AST 521 Fall 2025 **Homework #6**

Due Friday, December 12, 2025

Generals prep. Make sure you can provide brief definitions of the following terms: magnetic moment, Landau resonance, Landau damping, Barnes (or transit-time) damping, pressure anisotropy.

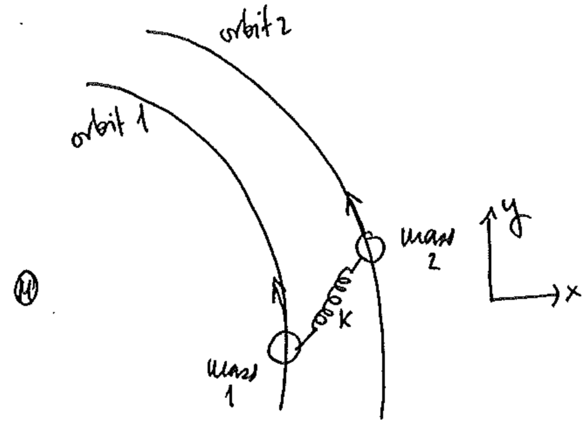
Nota bene! This problem set can be done in two parts. After today (Nov 24), you should be able to do parts (a)–(f). After Dec 3, you should be able to do parts (g)–(j). Don't wait until Dec ∼8 to start! Feel free to check your work with me along the way.

1. **Kinetic MRI, four ways**. The acknowledgement at the end of Balbus & Hawley (1992a) reads, "It is fitting and proper to acknowledge Alar Toomre for this important insight that the Hill equations had something to contribute to the MHD stability problem." This insight is what led S. Balbus and J. Hawley to develop the now-famous spring model of the MRI, which was then used to conjecture that the Oort A-value is the universal growth rate limit for accretion-disk shear instabilities. The Hill equations describe local disk dynamics in a rotating frame – *local* in that they describe small excursions $x \doteq R - R_0$ and $y \doteq R_0(\phi - \Omega_0 t)$ from a circular orbit $R = R_0$, $\phi = \Omega_0 t$. They are given by

$$\ddot{x} - 2\Omega_0 \dot{y} = -4A_0\Omega_0 x + f_x, \tag{1a}$$

$$\ddot{y} + 2\Omega_0 \dot{x} = f_y, \tag{1b}$$

where the overdot indicates a time derivative and f_x and f_y represent local forces in the x and y directions. The Oort A-value $A_0 = -(3/4)\Omega_0$ for Keplerian rotation.



The MRI analogy goes as follows. Consider the local force to be nondissipative and to act by restoring a displacement back to its equilibrium position. The leading-order contribution to f_x and f_y in a Taylor expansion about $(R_0, \Omega_0 t)$ is linear; for an *isotropic* force, we have $f_x = -Kx$ and $f_y = -Ky$, where $K > 0$ is some constant. (You could also profitably think of this force as being due to an ideal spring with spring constant K.) Then (1) becomes

$$\ddot{x} - 2\Omega_0 \dot{y} = -4A_0\Omega_0 x - Kx, \tag{2a}$$

$$\ddot{y} + 2\Omega_0 \dot{x} = -Ky. \tag{2b}$$

These equations have the solutions $x, y \propto \exp(\pm\mathrm{i}\omega t)$ with a frequency ω that satisfies

$$\omega^4 - \omega^2\big(\kappa^2 + 2K\big) + K\big(K + 4A_0\Omega_0\big) = 0, \tag{3}$$

where $\kappa^2 \doteq 4\Omega_0^2(1 + A_0/\Omega_0)$ is the square of the epicyclic frequency, which is positive for Keplerian rotation. Equation (3) should look familiar: replace K with $(\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{v}_{\mathrm{A}})^2$ and you get the axisymmetric MRI dispersion relation. But why assume that our spring-like force is isotropic? Let us explore what happens if we relax this assumption.

(a) Read S. A. Balbus and J. F. Hawley, *Astrophys. J.* **392**, 662 (1992). It's short. They "conjecture that the Oort A-value is an upper bound to the growth rate of any instability feeding upon the free energy of differential rotation." En route, they show that the maximum growth rate of the MRI (the Oort-A value) occurs at $K_{\max}/\Omega_0^2 = 1-(\kappa/2\Omega_0)^4$ and that the corresponding eigenvector satisfies $y/x = -1$, i.e., radial and azimuthal displacements are equal in size. Remember this last point as you proceed.

(b) Let $f_x = -K_x x$ and $f_y = -K_y y$ in (1), with $K_x \neq K_y$ being positive constants. Compute the new dispersion relation governing the time-evolution of small displacements. Is the growth rate larger or smaller than the Oort-A value for $K_x > K_y$? for $K_x < K_y$? From this result, find the maximum growth rate $\gamma_{\max}$ and the (hint: asymptotic) values of K_x and K_y at which $\gamma_{\max}$ is achieved. (It may help you to make a contour plot of the growth rate in the K_x–K_y plane using your dispersion relation.)

(c) It is not difficult to show – see equation (VII.3.7) of the Kunz lecture notes – that the Lagrangian change in the rotation frequency of a fluid element as it is displaced a distance (x, y) about the point $(R_0, \Omega_0 t)$ is given by $\Delta\Omega = \dot{y}/R_0$. Using your results from part (b), calculate y/x and show that the Lagrangian change in a fluid element's specific angular momentum $\ell = \Omega R^2$ is given by

$$\frac{\Delta\ell}{\ell_0} = \frac{2K_y}{K_y - \omega^2}\frac{x}{R_0}. \tag{4}$$

(In other words, if you displace a fluid element outwards by a distance $\Delta R = x$, its angular momentum would change by an amount $\Delta\ell$.) If you were in charge of these springs, what would you do to ensure that outwardly displaced fluid elements ($x > 0$) continue to move outwards? Is the resulting $|y/x|$ small, order unity (as in the standard MRI), or large? Use these facts to explain what is going on physically, in light of the discussion in §2.4 of BH92.

Toy models are nice, but let's do the real thing. The momentum equation in the shearing sheet for a magnetized plasma is

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} \doteq \left(\frac{\partial}{\partial t} + \boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{\nabla}\right)\boldsymbol{u} = -\frac{1}{\rho}\boldsymbol{\nabla}\boldsymbol{\cdot}\mathsf{P} + \frac{\boldsymbol{j}\times\boldsymbol{B}}{c\rho} - 2\Omega_0\hat{\boldsymbol{z}}\times\boldsymbol{u} - 4A_0\Omega_0 x\hat{\boldsymbol{x}}, \tag{5}$$

where $\boldsymbol{u}$ is the flow velocity, ρ is the mass density, $\boldsymbol{B}$ is the magnetic field, $\boldsymbol{j} = (c/4\pi)\boldsymbol{\nabla}\times\boldsymbol{B}$ is the current density, and the pressure tensor

$$\mathsf{P} = p_\perp\big(\mathsf{I} - \hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\big) + p_\parallel\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} \tag{6}$$

is anisotropic with respect to the magnetic-field direction $\hat{\boldsymbol{b}} = \boldsymbol{B}/B$. (Remember that $\mathsf{P} = \sum_\alpha \mathsf{P}_\alpha$ includes the thermal pressure of both the ions and the electrons!) Let's go through the same linear shearing-sheet MRI calculation as in HW04 #3, but this time using the ideal MHD induction equation and this anisotropic pressure tensor. With $\boldsymbol{B} = B_0\hat{\boldsymbol{z}}$ and $p_{\perp 0} = p_{\parallel 0} = p_0$ in the uniform equilibrium state (which has $\boldsymbol{u}_0 = 2A_0 x\hat{\boldsymbol{y}}$), and assuming $\boldsymbol{k} = k\hat{\boldsymbol{z}}$ for the fluctuations, the linearized version of (5) can be written in the following form:

$$\frac{\mathrm{i}kB_0}{4\pi\rho}\begin{pmatrix}\delta B_x\\ \delta B_y\end{pmatrix} = \begin{pmatrix}-\mathrm{i}\omega & -2\Omega_0\\ 2\Omega_0 + 2A_0 & -\mathrm{i}\omega\end{pmatrix}\begin{pmatrix}\delta u_x\\ \delta u_y\end{pmatrix}. \tag{7}$$

It's the same as before! Pressure fluctuations play no role in the MRI in this case, so it doesn't matter whether they're anisotropic or not. To get something different, the wavenumber must have a non-vertical component *or* the equilibrium magnetic field must have a non-vertical component. The latter case is a bit simpler to calculate and to interpret physically. It's also more interesting, so...

Let $\boldsymbol{B}_0 = B_{0y}\hat{\boldsymbol{y}} + B_{0z}\hat{\boldsymbol{z}}$.

(d) Show that the linearized momentum equation is now

$$\frac{\mathrm{i}kB_{0z}}{4\pi\rho}\begin{pmatrix}\delta B_x\\ \delta B_y\end{pmatrix} = \begin{pmatrix}-\mathrm{i}\omega & -2\Omega_0\\ 2\Omega_0 + 2A_0 & -\mathrm{i}\omega\end{pmatrix}\begin{pmatrix}\delta u_x\\ \delta u_y\end{pmatrix} - \frac{\mathrm{i}k}{\rho}\frac{B_{0y}B_{0z}}{B_0^2}\begin{pmatrix}0\\ \delta p_\perp - \delta p_\parallel\end{pmatrix}. \tag{8}$$

To go any further, we need a closure for the pressure anisotropy.

(e) In a collisionless plasma that evolves double-adiabatically, we have $p_\perp/\rho B = \text{const}$ and $p_\parallel B^2/\rho^3 = \text{const}$ (Chew, Goldberger & Low 1956). Assume that these relations hold for the perturbed ions, but that the electrons remain isotropic as they are perturbed (i.e., $\delta p_{\perp\mathrm{e}} = \delta p_{\parallel\mathrm{e}}$). The MRI is approximately incompressible for $\beta \gg 1$, so you may *ignore density fluctuations.* Plug the resulting expression for the pressure anisotropy into (8), use the perturbed ideal induction equation

$$\mathrm{i}kB_{0z}\begin{pmatrix}\delta u_x\\ \delta u_y\end{pmatrix} = \begin{pmatrix}-\mathrm{i}\omega & 0\\ -2A_0 & -\mathrm{i}\omega\end{pmatrix}\begin{pmatrix}\delta B_x\\ \delta B_y\end{pmatrix} \tag{9}$$

to eliminate velocities in favor of magnetic-field fluctuations, and calculate the dispersion relation. (A request on notation: write $p_{0\mathrm{i}}/\rho \simeq T_{0\mathrm{i}}/m_\mathrm{i} \doteq v_{\mathrm{thi}}^2/2$.) By comparing this dispersion relation with the one found in part (b), identify K_x and K_y. Show that the maximum growth rate $\gamma_{\max}$ found in part (b) is the same as for the dispersion relation found here and explain how it can be achieved. For these values, is $|\delta B_y/\delta B_x|$ small, order unity, or large?

(f) In a weakly collisional plasma, the pressure anisotropy represents a balance between double-adiabatic production and collisional relaxation (Braginskii 1965):

$$\frac{p_\perp - p_\parallel}{p} = \frac{1}{\nu}\frac{\mathrm{D}}{\mathrm{D}t}\ln\frac{B^3}{\rho^2}. \tag{10}$$

Because the collision frequency ν of the electrons is a factor $\sim(m_\mathrm{i}/m_\mathrm{e})^{1/2} \gg 1$ larger than that of the ions, you may consider the pressure anisotropy to be dominated by its ion contribution; thus, in (10), p is p_i and ν is ν_i. Now use this closure to determine $\delta p_\perp - \delta p_\parallel$, plug the result into (8), and calculate the resulting dispersion relation. Again, you may *neglect density fluctuations*. By comparing this dispersion relation with the one found in part (b), identify K_x and K_y. Show that the maximum growth rate γ_max found in part (b) is the same as for the dispersion relation found here and explain how it can be achieved. For these values, is $|\delta B_y/\delta B_x|$ small, order unity, or large? (Hint: it might help to write the dispersion relation with all the usual MRI terms from ideal MHD on the left-hand side, and all the new terms related to the Braginskii viscosity on the right-hand side. Then it's easier to take limits like $\Omega_0/\nu_\mathrm{i} \gg 1$ or $\ll 1$.)

Now, let's return to a collisionless plasma, but *without* the assumption of double-adiabaticity. Instead, we will adopt a drift-kinetic approach (Kulsrud 1964, 1983) to calculate the pressure anisotropy directly – with no restrictions other than that the frequencies of interest are much smaller than the Larmor frequency and the lengthscales of interest are much larger than the Larmor radius. I'll walk you through the process. If you're in a rush, you can skip directly to (17), but it would benefit you to work it out yourself alongside the following tutorial.

The drift-kinetic equation, tailored for the shearing sheet, is

$$\frac{\mathrm{d}f_\alpha}{\mathrm{d}t_\alpha} + \frac{\mathrm{d}\ln B}{\mathrm{d}t_\alpha}\frac{w_\perp}{2}\frac{\partial f_\alpha}{\partial w_\perp} + \left[\frac{q_\alpha E_\parallel}{m_\alpha} - \hat{\boldsymbol{b}}\boldsymbol{\cdot}\left(\frac{\mathrm{d}\boldsymbol{u}_\alpha}{\mathrm{d}t_\alpha} + 2\Omega_0\hat{\boldsymbol{z}}\times\boldsymbol{u}_\alpha + 4A_0\Omega_0 x\hat{\boldsymbol{x}}\right) - \frac{w_\perp^2}{2}\hat{\boldsymbol{b}}\boldsymbol{\cdot}\boldsymbol{\nabla}\ln B\right]\frac{\partial f_\alpha}{\partial w_\parallel} = 0, \quad (11)$$

where $\boldsymbol{w} \doteq \boldsymbol{v} - \boldsymbol{u}_\alpha(t,\boldsymbol{r})$ is the particle velocity peculiar to the mean ("fluid") velocity of species α, $\boldsymbol{u}_\alpha(t,\boldsymbol{r})$; $\hat{\boldsymbol{b}} \doteq \boldsymbol{B}/B$ is the unit vector in the direction of the magnetic field; $E_\parallel$ is the parallel component of the electric field (determined by enforcing quasi-neutrality); and the Lagrangian time derivative

$$\frac{\mathrm{d}}{\mathrm{d}t_\alpha} \doteq \frac{\partial}{\partial t} + \boldsymbol{u}_\alpha\boldsymbol{\cdot}\boldsymbol{\nabla} + w_\parallel\hat{\boldsymbol{b}}\boldsymbol{\cdot}\boldsymbol{\nabla} \quad (12)$$

accounts for advection by the mean velocity of species s and the parallel streaming of particles along magnetic-field lines. The colored terms are as follows. The orange term represents conservation of the magnetic moment – changes in magnetic-field strength as measured in the frame of the guiding center adiabatically induce changes in the perpendicular energy of the particles. Indeed, this term may be rewritten as $\mu_\alpha(\mathrm{d}B/\mathrm{d}t_\alpha)(\partial f_\alpha/\partial\varepsilon_{\alpha\perp})$, where $\mu_\alpha \doteq m_\alpha w_\perp^2/2B$ and $\varepsilon_{\alpha\perp} \doteq m_\alpha w_\perp^2/2$. The blue term represents the parallel acceleration of guiding centers by the electric field. The purple terms represent the parallel components of several inertial terms due to the acceleration of the fluid element on which the reference frame is anchored. Note the appearance of the Coriolis and tidal forces. Finally, the red term captures the acceleration/deceleration of guiding centers along field lines as they encounter regions of decreasing/increasing magnetic-field strength. This, again, is due to μ_α conservation.

Now, write $f_\alpha = f_{0\alpha}(w_\parallel, w_\perp) + \delta f_\alpha$, $\boldsymbol{B} = \boldsymbol{B}_0 + \delta\boldsymbol{B}$, and $\boldsymbol{u}_\alpha = 2A_0 x\hat{\boldsymbol{y}} + \delta\boldsymbol{u}_\alpha$, and assume a space-time dependence for the fluctuations $\propto \exp(\mathrm{i}kz - \mathrm{i}\omega t)$ with $k > 0$. One can show from

(11) that the perturbed distribution function then satisfies

$$\delta f_\alpha = -\frac{\delta B_\parallel}{B_0}\frac{w_\perp}{2}\frac{\partial f_{0\alpha}}{\partial w_\perp} + \delta u_{\parallel\alpha}\frac{\partial f_{0\alpha}}{\partial w_\parallel} + \frac{\mathrm{i}}{k_\parallel}\left[\frac{q_\alpha E_\parallel}{m_\alpha} - \frac{\kappa^2}{2\Omega_0} b_{0y}\delta u_{x\alpha} - 2A_0 b_{0y} w_\parallel \frac{\delta B_x}{B_0} - \frac{w_\perp^2}{2}\frac{\mathrm{i}k_\parallel \delta B_\parallel}{B_0}\right]\frac{\partial f_{0\alpha}/\partial w_\parallel}{w_\parallel - \omega/k_\parallel}, \tag{13}$$

where $k_\parallel = k b_{0z}$ is the field-parallel wavenumber and $\delta u_\parallel = \hat{\boldsymbol{b}}_0 \boldsymbol{\cdot} \delta\boldsymbol{u} = b_{0y}\delta u_y + b_{0z}\delta u_z$ is the field-parallel flow velocity. Because $\boldsymbol{k}\boldsymbol{\cdot}\delta\boldsymbol{B} = 0$ and $\boldsymbol{k} = k\hat{\boldsymbol{z}}$, we have $\delta B_z = 0$, and so the perturbed magnetic-field strength is simply $\delta B_\parallel = b_{0y}\delta B_y$.

Using (13), the perturbed perpendicular and parallel pressures are computed via

$$\delta p_{\perp\alpha} = m_\alpha \int \mathrm{d}^3\boldsymbol{w}\, \frac{w_\perp^2}{2}\,\delta f_\alpha \qquad \text{and} \qquad \delta p_{\parallel\alpha} = m_\alpha \int \mathrm{d}^3\boldsymbol{w}\, w_\parallel^2\,\delta f_\alpha, \tag{14}$$

where $\mathrm{d}^3\boldsymbol{w} = 2\pi w_\perp \mathrm{d}w_\perp \mathrm{d}w_\parallel$. Note the appearance of the Landau resonance in the final term of (13). In particular, when $f_{0\alpha}$ is an isotropic Maxwellian distribution function with temperature $T_{0\alpha} \doteq m_\alpha v_{\mathrm{th}\alpha}^2/2$, the integral

$$\begin{aligned}
\int \mathrm{d}^3\boldsymbol{w}\, \frac{\partial f_{0\alpha}/\partial w_\parallel}{w_\parallel - \omega/k_\parallel} &= -\frac{2}{v_{\mathrm{th}\alpha}^2}\int \mathrm{d}^3\boldsymbol{w}\, \frac{w_\parallel f_{0\alpha}}{w_\parallel - \omega/k_\parallel} \\
&= -\frac{2}{v_{\mathrm{th}\alpha}^2}\left[\int \mathrm{d}^3\boldsymbol{w}\, f_{0\alpha} + \frac{\omega}{k_\parallel}\int \mathrm{d}^3\boldsymbol{w}\, \frac{f_{0\alpha}}{w_\parallel - \omega/k_\parallel}\right] \\
&= -\frac{2n_{0\alpha}}{v_{\mathrm{th}\alpha}^2}\left[1 + \zeta_\alpha \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \mathrm{d}x\, \frac{\mathrm{e}^{-x^2}}{x - \zeta_\alpha}\right],
\end{aligned}$$

where $\zeta_\alpha \doteq \omega/k_\parallel v_{\mathrm{th}\alpha}$ is the dimensionless phase speed and I've set $x = w_\parallel/v_{\mathrm{th}\alpha}$ in cleaning up the final integral. As you know from class, this integral has a name:

$$Z(\zeta) \doteq \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \mathrm{d}x\, \frac{\mathrm{e}^{-x^2}}{x - \zeta} \tag{15}$$

is the plasma dispersion function. So,

$$\int \mathrm{d}^3\boldsymbol{w}\, \frac{\partial f_{0\alpha}/\partial w_\parallel}{w_\parallel - \omega/k_\parallel} = -\frac{2n_{0\alpha}}{v_{\mathrm{th}\alpha}^2}\big[1 + \zeta_\alpha Z(\zeta_\alpha)\big]. \tag{16}$$

While we're paused, here are a few handy integrals involving Z:

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \mathrm{d}x\, \frac{x\mathrm{e}^{-x^2}}{x - \zeta} = 1 + \zeta Z(\zeta), \qquad \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \mathrm{d}x\, \frac{x^2\mathrm{e}^{-x^2}}{x - \zeta} = \zeta\big[1 + \zeta Z(\zeta)\big],$$

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \mathrm{d}x\, \frac{x^3\mathrm{e}^{-x^2}}{x - \zeta} = \frac{1}{2} + \zeta^2\big[1 + \zeta Z(\zeta)\big].$$

You can – and should, for parts (g) and (h) below – familiarize yourself with the asymptotic forms of the Z function, e.g., in the NRL plasma formulary.

Okay. Returning to (13) and (14), after much work the result may be written as

$$\frac{\delta p_{\perp\alpha}}{p_{0\alpha}} = \frac{\delta n_\alpha}{n_{0\alpha}} + D_{1\alpha}\frac{\delta B_\|}{B_0} \quad \text{with} \quad D_{1\alpha} \doteq -\zeta_\alpha Z(\zeta_\alpha); \tag{17a}$$

$$\frac{\delta p_{\|\alpha}}{p_{0\alpha}} = \frac{\delta n_\alpha}{n_{0\alpha}} + D_{2\alpha}\left(\frac{\delta n_\alpha}{n_{0\alpha}} - \frac{\delta B_\|}{B_0}\right) \quad \text{with} \quad D_{2\alpha} \doteq 2\zeta_\alpha^2 - \frac{\zeta_\alpha Z(\zeta_\alpha)}{1+\zeta_\alpha Z(\zeta_\alpha)}. \tag{17b}$$

These equations provide the equation of state of the drift-kinetic plasma.

(g) First, show that $\zeta_\alpha \gg 1$ implies $D_{1\alpha} \approx 1$ and $D_{2\alpha} \approx 2$, and thus

$$\frac{\delta p_{\perp\alpha}}{p_{0\alpha}} \approx \frac{\delta n_\alpha}{n_{0\alpha}} + \frac{\delta B_\|}{B_0} \quad \text{and} \quad \frac{\delta p_{\|\alpha}}{p_{0\alpha}} \approx 3\frac{\delta n_\alpha}{n_{0\alpha}} - 2\frac{\delta B_\|}{B_0}. \tag{18}$$

These should look familiar from part (e). What are they? Why (physically) should these relations emerge from the $\zeta_\alpha \gg 1$ limit?

(h) Second, show that $\zeta_\alpha \ll 1$ implies $D_{1\alpha} \approx D_{2\alpha} \approx -\mathrm{i}\sqrt{\pi}\zeta_\alpha$, and thus

$$\frac{\delta p_{\perp\alpha}}{p_{0\alpha}} \approx \frac{\delta n_\alpha}{n_{0\alpha}} - \mathrm{i}\sqrt{\pi}\zeta_\alpha\frac{\delta B_\|}{B_0} \quad \text{and} \quad \frac{\delta p_{\|\alpha}}{p_{0\alpha}} \approx \frac{\delta n_\alpha}{n_{0\alpha}} - \mathrm{i}\sqrt{\pi}\zeta_s\left(\frac{\delta n_\alpha}{n_{0\alpha}} - \frac{\delta B_\|}{B_0}\right). \tag{19}$$

Note that these are nearly isothermal fluctuations! Why (physically) should these relations emerge from the $\zeta_\alpha \ll 1$ limit?

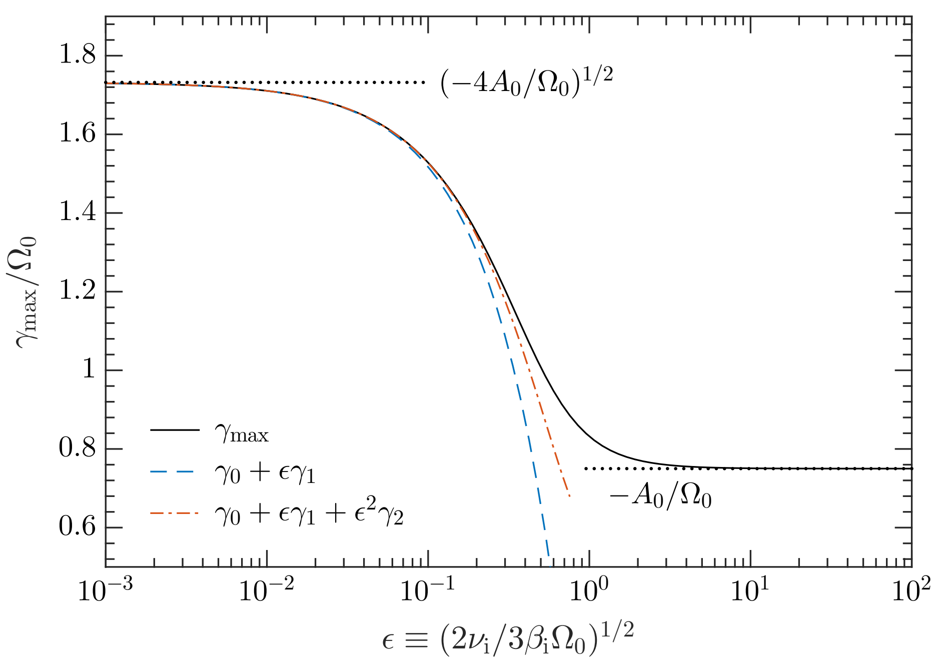
(i) Equations (19) are the appropriate limit for the MRI, since the MRI has $\omega \lesssim k_\| v_{\mathrm{A}} = k_\| v_{\mathrm{thi}}/\beta_{\mathrm{i}}^{1/2} \ll k_\| v_{\mathrm{thi}}$ for $\beta_{\mathrm{i}} \gg 1$. Neglecting density fluctuations[1] and assuming $\delta p_{\perp\mathrm{e}} = \delta p_{\|\mathrm{e}}$ (as above), plug the resulting pressure anisotropy into equation (8), use the linearized induction equation (9), and compute the dispersion relation of the *drift-kinetic MRI*. Using $\zeta_{\mathrm{i}} \sim \beta_{\mathrm{i}}^{-1/2} \ll 1$, find the maximum growth rate and compare this to your answers from parts (e) and (f).

(j) Earlier in this homework, you showed that the MRI in a magnetized, weakly collisional (i.e., Braginskii) plasma can grow faster than the Oort A-value, and that this property derives from certain components of the momentum (those that produce $\delta B_\| \neq 0$) being viscously damped in an advantageous way. Here, there are no collisions at all, and yet we find a similar result. So, without collisions, what is responsible for damping the motions that generate $\delta B_\|$? In the Braginskii case, the kinetic energy stored in those damped motions get transferred into heat via collisional dissipation. In the collisionless case, where does the damped energy go?

[1] This is a good approximation when $\beta_{\mathrm{i}} \gg 1$. See next page.

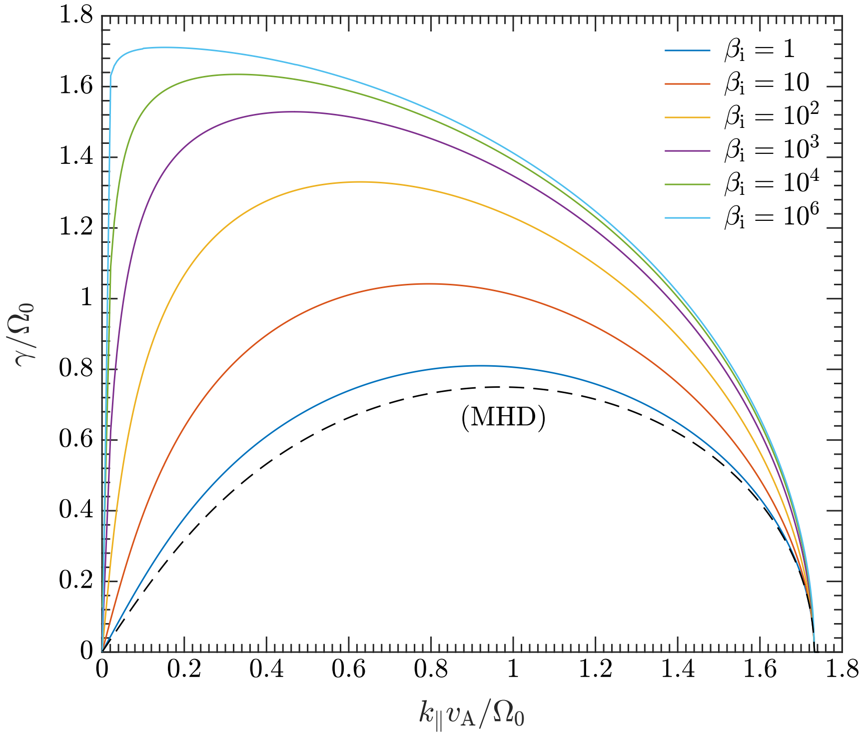
If you're bored and would like to play with some mathematics, here are two **optional** calculations that you can try.

1. Introduce $\epsilon^2 \doteq (2/3)(v_{\rm A}^2/v_{\rm thi}^2)(\nu_{\rm i}/\Omega_0) \ll 1$ and write the dispersion relation found in part (f) as a function of it and the dimensionless parameters $s \doteq -{\rm i}\omega/\Omega_0$ and $X \doteq k^2 v_{{\rm A}z}^2/\Omega_0^2$. By differentiating this dispersion relation with respect to X, find an expression for the location of the maximum growth rate $X_{\rm max}$ and thereby an equation for the maximum growth rate $s_{\rm max}$ itself. You won't be able to obtain an exact analytic solution from this, but you can get a pretty accurate asymptotic solution, as follows. Write $s_{\rm max} = s_0 + \epsilon s_1 + \epsilon^2 s_2 + \dots$ and collect orders of ϵ in your equation for $s_{\rm max}$ to solve for s_0 and its first-order correction s_1. (If you're ambitious, you can try to obtain the second-order correction s_2; the result is surprisingly simple.) Use these to calculate the leading-order solutions for $X_{\rm max}$ and $\delta B_y/\delta B_x$; this is actually a rigorous way of checking your answers to part (f). When you have finished obtaining those, adopt $B_{0y}/B_{0z} = 1$ and numerically solve the exact dispersion relation for $s_{\rm max}$ as a function of ϵ. Plot $s_{\rm max}$ vs $\log_{10}(\epsilon) \in [10^{-3}, 10^2]$. Overplot the analytic approximations s_0 and $s_0+\epsilon s_1$ (and $s_0+\epsilon s_1+\epsilon^2 s_2$ if you found s_2), as well as the dimensionless maximum MRI growth rate, $-A_0/\Omega_0$. Here's what you should find:



2. Use equations (17) to write down the general form of the ion pressure anisotropy and use *this* in equation (8) to compute the general drift-kinetic MRI dispersion relation. (You may assume $\delta p_{\perp {\rm e}} = \delta p_{\parallel {\rm e}}$.) You can't solve this analytically, so solve it numerically. Because this equation is implicit, you might consider using an iterative solver. I recommend simple Newton–Raphson iteration, with the initial guess for the iterations based on the approximate dispersion relation you found in part (i). You can find various numerical implementations of the plasma dispersion function online. Usually $Z(\zeta) = \sqrt{\pi}\exp(-\zeta^2)[{\rm i} - \texttt{erfi}(\zeta)]$, where `erfi` is the complex error function, works pretty well. Python can calculate $Z(\zeta)$ using `complex(0,1.0)*np.sqrt(np.pi)*sp.wofz(zeta)`. Plot the dimensionless growth rate ${\rm Im}(\omega/\Omega_0)$ vs $k_\parallel v_{\rm A}/\Omega_0$ for $b_{0y} = b_{0z} = 1/\sqrt{2}$ (i.e., a mean magnetic field tilted at 45°) and

$\beta_{\rm i} \doteq v_{\rm thi}^2/v_{\rm A}^2 = 1, 10, 10^2,\ 10^3, 10^4$, and 10^6. Overplot the solution for $B_{0y} = 0$ as a dashed line; this should be independent of $\beta_{\rm i}$ and equivalent to the standard MHD solution. Here's what you should find:



If you retain the density fluctuations in (17) in your calculation, then you'll have to express δn in terms of $\delta B_\parallel$ using the linearized continuity equation and the z-component of the linearized momentum equation. (You may assume $\delta p_{\perp\rm e} = \delta p_{\parallel\rm e} = T_{0\rm e}\,\delta n$.) My advice to keep your calculation tidy is to write $\delta n/n_0 = \chi\,\delta B_\parallel/B_0$ and try to solve for $\chi = \chi(\zeta_{\rm i}, \beta_{\rm i}, T_{0\rm e}/T_{0\rm i})$. The solution plotted above is for $T_{0\rm e}/T_{0\rm i} = 1$. Neglecting the density fluctuations involves considerably less algebra and, besides, is quite accurate for $\beta_{\rm i} \gg 1$.