☺ Due Wednesday, 2 Nov 2022 ☺

Please indicate on your solutions with whom you worked, what resources you consulted (if any), and what were the contributions of each team member. Your code should be well-documented; include it with your solutions. All plot axes and lines should be labeled clearly.

1. **Thermal equilibrium in the interstellar medium (ISM).** The ISM constitutes a veritable smorgasbord of thermodynamics, chemistry, radiation, and magnetohydrodynamics. But let's curb our ambition and focus only on the thermodynamics – and in a very simplified form at that. In this problem, you'll use Newton–Raphson iteration to determine the thermal equilibrium curve in the ISM using simplified models for the amount of energy per unit time lost and gained by the ISM gas due to cooling and heating processes.

The primary sources of heating in the general ISM are FUV photoelectric heating from dust grains, cosmic rays, and ∼0.1–1 keV ("soft") X-rays. Let us model the resulting heating rate rather crudely as

$$\Gamma = 2 \times 10^{-26} \text{ ergs s}^{-1}. \tag{1}$$

Cooling in the general ISM is primarily due to collisions between ionized carbon (C^+) and atomic hydrogen (H I), and collisional excitation of H I by electrons followed by subsequent emission of a photon. These two processes may be approximated by the following temperature-dependent cooling rate:

$$\frac{\Lambda(T)}{\Gamma} = \left[10^7 \exp\left(\frac{-1.184 \times 10^5}{T + 1000}\right) + 1.4 \times 10^{-2} \sqrt{T} \exp\left(\frac{-92}{T}\right)\right] \text{cm}^3, \tag{2}$$

where the temperature T is measured in Kelvin. Thermal equilibrium is given by a balance between cooling and heating:

$$\mathcal{L}(n, T) \equiv n\Lambda(T) - \Gamma = 0, \tag{3}$$

where n is the number density of the gas (in cm^{-3}).[1] Note that (3) is an *implicit* equation for the equilibrium temperature T_{eq} as a function of density, and so must be solved numerically.

There are a variety of recipes for numerically solving an equation like (3), which invariably proceed by iteration from an initial trial solution until some predetermined convergence criterion is satisfied. This means that you must guess a trial solution (say, T_0) corresponding to some density n. Hmm. . .

(a) I could give you a trial solution with which to start, but I won't. That'd be too easy. "Ahhh," you say, "but I can just plot (3) for a particular density to visually obtain my trial solution." I'm not going to let you do that either. Instead, I'd like you to write a program to figure out approximately where the root lies using the method of **bracketing**. The idea is simple. Choose two endpoints, T_1 and T_2; if $\mathcal{L}(n, T_1)$

[1] These approximate heating and cooling rates are taken from Nagashima, Inutsuka & Koyama (2006). Their origins may be traced to pioneering papers by Goldsmith, Field & Habing (1969), Zel'dovich & Pikelner (1969), and Penston & Brown (1970). An updated calculation of these rates may be found in Wolfire et al. (1995).

and $\mathcal{L}(n, T_2)$ have opposite signs, then at least one root must lie in that interval. My advice is to start by setting $n = 10^{-2}$, $T_1 = 6000$, and $T_2 = 6001$; it's easy to check by substitution that $\mathcal{L} < 0$ in this tiny range. Your program should increase T_2 progressively—Numerical Recipes iteratively adds $1.6(T_2 - T_1)$ to it—until it finds $\mathcal{L}(n, T_1)\mathcal{L}(n, T_2) < 0$. Record this T_2; a root lies somewhere between it and T_1. **[1 pt]**

(b) Using the bracketed interval you found in part (a), write a program to solve equation (3) for $T_{\rm eq}$ when $n = 10^{-2}$ using the **bisection method**. For your convergence criterion, accept a root as the "solution" once its value changes by $<10^{-10}$ from one iteration to the next. In other words, $T_{\rm eq} = T_i$ if after iteration i you have $\epsilon_i \equiv |T_i - T_{i-1}|/T_i < \epsilon = 10^{-10}$. Record how many iterations it took for your initial "guess" at the bracket midpoint $T_0 = (T_1 + T_2)/2$ to converge to the "true" solution $T_{\rm eq}$; call it $N_{\rm iter}^{\rm bis}$. Compare $N_{\rm iter}^{\rm bis}$ to the expected value of $\log_2(\epsilon_0/\epsilon)$ steps to converge, where $\epsilon_0 \equiv |T_2 - T_1|/T_{\rm eq}$ is the fractional size of the initial bracket. **[2 pts]**

(c) Now that you have an accurate solution for $T_{\rm eq}$ at $n = 10^{-2}$, write a program to solve for $T_{\rm eq}$ as a function of $n \in [10^{-2}, 10^3]$ using **Newton-Raphson iteration** with a tolerance of $\epsilon < 10^{-10}$. Note that you may use your converged answer $T_{\rm eq}$ at $n = 10^{-2}$ as an initial guess for determining the equilibrium temperature at a slightly different density, say, $n = 10^{-1.96}$. This procedure of using the previously converged $T_{\rm eq}$ as an initial guess for $T_{\rm eq}$ at the next n can be repeated to scan across the full range of n, provided that $T_{\rm eq}$ does not change too rapidly from one n to the next. When you finish, make two log-log plots: one for $T_{\rm eq}$ vs. n, and one for the equilibrium pressure $p_{\rm eq} \equiv nT_{\rm eq}$ vs. n. There's something pretty neat in the latter plot that I discuss below... for now, just note that $\mathcal{L} > 0$ (< 0) above (below) this curve. **[5 pts]**

(d) Newton-Raphson iteration converges quadratically, with the error after iteration i being $\epsilon_i \approx |T_{\rm eq}\mathcal{L}''(T_{\rm eq})/2\mathcal{L}'(T_{\rm eq})|\,\epsilon_{i-1}^2 \equiv C\epsilon_{i-1}^2$. In other words, near a root, the number of significant digits approximately doubles with each step. That's nice, and it means that your Newton-Raphson program should have been extremely fast: after just

$$N_{\rm iter}^{\rm NR} = \log_2\left(\frac{\log C\epsilon}{\log C\epsilon_0}\right) \tag{4}$$

iterations, the error is $\lesssim\epsilon$. Let's check this. Perform a N-R iteration with $\epsilon = 10^{-10}$ for $n = 10^{-2}$ at the T_0 from part (b). How many iterations did it take to converge? Was it roughly equal to equation (4)? (Isn't that so much nicer than bisection?!) **[1 pt]**

(e) Now start a Newton-Raphson iteration for $n = 10^{-2}$ at $T_0 = 5000$. What happened? Why? **[1 pt]**

One astrophysics lesson here is that the ISM has two thermally stable phases: a warm phase at $T \sim 10^4$ K and a cold phase at $T \lesssim 100$ K. (The cold phase may be more familiar to you as the parts of the ISM where stars are born.) In between, the ISM is thermally unstable, i.e., a small perturbation to the temperature of some parcel of ISM in this intermediate phase will send it rapidly towards one of the two stable phases. From your $p_{\rm eq}$ vs. n plot in part (c), you should notice that a substantial range of densities in these two phases exist at the same pressure (for $p_{\rm eq} \approx 1600$–5000 ergs cm^{-3}). Thus, the ISM has two phases in pressure equilibrium, with phase changes transporting gas thermodynamically between them. Neat.

2. **Solving Kepler's equation.** The study of orbital mechanics brings with it a lovely assortment of opaque 17th-century words like "eccentric anomaly", "true anomaly", "mean anomaly", "longitude of the ascending node", "argument of periapsis" ... the list goes on. If you'd like know what some of these terms actually mean and assure yourself that knowledge procured more than four centuries ago has not been lost on the collegiate class of the modern human species, you can find a discussion of the so-called "two-body problem" and its solution at the end of this problem set. But if you're eager to skip the physics lesson and get right to scoring some homework points, here's the problem as succinctly as I can state it...

The motion of two gravitationally interacting bodies with total mass M in an elliptical orbit with semi-major axis a may be described parametrically in polar coordinates $[r(t), \theta(t)]$ once the following equation ("Kepler's equation") is solved for the angle $E = E(t)$:

$$E - e \sin E = \Omega(t - \tau), \tag{5}$$

where $0 \leq e < 1$ is the eccentricity of the orbit, $\Omega = (GM/a^3)^{1/2}$ is the orbital frequency (via Kepler's third law), and τ is the time of periapsis passage (if that means anything to you). For $e = 0$, equation (5) traces out a circle in the orbital plane: $r(t) = a$ and $\theta(t) = E = \Omega(t - \tau)$. Otherwise, equation (5) is implicit and thus must be solved numerically.

(a) Go here: https://exoplanets.nasa.gov/exoplanet-catalog/1343/hat-p-21-b/. You'll find information about HAT-P-21b, a gas giant exoplanet orbiting a 0.947-$M_\odot$ G-type star that was discovered by our very own Prof. Gaspar Bakos and Dr. Joel Hartman in 2010 using the Hungarian-made Automated Telescope Network (HATNet).[2] Record the observationally inferred values of e, a, and Ω for this two-body system. Set $\tau = 0$, solve Kepler's equation using Newton-Raphson iteration, and plot the resulting values of $E/2\pi$ vs. t (in yr). (Don't use some pre-packaged routine like scipy.optimize.newton; roll your own.) Be sure to explain your choice of initial guess for the iteration. **[5 pts]**

(b) The polar angle θ is related to E via

$$\tan \frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2}. \tag{6}$$

The radial distance r may then be computed from Kepler's first law:

$$r = \frac{a(1-e^2)}{1 + e \cos \theta}. \tag{7}$$

Use these to plot the orbit. Be sure to label all axes, and mind the aspect ratio of your plot so that a circle looks like a circle and an ellipse looks like an ellipse. **[2 pts]**

(c) Go here: https://exoplanets.nasa.gov/exoplanet-catalog/6636/hd-43197-b/. Repeat parts (a) and (b) for this two-body system. The orbit of this exoplanet has a large eccentricity, so you should re-evaluate your choice of initial guess for the iteration. To do so, answer the following: What happens to the N-R iteration at large values of eccentricity when $E/2\pi$ is close to an integer? Why might this be bad? **[3 pts]**

[2] The original discovery paper is here: https://ui.adsabs.harvard.edu/abs/2011ApJ...742..116B/abstract.

The two-body problem. Consider N bodies in orbit about their mutual center of mass,

$$\boldsymbol{r}_{\rm cm} \equiv \sum_{i=0}^{N-1} m_i \boldsymbol{r}_i \Bigg/ \sum_{i=0}^{N-1} m_i,$$

where m_i and $\boldsymbol{r}_i$ are the mass and spatial location of the ith body, respectively. The latter, of course, is a function of time:

$$\frac{{\rm d}\boldsymbol{r}_i}{{\rm d}t} = \boldsymbol{v}_i \quad \text{and} \quad \frac{{\rm d}\boldsymbol{v}_i}{{\rm d}t} = \sum_{i\neq j}^{N} \frac{G m_j \boldsymbol{r}_{ij}}{r_{ij}^3}, \tag{8}$$

where $\boldsymbol{r}_{ij} \equiv \boldsymbol{r}_j - \boldsymbol{r}_i$. Thus, according to Newton, all we must do to determine the location of each body for all time is to adopt some "initial" conditions and integrate (8). Easy!

Not so fast. There is no general analytical solution for $N \geq 3$ given in terms of simple algebraic expressions and integrals, which means that (8) must be integrated numerically. (Foreshadowing...) But, for $N = 2$, there is a well-known analytic solution. To obtain it, begin by writing (8) with $N = 2$ in terms of $\boldsymbol{r}_{\rm cm}$ and the separation $\boldsymbol{r}_{12} \equiv \boldsymbol{r}_2 - \boldsymbol{r}_1$:

$$\frac{{\rm d}^2\boldsymbol{r}_{\rm cm}}{{\rm d}t^2} = \frac{1}{m_1 + m_2}\left(m_1 \frac{{\rm d}\boldsymbol{v}_1}{{\rm d}t} + m_2 \frac{{\rm d}\boldsymbol{v}_2}{{\rm d}t} \right) = 0, \tag{9}$$

$$\frac{{\rm d}^2\boldsymbol{r}_{12}}{{\rm d}t^2} = \frac{{\rm d}\boldsymbol{v}_2}{{\rm d}t} - \frac{{\rm d}\boldsymbol{v}_1}{{\rm d}t} = -\frac{G(m_1+m_2)\boldsymbol{r}_{12}}{r_{12}^3}. \tag{10}$$

Introduce the total mass $M \equiv m_1 + m_2$ and the reduced mass $\mu \equiv m_1 m_2/(m_1 + m_2)$. Take the dot product of (9) with $M{\rm d}\boldsymbol{r}_{\rm cm}/{\rm d}t$ and of (10) with $\mu{\rm d}\boldsymbol{r}_{12}/{\rm d}t$, and then sum the results to find energy conservation for the system:

$$\frac{{\rm d}\mathcal{E}}{{\rm d}t} \equiv \frac{\rm d}{{\rm d}t}\left[\frac{1}{2} m_1 \left(\frac{{\rm d}\boldsymbol{r}_1}{{\rm d}t} \right)^2 + \frac{1}{2} m_2 \left(\frac{{\rm d}\boldsymbol{r}_2}{{\rm d}t} \right)^2 - \frac{G m_1 m_2}{r_{12}} \right] \tag{11a}$$

$$= \frac{\rm d}{{\rm d}t}\left[\frac{1}{2} M \left(\frac{{\rm d}\boldsymbol{r}_{\rm cm}}{{\rm d}t} \right)^2 + \frac{1}{2} \mu \left(\frac{{\rm d}\boldsymbol{r}_{12}}{{\rm d}t} \right)^2 - \frac{G m_1 m_2}{r_{12}} \right] = 0. \tag{11b}$$

Likewise, take the cross product of (9) with $M\boldsymbol{r}_{\rm cm}$ and of (10) with $\mu\boldsymbol{r}_{12}$, and then sum the results to find angular-momentum conservation for the system:

$$\frac{{\rm d}\boldsymbol{L}}{{\rm d}t} \equiv \frac{\rm d}{{\rm d}t}\left[m_1 \boldsymbol{r}_1 \times \frac{{\rm d}\boldsymbol{r}_1}{{\rm d}t} + m_2 \boldsymbol{r}_2 \times \frac{{\rm d}\boldsymbol{r}_2}{{\rm d}t} \right] \tag{12a}$$

$$= \frac{\rm d}{{\rm d}t}\left[M \boldsymbol{r}_{\rm cm} \times \frac{{\rm d}\boldsymbol{r}_{\rm cm}}{{\rm d}t} + \mu \boldsymbol{r}_{12} \times \frac{{\rm d}\boldsymbol{r}_{12}}{{\rm d}t} \right] = 0. \tag{12b}$$

Since we may arbitrarily erect a coordinate system such that $\boldsymbol{r}_{\rm cm} = 0$, the first term in (12b) can be made to vanish. Then, $\boldsymbol{L} = \mu\boldsymbol{r}_{12} \times {\rm d}\boldsymbol{r}_{12}/{\rm d}t = \boldsymbol{r}_{12} \times \boldsymbol{p}$, where $\boldsymbol{p} \equiv \mu{\rm d}\boldsymbol{r}_{12}/{\rm d}t$ is the momentum of the reduced mass.

Now, because $\boldsymbol{L}$ is always perpendicular to the plane defined by the position and momentum of the reduced mass, $\boldsymbol{r}_{12}$ and $\boldsymbol{p}$ must always lie in the plane perpendicular to $\boldsymbol{L}$, and can therefore be described using polar coordinates (r, θ) with the origin at the fixed center $\boldsymbol{r}_{\rm cm}$. Equations (11) and (12) then become

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} - \frac{L^2}{\mu^2 r^3} = -\frac{GM}{r^2}, \tag{13}$$

$$L = \mu r^2 \frac{\mathrm{d}\theta}{\mathrm{d}t} = \text{const}, \tag{14}$$

respectively. The solution to (13) may be written parametrically as $r = r(\theta)$; in other words, given an angular position θ in the polar plane, the separation between the two bodies at that θ is r. The angle θ is referred to as the *true anomaly.* Using the chain rule $\mathrm{d}/\mathrm{d}t = (\mathrm{d}\theta/\mathrm{d}t)(\mathrm{d}/\mathrm{d}\theta) = (L/\mu r^2)(\mathrm{d}/\mathrm{d}\theta)$ to replace the time derivatives by θ derivatives, and making the variable substitution $u = 1/r$, equation (13) becomes

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} + u = \frac{GM\mu^2}{L^2},$$

which may be readily integrated to find

$$u = \frac{1}{r} = \frac{GM\mu^2}{L^2}\big(1 + e\cos\theta\big). \tag{15}$$

Here, e is a constant representing the eccentricity of the orbit. For $0 \le e < 1$, equation (15) describes an ellipse with semi-latus rectum $\ell = L^2/GM\mu^2$, semi-major axis $a = \ell/(1-e^2)$, and semi-minor axis $b = a\sqrt{1-e^2} = \ell/\sqrt{1-e^2}$. Thus, Kepler's first law.

As for Kepler's second law, note that the differential area swept out by $\boldsymbol{r}_{12}$ per unit time is

$$\mathrm{d}A = \frac{1}{2} r (r\mathrm{d}\theta) \quad \Longrightarrow \quad \frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2} r^2 \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{L}{2\mu},$$

where the last step follows from (12). Integrating this equation over one orbit and knowing that the area of an ellipse is πab determines the orbital period P:[3]

$$\pi ab = \frac{L}{2\mu} P \quad \Longrightarrow \quad P^2 = \frac{(2\pi)^2}{GM} a^3. \tag{16}$$

The square of the orbital period of a planet is proportional to the cube of the semi-major axis of its orbit, i.e., Kepler's third law.

Finally, given (15) and the condition that $\theta = 0$ occurs at some time $t = \tau$, equation (14) may be integrated to obtain the angular position of the planet as a function of time:

$$\int_0^{\theta(t)} \frac{\mathrm{d}\theta'}{(1 + e\cos\theta')^2} = \frac{L}{\mu\ell^2}(t - \tau)\,; \tag{17}$$

that is, provided you can actually do the integral on the left-hand side. Here's a not-so-

[3] Kepler's third law implies $G = (2\pi)^2$ in units where time is measured in yr, length is measured in au, and mass is measured in $\mathrm{M}_\odot = 1047.59421\,\mathrm{M_J}$, where $\mathrm{M_J}$ is the mass of Jupiter. These may prove useful to you in the future.

obvious trick: introduce the *eccentric anomaly* E, defined implicitly via

$$\tan\frac{\theta}{2} = \begin{cases} \sqrt{\dfrac{1+e}{1-e}}\tan\dfrac{E}{2} & \text{if } e<1 \\ \sqrt{\dfrac{e+1}{e-1}}\tanh\dfrac{E}{2} & \text{if } e>1 \\ \dfrac{E}{\ell} & \text{if } e=1. \end{cases} \tag{18}$$

(Note that E is an angle.) Let's confine our interest to $0 \le e < 1$, i.e., elliptical orbits. Using (18) to change variables in (17), exploiting two half-angle formulas, and expending some effort to perform the integration, one obtains *Kepler's equation*,

$$E - e\sin E = \frac{2\pi(t-\tau)}{P}. \tag{19}$$

For $e = 0$, $E = \theta$, and so (19) traces out a circle: $\theta = 2\pi(t-\tau)/P$. Otherwise, equation (19) is implicit and thus must be solved numerically (thus, this problem set). It's essentially a restatement of Kepler's second law: orbital time is measured in area.

So that's the analytic solution of the two-body problem. Equation (19) with (18) tells you how $\theta = \theta(t)$ advances in time, and (15) tells you $r = r(\theta)$. Now you know!