AST 521 Fall 2025 **Homework #5**

Due Monday, November 24, 2025

Generals prep. Make sure you can provide brief definitions of the following terms: Cowling's and Zel'dovich's anti-dynamo theorems, fluctuation dynamo, Kolmogorov turbulence, Iroshnikov–Kraichnan turbulence, Goldreich–Sridhar turbulence, critical balance.

1. **Compressive fluctuations in RMHD**. In class and in my lecture notes (see §V.4), I went through the derivation of the RMHD equations governing nonlinear Alfvén-wave fluctuations. These equations have been used in a wide variety of contexts, from modeling elongated structures in tokamaks (Kadomtsev & Pogutse 1974; Strauss 1976, 1977), to describing solar-wind turbulence (Zank & Matthaeus 1992*a*,*b*; Bhattacharjee, Ng & Spangler 1998), to performing tearing-mode calculations like the one we did in class. You may have noticed that compressive fluctuations (e.g., slow magnetosonic waves, entropy modes) made no appearance in those equations; in RMHD, Alfvénic turbulence doesn't care whether or not the density is fluctuating. In this problem, we'll ask the converse – whether or not the compressive fluctuations care about the Alfvénic ones. This turns out to be important in the context of Alfvénic turbulence.

Following the lecture, consider a uniform static equilibrium with a straight mean field in the z direction, subject to small-amplitude fluctuations in all quantities:

$$\rho = \rho_0 + \delta\rho, \quad p = p_0 + \delta p, \quad \boldsymbol{u} = \boldsymbol{u}_\perp + u_\parallel \hat{\boldsymbol{z}}, \quad \boldsymbol{B} = B_0\hat{\boldsymbol{z}} + \delta\boldsymbol{B}_\perp + \delta B_\parallel \hat{\boldsymbol{z}}, \tag{1}$$

where

$$\frac{\delta\rho}{\rho_0} \sim \frac{\delta p}{p_0} \sim \frac{u_\perp}{v_\mathrm{A}} \sim \frac{u_\parallel}{v_\mathrm{A}} \sim \frac{\delta B_\perp}{B_0} \sim \frac{\delta B_\parallel}{B_0} \sim \epsilon \doteq \frac{k_\parallel}{k_\perp} \ll 1 \tag{2}$$

and $v_\mathrm{A} \doteq B_0/\sqrt{4\pi\rho_0}$ is the Alfvén speed. The fluctuations are assumed to be anisotropic, having characteristic scales along the mean field ($\|$) much larger than those across it ($\perp$). Note that the ordering (2) means that the Mach number

$$\mathrm{Ma} \sim \frac{u}{c_s} \sim \frac{\epsilon}{\sqrt{\beta}},$$

where $c_\mathrm{s} \doteq (\gamma p_0/\rho_0)^{1/2}$ is the speed of sound and

$$\beta \doteq \frac{8\pi p_0}{B_0^2} = \frac{2}{\gamma}\frac{c_\mathrm{s}^2}{v_\mathrm{A}^2} \sim 1.$$

Subsidiary limits in high and low β can be taken after the ϵ expansion is performed.

We showed in class that the Alfvénic fluctuations $\boldsymbol{u}_\perp$ and $\delta\boldsymbol{B}_\perp$ may be expressed in terms of scalar stream and flux functions,

$$\boldsymbol{u}_\perp = \hat{\boldsymbol{z}} \times \boldsymbol{\nabla}_\perp \Phi \quad \text{and} \quad \frac{\delta\boldsymbol{B}_\perp}{\sqrt{4\pi\rho_0}} = \hat{\boldsymbol{z}} \times \boldsymbol{\nabla}_\perp \Psi, \tag{3}$$

respectively, and that the leading-order equations governing Φ and Ψ are

$$\frac{\partial}{\partial t}\nabla_\perp^2\Phi + \bigl\{\Phi, \nabla_\perp^2\Phi\bigr\} = v_\mathrm{A}\frac{\partial}{\partial z}\nabla_\perp^2\Psi + \bigl\{\Psi, \nabla_\perp^2\Psi\bigr\}, \tag{4}$$

$$\frac{\partial\Psi}{\partial t} + \{\Phi, \Psi\} = v_\mathrm{A}\frac{\partial\Phi}{\partial z}, \tag{5}$$

where $\{\Phi, \Psi\} = \hat{\boldsymbol{z}} \cdot (\boldsymbol{\nabla}_\perp \Phi \times \boldsymbol{\nabla}_\perp \Psi)$ is the Poisson bracket capturing the nonlinear interactions between Φ and Ψ. These equations take into account that, to lowest order,

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \boldsymbol{u}_\perp \cdot \boldsymbol{\nabla}_\perp = \frac{\partial}{\partial t} + \{\Phi, \dots\}, \tag{6}$$

$$\hat{\boldsymbol{b}} \cdot \boldsymbol{\nabla} = \frac{\partial}{\partial z} + \frac{\delta \boldsymbol{B}_\perp}{B_0} \cdot \boldsymbol{\nabla}_\perp = \frac{\partial}{\partial z} + \frac{1}{v_\mathrm{A}}\{\Psi, \dots\}, \tag{7}$$

where $\hat{\boldsymbol{b}} \doteq \boldsymbol{B}/B_0$ is the unit vector along the perturbed field line. We also showed from (4) and (5) that the Elsässer potentials $\zeta^\pm \doteq \Phi \pm \Psi$ satisfy

$$\frac{\partial}{\partial t}\nabla_\perp^2 \zeta^\pm \mp v_\mathrm{A} \frac{\partial}{\partial z}\nabla_\perp^2 \zeta^\pm = -\frac{1}{2}\Big(\{\zeta^+, \nabla_\perp^2 \zeta^-\} + \{\zeta^-, \nabla_\perp^2 \zeta^+\} \mp \nabla_\perp^2 \{\zeta^+, \zeta^-\}\Big). \tag{8}$$

Note that, if either $\zeta^+ = 0$ or $\zeta^- = 0$, all of the nonlinear terms on the right-hand side of this equation vanish, and the non-vanishing Elsässer potential describes an Alfvén-wave packet of arbitrary shape and amplitude propagating along the mean field at the Alfvén speed. Only counterpropagating waves can interact and give rise to an Alfvén-wave cascade (Kraichnan 1965). Also note that these equations say nothing about the ratio of the energies in each of the $\zeta^\pm$ fluctuations, $\varepsilon^\pm$, each of which is independently conserved by (8). This leads to the notion of *imbalanced turbulence*, having $\varepsilon^+ \neq \varepsilon^-$, a situation routinely measured in the fast solar wind (Tu & Marsch 1995; Bruno & Carbone 2005 and references therein).

But enough of that. On to the compressive fluctuations...

(a) Revisit the perpendicular part of the momentum equation

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\boldsymbol{\nabla}\left(p + \frac{B^2}{8\pi}\right) + \frac{\boldsymbol{B} \cdot \boldsymbol{\nabla} \boldsymbol{B}}{4\pi} \tag{9}$$

and use the ordering (2) to show that to lowest order, $\mathcal{O}(1)$, we have pressure balance:

$$\boldsymbol{\nabla}_\perp \left(\delta p + \frac{B_0 \delta B_\parallel}{4\pi}\right) = 0 \quad \Longrightarrow \quad \frac{\delta p}{p_0} = -\gamma \frac{v_\mathrm{A}^2}{c_\mathrm{s}^2}\frac{\delta B_\parallel}{B_0}. \tag{10}$$

Also show that the continuity equation implies

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = -\frac{\mathrm{D}}{\mathrm{D}t}\frac{\delta \rho}{\rho_0} \sim \mathcal{O}(\epsilon). \tag{11}$$

(b) Use (11) and the parallel component of the induction equation to show that

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\delta B_\parallel}{B_0} - \frac{\delta \rho}{\rho_0}\right) - \hat{\boldsymbol{b}} \cdot \boldsymbol{\nabla} u_\parallel = 0. \tag{12}$$

(c) Use (10), (12), and the entropy equation $\mathrm{D}s/\mathrm{D}t \doteq \mathrm{D} \ln p\rho^{-\gamma}/\mathrm{D}t = 0$ to show that

$$\frac{\mathrm{D}}{\mathrm{D}t}\frac{\delta s}{s_0} = 0, \tag{13}$$

$$\frac{\mathrm{D}}{\mathrm{D}t}\frac{\delta \rho}{\rho_0} = -\frac{1}{1 + c_\mathrm{s}^2/v_\mathrm{A}^2}\hat{\boldsymbol{b}} \cdot \boldsymbol{\nabla} u_\parallel, \tag{14}$$

$$\frac{\mathrm{D}}{\mathrm{D}t}\frac{\delta B_\parallel}{B_0} = \frac{1}{1 + v_\mathrm{A}^2/c_\mathrm{s}^2}\hat{\boldsymbol{b}} \cdot \boldsymbol{\nabla} u_\parallel. \tag{15}$$

(d) Use the parallel component of the momentum equation (9) to show that

$$\frac{\mathrm{D}u_\parallel}{\mathrm{D}t} = v_\mathrm{A}^2 \hat{\boldsymbol{b}} \boldsymbol{\cdot} \boldsymbol{\nabla} \frac{\delta B_\parallel}{B_0}. \tag{16}$$

(e) Equations (14)–(16) describe the slow-wave-polarized fluctuations; equation (13) describes the zero-frequency entropy mode. Note that the only nonlinearities in these equations are via the derivatives defined in (6) and (7). What does this imply about the relationship between the compressive fluctuations and the Alfvénic ones? Also, recall that MHD supports fast magnetosonic waves – where are they in RMHD?

(f) The original Elsässer fields $\boldsymbol{z}^\pm = \boldsymbol{u} \pm \delta\boldsymbol{B}/\sqrt{4\pi\rho_0}$ were derived from the incompressible MHD equations. Define the generalized Elsässer fields for the compressive fluctuations,

$$z_\parallel^\pm \doteq u_\parallel \pm \frac{\delta B_\parallel}{\sqrt{4\pi\rho_0}} \left(1 + \frac{v_\mathrm{A}^2}{c_\mathrm{s}^2}\right)^{1/2}, \tag{17}$$

and use (15) and (16) to derive evolution equations for $z_\parallel^\pm$. What do you learn from these equations about the nonlinear dynamics of compressive fluctuations? (In may help if you explicitly write out the nonlinearities in terms of the Alfvénic Elsässer potentials $\zeta^\pm \doteq \Phi \pm \Psi$.) In particular, what happens in the high-beta limit, $v_\mathrm{A} \ll c_\mathrm{s}$?

(g) Use your answers to parts (e) and (f), alongside the standard Kolmogorov turbulence arguments, to explain qualitatively and, if possible, quantitatively why the scalar fluctuations associated with the compressive fluctuations should have the same turbulence scaling law as the Alfvénic fluctuations, corresponding to a $k_\perp^{-5/3}$ spectrum.

2. **Simple solutions for shrinking sheets (and their tearing).** As was shown in class and in my lecture notes (see §VII.1), the RMHD equations provide a useful theoretical framework for studying linear tearing of a current sheet (CS). In particular, equations (4) and (5) above were used to study the evolution of small perturbations to an equilibrium magnetic configuration specified by $\Psi(x)$; the case of a Harris-sheet profile, for which $\Psi = av_\mathrm{A} \ln[\cosh(x/a)]$ with characteristic thickness a, was the focus of §VII.1.5. Under certain restrictions, one can used the RMHD equations to describe a *time-dependent* magnetic configuration and its linear stability. In this problem, you'll use (4) and (5) to obtain simple solutions describing a thinning CS, whose width $a(t)$ shrinks in time and causes the tearing-mode stability parameter $\Delta'(t,k)$ to increase in time. These time-dependent solutions are based on S. Chapman & P. C. Kendall, *Proc. Roy. Soc. London Ser. A*, **271**, 435 (1963), and were used recently by N. F. Loureiro and D. A. Uzdensky, *Phys. Rev. Lett.* **116**, 105003 (2016) and E. A. Tolman *et al.*, *J. Plasma Phys.* **84**, 905840115 (2018) in their studies of the onset of reconnection in a thinning CS.[1] Here we'll sketch out those authors' basic idea.

(a) Consider the following time-dependent stream and flux functions:

$$\Phi(t,x,y) = \Lambda(t)xy \quad \text{and} \quad \Psi(t,x,y) = \frac{B_0}{2}\left[\frac{x^2}{a(t)} - \frac{y^2}{L(t)}\right]. \tag{18}$$

[1] Libby Tolman was recently a postdoctoral member of the IAS; she's now at CCA but still visits Princeton.

These describe a local incompressible flow that is thinning and lengthening a CS about an X-point. For these potentials to be solutions of the RMHD equations, what must the CS width $a(t)$ and length $L(t)$ satisfy? Plot iso-contours of $\Phi/(\Lambda a^2)$ and $\Psi/(B_0 a)$ in the (x/a)–(y/a) plane for $L/a = 10$ and describe what you see. (You might find it helpful to calculate the flow velocity $\boldsymbol{u} = \hat{\boldsymbol{z}} \times \boldsymbol{\nabla}_\perp \Phi$ and the magnetic field $\boldsymbol{B} = \hat{\boldsymbol{z}} \times \boldsymbol{\nabla}_\perp \Psi$ corresponding to these functions and plot their vector fields.)

(b) Set $\Lambda(t) = \tau^{-1}$ with $\tau = \text{const}$ and solve your equations for $a(t)$ and $L(t)$. (Name the initial values of the CS thickness and length a_0 and L_0, respectively.) Briefly describe in words the evolution of this CS.

(c) Suppose $L(t) = L_0(1 + t/\tau)$ with $\tau = \text{const}$. Obtain the corresponding $\Lambda(t)$ and $a(t)$. Briefly describe in words the evolution of this CS.

(d) Let's adopt the CS model from part (c) and set $\tau \doteq (L_0/v_\text{A})M_\text{A}^{-1}$, where M_A is the Alfvén Mach number of the incompressible flow. We now ask how linear tearing modes grow on top of this time-dependent background and determine which of these linear modes grows the fastest at any given time in the CS evolution. For that, give the CS some resistivity η, and assume that the outer solution for the CS provides $\Delta'(k) \sim 1/ka^2$. (The "$\sim$" here means that we are dropping factors of order unity.) The number of tearing-induced magnetic islands with wavenumber k that can fit inside the length of this CS at any given time is $\sim kL \doteq N$. Because each tearing-mode wavelength k^{-1} is stretched by the flow in the same way as is L, each tearing mode can be labeled by its own unique value of N. With this borne in mind, answer the following:

(i) Take the long-time limit $t \gg \tau$, such that $L(t) \sim L_0(t/\tau)$. Write down how $\Delta'(N)$ evolves in time for this CS. Your answer should involve N, L_0, a_0, and t/τ only.

(ii) Show that, in the FKR regime, the time-dependent growth rate γ_FKR satisfies

$$\gamma_\text{FKR}(t)\tau_0 \sim N^{-2/5} M_\text{A}^{12/5} S_0^{-3/5} \left(\frac{t}{\tau_0}\right)^{12/5}, \tag{19}$$

where $\tau_0 \doteq (a_0 L_0)^{1/2}/v_\text{A}$ and $S_0 \doteq v_\text{A}(a_0 L_0)^{1/2}/\eta$. Thus, the fastest-growing FKR mode is the $N = 1$ mode.[2]

(iii) Use (19) to determine the approximate time at which the $N = 1$ FKR mode grows faster than the rate at which the CS thickness is shrinking. (Don't be too fancy here – I'm only looking for a scaling argument.) Name this time t_cr and express it in terms of τ_0, M_A, and S_0.

(iv) Determine the approximate time at which this $N = 1$ mode transitions into the Coppi regime. Name this time t_tr and express it in terms of τ_0, M_A, and S_0.

(v) For what combination of M_A and S_0 is $t_\text{tr} \sim t_\text{cr}$? In this situation, the maximally growing FKR mode enters the Coppi regime just as it begins growing fast enough to disrupt the evolving CS. Loureiro & Uzdensky argued that, under these conditions, this time marks the onset of reconnection and the disruption of the CS.[3]

[2] In writing (19), we are implicitly assuming that the secular evolution of the CS does not greatly affect the instantaneous exponential growth of the tearing modes – only that it changes the instantaneous values of τ_A, τ_η, and $\Delta' a$ that figure into the usual FKR growth rate. This is a good approximation when $\gamma_\text{FKR}(t) \gg |\dot{a}/a|$ – see Tolman *et al.* (2018) if you're interested in the more rigorous details.

[3] They also considered the cases $t_\text{tr} > t_\text{cr}$ and $t_\text{cr} < t_\text{tr}$; I picked $t_\text{tr} \sim t_\text{cr}$ just to keep this problem short(ish).

(vi) Consider a solar flare powered by a reconnecting CS whose $L_0 \sim a_0 \sim 10^4$ km and which evolves according to our crude model here. Typical photospheric values are $v_{\rm A} \sim 2000$ km s^{-1}, $M_{\rm A} \sim 10^{-3}$, and $S_0 \sim 10^{13}$. If you plug these numbers in to your answer from part (v), you should find that $t_{\rm tr} \sim t_{\rm cr}$. Use this to estimate the time at which reconnection onsets, as well as the aspect ratio of the CS at this time. The former turns out to be reasonably consistent with the observed pre-flare energy-buildup times in the solar photosphere. Neat.

3. **Critical balance in rotating, hydrodynamic turbulence.** In a rigidly rotating, hydrodynamic, incompressible fluid, the characteristic linear frequency of waves is $\omega = \pm(k_\parallel/k)\Omega$, where $\boldsymbol{\Omega} = \Omega\hat{\boldsymbol{z}}$ is the angular velocity of the flow and $k_\parallel = k_z$ is component the wavenumber oriented parallel to the rotation axis. Suppose that such a fluid is turbulent, with velocity fluctuations satisfying $k_\parallel/k_\perp \ll 1$, i.e., the fluctuations are anisotropic with respect to the rotation axis and elongated in that direction. Assume the turbulence to be strong and critically balanced. Obtain the resulting perpendicular and parallel power spectra of the turbulent velocities and the scaling relation linking $k_\parallel$ and $k_\perp$. (*This should take no more than a few lines!*) Does the anisotropy of the fluctuations increase or decrease as the cascade goes to smaller scales? In other words, does the assumption of anisotropy get better or worse at progressively smaller scales? Is this similar to or different than the situation in Goldreich–Sridhar turbulence? Physically, why?