \left. - \boldsymbol{\xi}_\perp\cdot\left[\frac{\boldsymbol{j}_\parallel\times\delta\boldsymbol{B}}{c} + 2\boldsymbol{\kappa}_\mathrm{c}(\boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}P)\right]\right\}, \end{aligned} \tag{11}$$

where the magnetic perturbation $\delta\boldsymbol{B} = \boldsymbol{\nabla}\times(\boldsymbol{\xi}\times\boldsymbol{B})$, the magnetic curvature $\boldsymbol{\kappa}_\mathrm{c} = \hat{\boldsymbol{b}}\cdot\boldsymbol{\nabla}\hat{\boldsymbol{b}}$, and the parallel current density $\boldsymbol{j}_\parallel = \hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\cdot(c/4\pi)\boldsymbol{\nabla}\times\boldsymbol{B}$; the magnetic unit vector $\hat{\boldsymbol{b}} \doteq \boldsymbol{B}/B$. The first line of (11) contains terms arising from perturbations to the surface of the plasma; these are usually complicated terms that depend mostly on engineering choices, and will be dropped for the remainder of the problem with little consequence for the instabilities of interest here. The second line contains non-negative terms responsible for stabilizing the plasma. The third line contains sign-indefinite terms that can either stabilize or destabilize the plasma, depending on the equilibrium profiles. You can (and should) read further about δW_2 in §VI.10 of the lecture notes.

(a) How do you know that any instability found by examining (11) will give rise to purely growing modes, rather than to growing oscillations? (A single sentence will do.)

(b) Using words and/or diagrams, explain in physical terms and/or illustrate how each of the terms on the second line of (11) contribute to the plasma's stability. It might help to note that

$$\boldsymbol{\xi}_\perp \boldsymbol{\cdot} \boldsymbol{\kappa}_{\rm c} = \frac{4\pi}{B^2}\,\boldsymbol{\xi}_\perp \boldsymbol{\cdot} \boldsymbol{\nabla}\left(P + \frac{B^2}{8\pi}\right)$$

in magnetohydrostatic equilibrium.

A brief educational aside before proceeding to part (c). *You are not expected to reproduce any of the calculations in this aside.* Consider an equilibrium screw pinch in (R, φ, z) coordinates whose magnetic field satisfies

$$\boldsymbol{B} = B_\varphi(R)\,\hat{\boldsymbol{\varphi}} + B_z(R)\,\hat{\boldsymbol{z}}.$$

In this configuration, the parallel current, magnetic curvature, and pressure gradient are given, respectively, by

$$\boldsymbol{j}_\parallel = \frac{c}{4\pi}\left[\frac{b_z}{R}\frac{{\rm d}(RB_\varphi)}{{\rm d}R} - b_\varphi\frac{{\rm d}B_z}{{\rm d}R}\right]\hat{\boldsymbol{b}}, \quad \boldsymbol{\kappa}_{\rm c} = -\frac{b_\varphi^2}{R}\hat{\boldsymbol{R}}, \quad \boldsymbol{\nabla}P = -\left[\frac{\rm d}{{\rm d}R}\frac{B_z^2}{8\pi} + \frac{B_\varphi}{4\pi R}\frac{{\rm d}(RB_\varphi)}{{\rm d}R}\right]\hat{\boldsymbol{R}}.$$

Inserting these into (11), writing $\boldsymbol{\xi} \to \sum_{m,k_z} \xi(R)\exp(-{\rm i}m\varphi + {\rm i}k_z z)$, and optimizing to select out the least stable perturbations (see equations (VI.12.17)–(VI.12.22) in the lecture notes) leads to perturbations that satisfy $\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\xi} = 0$ and

$$\delta W_2 = \sum_{m,k_z} \pi L_z \int_0^\infty {\rm d}R\,R\rho \left\{ v_{\rm A}^2 \left| \frac{k_\parallel}{kR}\frac{{\rm d}(R\xi_R)}{{\rm d}R} + \frac{2b_\varphi m}{kR}\frac{\xi_R}{R}\right|^2 + \left[k_\parallel^2 v_{\rm A}^2 - \frac{2v_{{\rm A}\varphi}^2}{R^2}\left(1 + \frac{{\rm d}\ln B_\varphi}{{\rm d}\ln R}\right)\right]|\xi_R|^2 \right\}, \tag{12}$$

where $k_\parallel = (m/R)b_\varphi + k_z b_z$ and $k^2 \doteq k_\parallel^2 + (m/R)^2$. The term on the first line here is non-negative and thus stabilizing; the term on the second line can be positive or negative, depending on $k_\parallel^2 R^2$ and the equilibrium profiles. The key point for what follows is that modes with $k_\parallel = 0$ (so-called *flute modes*), which involve no bending of the magnetic-field lines, satisfy

$$\delta W_2 = \sum_{m,k_z} \pi L_z \int_0^\infty {\rm d}R\,R\rho \left[\frac{2v_{{\rm A}\varphi}^2}{R^2}\left(1 - \frac{{\rm d}\ln B_\varphi}{{\rm d}\ln R}\right)|\xi_R|^2\right].$$

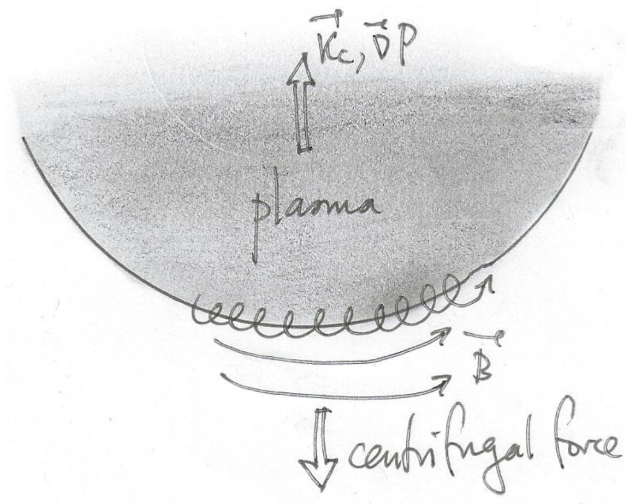
(Take a look back at Problem 4(c) in HW03 and you'll see something familiar.) In this case, the curvature drive

$$-\boldsymbol{\xi}_\perp \boldsymbol{\cdot} 2\boldsymbol{\kappa}_{\rm c}(\boldsymbol{\xi}_\perp \boldsymbol{\cdot} \boldsymbol{\nabla}P) = \frac{2b_\varphi^2}{R}\frac{{\rm d}P}{{\rm d}R}\,\xi_R^2$$

competes with the stabilizing advective term

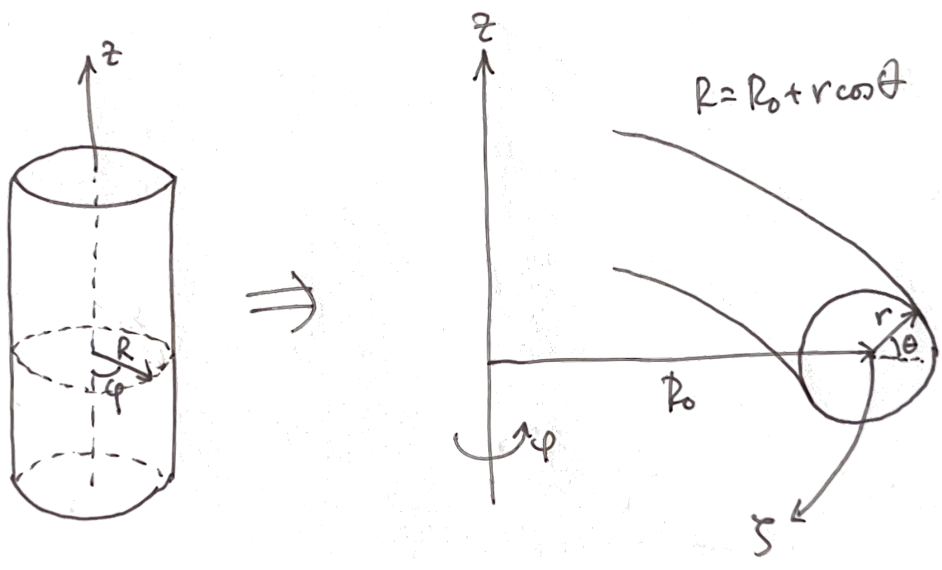
$$\frac{B^2}{4\pi}(2\boldsymbol{\xi}_\perp \boldsymbol{\cdot} \boldsymbol{\kappa}_{\rm c})^2 = \frac{4b_\varphi^2}{R}\frac{B_\varphi^2}{4\pi R}\,\xi_R^2$$

to determine stability. Namely, when the magnetic curvature vector points in the same direction as the pressure gradient (typically inwards), so-called *bad curvature* results and interchange modes are possible. It is then instructive to think of magnetic curvature as leading to a centrifugal force on the field-line-following particles that mimics gravity:



The analogy with the interchange instability investigated in Problem 1 should be apparent. Of course, magnetic tension can stabilize these modes if $k_\parallel \neq 0$ everywhere – thus, the $k_\parallel^2 v_\mathrm{A}^2$ term inside the brackets on the second line of (12). But what about those rational surfaces. . .

(c) Now that you're educated, let us follow up on the latter half of Problem 4 of HW03. Take this screw pinch and wrap it into a periodic, large-aspect-ratio tokamak with circular, concentric flux surfaces:



As a reminder, the equilibrium magnetic field and the safety factor in toroidal (r, θ, ζ) coordinates are

$$\boldsymbol{B} \simeq B_\theta(r)\hat{\boldsymbol{\theta}} + B_\zeta(R_0)\left(1 - \frac{r\cos\theta}{R_0}\right)\hat{\boldsymbol{\zeta}} \qquad \text{and} \qquad q(r) \doteq \frac{r}{R_0}\frac{B_\zeta(R_0)}{B_\theta(r)},$$

where $R = R_0 + r\cos\theta$ and $r/R_0 \sim B_\theta(r)/B_\zeta(R_0) \sim \epsilon \ll 1$. With $P = P(r)$ being a flux function satisfying $\mathrm{d}P/\mathrm{d}r < 0$, calculate the leading-order contribution to the curvature drive

$$-\boldsymbol{\xi}_\perp \boldsymbol{\cdot} 2\boldsymbol{\kappa}_\mathrm{c}(\boldsymbol{\xi}_\perp \boldsymbol{\cdot} \boldsymbol{\nabla} P)$$

in this geometry. Identify on a sketched cross-section of the tokamak the regions of good- and bad-curvature. Hint: $\hat{\boldsymbol{R}} = \hat{\boldsymbol{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$.

(d) In a tokamak, magnetic-field lines pass through regions of both good and bad curvature as they wrap around the torus. As a result, perturbations need not be uniform along the field line, and magnetic tension can provide stabilization via $k_\parallel^2 v_\mathrm{A}^2$. In this sense, the twist in the magnetic field can help stabilize curvature-driven instabilities by forcing perturbations to vary along the field line, thereby introducing magnetic tension. By comparing the curvature drive with the stabilizing magnetic-tension term $\rho k_\parallel^2 v_\mathrm{A}^2 \xi_r^2$, obtain an approximate threshold for interchange instability. Take $k_\parallel \sim (qR_0)^{-1}$.

(e) In a large-aspect-ratio tokamak, the parallel wavenumber $k_\parallel$ that results from applying $\hat{\boldsymbol{b}} \boldsymbol{\cdot} \boldsymbol{\nabla}$ to the displacement $\xi(r,\theta,\zeta) = \sum_{m,n} \xi_{m,n}(r)\exp(-\mathrm{i}m\theta + \mathrm{i}n\zeta)$ is a function of minor radius through the safety factor (see HW03):

$$k_\parallel(r) \approx \frac{1}{R_0}\left(n - \frac{m}{q(r)}\right),$$

where m and n are the poloidal and toroidal mode numbers. If $q(r) = \text{const}$, one can have global interchange modes with $k_\parallel = 0$ (as in Problem 1). But if there is magnetic shear ($\mathrm{d}q/\mathrm{d}r \neq 0$), one can have $k_\parallel = 0$ only at isolated radii where $q(r) = m/n$ – so-called rational (or resonant) surfaces. Argue that magnetic tension is weak near these surfaces and interchange-like perturbations can develop there.

(f) Since the leading-order expression for $k_\parallel$ predicts vanishing tension at rational surfaces, we must retain next-order geometric effects. Using $R = R_0 + r\cos\theta$ and expanding to first order in ϵ, show that $k_\parallel(r)$ in part (e) is actually the leading-order piece of

$$k_\parallel(r,\theta) = \frac{1}{R_0}\left(n - \frac{m}{q(r)}\right) - \frac{nr}{R_0^2}\cos\theta + \mathcal{O}(\epsilon^2).$$

Hence, the parallel structure of a perturbation can vary even at fixed r. Expand this expression for $k_\parallel(r,\theta)$ about a resonant surface at $r = r_0$ by writing $q(r) \simeq q(r_0) + q'(r_0)(r - r_0)$. Determine where the magnetic tension is smallest. Can a perturbation confined to a single flux tube stay in the minimum-tension region everywhere along the field? How do you think this affects interchange modes? Given your answer, where do you expect curvature-driven perturbations to localize in r and θ?

Such localized curvature-driven perturbations are called *ideal ballooning modes*. Their behavior is governed by the interplay between magnetic shear $s \doteq \mathrm{d}\ln q/\mathrm{d}\ln r$ and pressure–curvature drive $\alpha \doteq -q^2\beta R(\mathrm{d}\ln P/\mathrm{d}r)$, both of which are key parameters in the so-called "s-α stability diagram" for ballooning modes.