Introduction to Plasma Physics:
A 1-hour taste* of key concepts & results for astrophysics graduate students

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* Can’t possibly cover all interesting topics in plasma courses AST55n..., General Plasma Physics I & II, Waves, Irreversible Processes, Turbulence…

* Can’t possible cover all of these slides: some skipped, some briefly skimmed…

acknowledgements: some slides borrowed from Profs. Stone, Fisch, others
Introduction to Plasma Physics

- **Fundamentals of plasmas,**
  - 4th state of matter
  - weak coupling between pairs of particles, but
  - strong collective interactions: Debye shielding, electron plasma oscillations

- **Fundamental Length & Time Scales**
  - Debye length, mean free path, plasma frequency, collision frequency
  - hierarchy of length/ & time scales, related to fundamental plasma parameter:
    \[ \Lambda = \# \text{ of particles in a Debye sphere} \]

- **Single Particle Motion:**
  - ExB, grad(B), other drifts, conservation of adiabatic invariant \( \mu \), magnetic mirrors

- **Kinetic starting point:** Vlasov/Boltzmann equation

- **MHD Eqs.**
  - Braginskii/Chapman-Enskog fluid equations
  - approximations made in getting MHD, properties of MHD
  - Flux-freezing
  - Alfven waves

- **Collective kinetic effects:** Plasma waves, wave-particle interactions, Landau-damping
Plasma--4th State of Matter

solid

Heat

More Heat

Liquid

More Heat

Gas

Yet More Heat

Plasma
“Plasma” named by Irving Langmuir in 1920’s

The standard definition of a plasma is as the 4th state of matter (solid, liquid, gas, plasma), where the material has become so hot that (at least some) electrons are no longer bound to individual nuclei. Thus a plasma is electrically conducting, and can exhibit collective dynamics.

I.e., a plasma is an ionized gas, or a partially-ionized gas.

Implies that the potential energy of a particle with its nearest neighboring particles is weak compared to their kinetic energy (otherwise electrons would be bound to ions). → Ideal “weakly-coupled plasma” limit. (There are also more-exotic strongly-coupled plasmas, but we won’t discuss those.)

Even though the interaction between any pair of particles is typically weak, the collective interactions between many particles is strong. 2 examples: Debye Shielding & Plasma Oscillations.
States of Matter

1. Just an approximation, not a material property.

2. Depends on time scales, space scales, and physics of interest (is gravel a solid or a liquid?)
Broader Definition of Plasma

- The electron temperature needs to be above ~0.3-1 eV in order to have most hydrogen ionized in thermal equilibrium. However, at lower temperatures can have weakly ionized plasmas (where plasma effects are still important), single species plasmas (pure electrons or pure ions, so there is no recombination), or non-equilibrium plasmas (at low density it takes a long time to recombine).

- Single-species non-neutral plasmas include intense charged particle beams where the self-interactions of the beam become important relative to external forces.

- A broader definition of a plasma could include matter which is electrically conducting even if the weak-coupling approximation doesn’t hold. There are “strongly-coupled plasmas”, “plasma crystal” states….

- Unconventional plasma at extreme conditions involving collective effects through the strong nuclear force and not just electric forces: quark-gluon plasma (“Big Bang Goo”, NYT headline for article on RHIC, by J. Glanz, plasma physicist turned journalist).

- However, here we will focus on the conventional or ideal limit of “weakly-coupled plasmas”
Interesting Phenomena in Plasmas

- Electrically conducting: try to short out electric fields & give quasineutrality
- Low resistivity $\rightarrow$ can often approximate plasma as in ideal conducting fluid $\rightarrow$ magnetic fields are trapped in plasmas. Frozen-flux $\rightarrow$ magnetic fields and plasmas move together.
- Except when they don’t: breaking magnetic field lines by reconnection
- Dynamo mechanisms can generate magnetic fields
- Thermal conduction & viscosity very different parallel & perpendicular to magnetic field.
- Collective effects: various types of wave-particle interactions can occur
  - Landau damping: damping of electric fields in a conservative Hamiltonian system: waves lose energy to particles (2010 Fields medal)
  - Particle acceleration mechanisms (cosmic rays)
  - Kinetic instabilities: particles put energy into waves (collisionless shocks)
- All the interesting phenomena of neutral fluid dynamics: instabilities, shocks, turbulence, & chaos, plus electric and magnetic fields (with some surprising differences: e.g., Magneto-Rotational Instability in accretion disks).
(Standard) Plasmas are Weakly Coupled

Electron density \( n \approx N_e \approx N_i \)

Typical inter-particle spacing \( L_i \sim n^{-1/3} \)

(i.e., a box of size \( L_i^3 \) contains 1 particle on average, \( L_i^3 n = 1 \)).

Interaction between nearest neighbors is typically weak:

Typical Potential Energy of nearest neighbors

Typical Kinetic Energy \( \ll \)

\[
\frac{\text{P.E.}}{\text{K.E.}} \sim \frac{\frac{e^2}{L_i}}{\frac{1}{2}mv^2} \approx \frac{e^2 n^{4/3}}{T} \ll 1
\]

(Plasma literature measures \( T \) in energy units, so Boltzmann's constant \( k_B = 1 \).)

for a standard ("weakly-coupled") plasma.

To be a plasma, must be sufficiently hot and/or sufficiently rarified.
<table>
<thead>
<tr>
<th>Solar Wind @ 1 AU</th>
<th>Galactic Center @ Bondi Accretion Radius (measured by Chandra X-ray telescope)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{Ne} \ (\text{cm}^{-3}) )</td>
<td>( 10^{14} )</td>
</tr>
<tr>
<td>( T_e )</td>
<td>( 10 \text{ keV} )</td>
</tr>
<tr>
<td>( \frac{\text{P.E.}}{\text{K.E.}} \sim e^2 n^3 T^{-1} )</td>
<td>( 10^{-6} )</td>
</tr>
</tbody>
</table>
Collective Long Range Forces Dominate

\[ \text{Force } \propto \frac{\Delta r}{r^2} \propto (\Delta r)^3 \text{ for fixed solid angle } \]
\[ \frac{\Delta r}{r} \propto \text{const.} \]
\[ \propto r \]

Many particles far away dominate over few particles near by. (Mixture of electrons, ions, w/ Debye shielding, will reduce this.)

Don't need to know exactly where the distant particles are with great precision. Sufficient to treat them as an averaged, smooth charge density (a kind of smeared-out cloud), \( \propto \) current density, instead of discrete particles.

(From F.F. Chen, Intro to Plasma Physics)
\[ N_e(x, t) = \int d^3v \, f_e(x, v, t) \]

electron density

("fluid theory")

electron distribution function

("kinetic theory")

\[ f_e(x, v, t) (\Delta x)^3 (\Delta v)^3 \]

\[ = \text{# of electrons within } \Delta x \text{ of } x \text{ and } \]

\[ \text{within } \Delta v \text{ of } v \]

\[ \text{at time } t. \]
Debye Shielding

Add a charge $+Q$. Will attract (slightly) electrons and repel (slightly) ions producing a negative cloud that shields the charge $+Q$.

Electrons and ions in Boltzmann/Gibbs thermal equilibrium:

$$-\frac{H}{T} = -\frac{1}{2}m_0v^2 + q_s \Phi(x)$$

$$f_s(x,v) \propto e^{-\frac{1}{T}(\frac{1}{2}m_0v^2 + q_s \Phi(x))}$$

$$n_s(x) \propto e^{-\frac{q_s \Phi(x)}{T}}$$

$$n_s(x) \approx n_{so} \left(1 - \frac{q_s \Phi}{T}\right) \quad \left(\frac{q_s \Phi}{T} \ll 1\right)$$

$$n_0 = n_o = n_{so}$$

$$\Phi_{ext} = Q/|x|$$
\[ \nabla \cdot E = 4\pi \sum \frac{q_s n_s}{s} + 4\pi Q \delta(x) \]

\[-\nabla^2 \Phi = -\sum \frac{4\pi n_s q_s^2}{T} \frac{1}{\lambda_D^2} \Phi + 4\pi Q \delta(x) \]

\[ \frac{1}{\lambda_D^2} \]

\[ \text{often just the electron contribution is used, } \lambda_D \text{, de} \]

Fourier transform:

\[ \left( \kappa^2 + \frac{1}{\lambda_D^2} \right) \Phi_k = 4\pi Q \]

Inverse transform (or verify by direct subst.):

\[ \Phi(x) = \frac{Q}{r} e^{-r/\lambda_D} \]

\[ \text{strong shielding for } r > \lambda_D \]

Unshielded \( \Phi_{\text{ext}} = \frac{Q}{r} \)
Debye shielding very effective,

Debye shielding radius $\lambda_D$ usually very tiny:

<table>
<thead>
<tr>
<th></th>
<th>Magnetic Fusion</th>
<th>Solar Wind (SW)</th>
<th>Galactic Center (GC)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_D$ (cm)</td>
<td>$7 \times 10^{-3}$</td>
<td>$7 \times 10^2$</td>
<td>$3 \times 10^3$</td>
</tr>
<tr>
<td>$\omega_{pe}$ ($s^{-1}$)</td>
<td>$6 \times 10^{11}$</td>
<td>$2 \times 10^5$</td>
<td>$6 \times 10^5$</td>
</tr>
</tbody>
</table>

Associated frequency scale $\omega_{pe} = \frac{v_{te}}{\lambda_{De}} = \sqrt{\frac{4\pi n_e e^2}{m_e}} \propto \sqrt{n_e}$

= inverse of time scale for electrons to travel $\lambda_{De}$ to set up Debye shielding.

$\omega_{pe}$ = freq. of plasma oscillations.

$\omega_{pe}$ or cyclotron frequency $\omega_{Ce}$ are usually the highest frequency for collective dynamics or micromotions.
Aside: Every particle in a plasma is simultaneously being shielded by other particles, and is (weakly) participating in the shielding of other particles.

(BBGKY hierarchy $\Rightarrow$ Liouville Eq. or Klimontovich Eq. expand to give Vlasov Eq. + correction term (Collision operator) for weak correlations between particles.)
The Plasma Parameter

In order for smooth density \( n(\tilde{x}) \) approximation to hold (or equivalently, for the weak-coupling assumption to hold), need many particles within a Debye sphere:

\[
\Lambda = n_e \frac{4\pi}{3} \lambda_{De}^3 = 1.7 \times 10^9 T_{eV}^{3/2} n_e^{-1/2}
\]

\( \lambda_{De} \) in cm (handy formulas: NRL p. 28...)

\[
\Lambda = \text{"The number of particles in a Debye Sphere" a.k.a. "The Plasma Parameter"
}

<table>
<thead>
<tr>
<th>Magnetic Fusion (MFE)</th>
<th>Solar Wind (SW)</th>
<th>Galactic Center (GC)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Lambda = )</td>
<td>( \sim 10^8 )</td>
<td>( \sim 10^{10} )</td>
</tr>
</tbody>
</table>

It turns out that many key parameters can be expressed in terms of the number of particles in a Debye sphere. For example, the ratio of the potential energy between typical nearest neighbor particles to their typical kinetic energy:

\[
\frac{\text{Potential Energy of nearest neighbors}}{\text{Kinetic Energy}} \approx \frac{e^2 / n^{-1/3}}{T} = \frac{1}{(36 \pi)^{1/3} \Lambda^{2/3}}
\]

We will find that \( \Lambda >> 1 \) also implies that the mean free path between collisions is long compared to the Debye length.
"Plasmas that occur naturally or can be created in the laboratory are shown as a function of density (in particles per cubic centimeter) and temperature. The boundaries are approximate and indicate typical ranges of plasma parameters. Distinct plasma regimes are indicated. For thermal energies greater than that of the rest mass of the electron ($T > m_e c^2$), relativistic effects are important. At high densities, where the Fermi energy is greater than the thermal energy ($E_F > k_B T$), quantum effects are dominant [i.e., electron degeneracy pressure exceeds thermal pressure]. In strongly coupled plasmas (i.e., $n \lambda_D^{-3} < 1$, where $\lambda_D$ is the Debye screening length), the effects of the Coulomb interaction dominate thermal effects; and when $E_F > e^2 n^{1/3}$, quantum effects dominate those due to the Coulomb interaction (i.e., the Fermi energy exceeds the potential energy of typical nearest-neighbor particles], resulting in nearly ideal quantum plasmas. At temperatures less than about $10^5$K, recombination of electrons and ions can be significant, and the plasmas are often only partially ionized." [From National Research Council Decadal Review, Plasma Science: From Fundamental Research to Technological Applications (1995) [explanations added] http://www.nap.edu/catalog.php?record_id=4936]
Nonrelativistic Plasma

- Intergalactic Medium
- Interstellar Medium
- Solar Wind Magnetosphere
- Ionosphere
- Gas Discharge
- Recombined
- Nondegenerate
- Degenerate

Relativistic
Nonrelativistic

log n, m⁻³

log T, K

log ρ, kg/m³

log kT, kev

Wide range of possible plasma parameters. Plasmas above the line marked “Uncorrelated-Correlated” correspond to $\Lambda \gg 1$. See also NRL Formulary version of this plot.
## APPROXIMATE MAGNITUDES IN SOME TYPICAL PLASMAS

<table>
<thead>
<tr>
<th>Plasma Type</th>
<th>$n$ cm$^{-3}$</th>
<th>$T$ eV</th>
<th>$\omega_{pe}$ sec$^{-1}$</th>
<th>$\lambda_D$ cm</th>
<th>$n\lambda_D^3$</th>
<th>$\nu_{ei}$ sec$^{-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interstellar gas</td>
<td>1</td>
<td>1</td>
<td>$6 \times 10^4$</td>
<td>$7 \times 10^2$</td>
<td>$4 \times 10^8$</td>
<td>$7 \times 10^{-5}$</td>
</tr>
<tr>
<td>Gaseous nebula</td>
<td>$10^3$</td>
<td>1</td>
<td>$2 \times 10^6$</td>
<td>20</td>
<td>$8 \times 10^6$</td>
<td>$6 \times 10^{-2}$</td>
</tr>
<tr>
<td>Solar Corona</td>
<td>$10^9$</td>
<td>$10^2$</td>
<td>$2 \times 10^9$</td>
<td>$2 \times 10^{-1}$</td>
<td>$8 \times 10^6$</td>
<td>60</td>
</tr>
<tr>
<td>Diffuse hot plasma</td>
<td>$10^{12}$</td>
<td>$10^2$</td>
<td>$6 \times 10^{10}$</td>
<td>$7 \times 10^{-3}$</td>
<td>$4 \times 10^5$</td>
<td>40</td>
</tr>
<tr>
<td>Solar atmosphere, gas discharge</td>
<td>$10^{14}$</td>
<td>1</td>
<td>$6 \times 10^{11}$</td>
<td>$7 \times 10^{-5}$</td>
<td>40</td>
<td>$2 \times 10^9$</td>
</tr>
<tr>
<td>Warm plasma</td>
<td>$10^{14}$</td>
<td>10</td>
<td>$6 \times 10^{11}$</td>
<td>$2 \times 10^{-4}$</td>
<td>$8 \times 10^2$</td>
<td>$10^7$</td>
</tr>
<tr>
<td>Hot plasma</td>
<td>$10^{14}$</td>
<td>$10^2$</td>
<td>$6 \times 10^{11}$</td>
<td>$7 \times 10^{-4}$</td>
<td>$4 \times 10^4$</td>
<td>$4 \times 10^6$</td>
</tr>
<tr>
<td>Thermonuclear plasma</td>
<td>$10^{15}$</td>
<td>$10^4$</td>
<td>$2 \times 10^{12}$</td>
<td>$2 \times 10^{-3}$</td>
<td>$8 \times 10^6$</td>
<td>$5 \times 10^4$</td>
</tr>
<tr>
<td>Theta pinch</td>
<td>$10^{16}$</td>
<td>$10^2$</td>
<td>$6 \times 10^{12}$</td>
<td>$7 \times 10^{-5}$</td>
<td>$4 \times 10^3$</td>
<td>$3 \times 10^8$</td>
</tr>
<tr>
<td>Dense hot plasma</td>
<td>$10^{18}$</td>
<td>$10^2$</td>
<td>$6 \times 10^{13}$</td>
<td>$7 \times 10^{-6}$</td>
<td>$4 \times 10^2$</td>
<td>$2 \times 10^{10}$</td>
</tr>
<tr>
<td>Laser Plasma</td>
<td>$10^{20}$</td>
<td>$10^2$</td>
<td>$6 \times 10^{14}$</td>
<td>$7 \times 10^{-7}$</td>
<td>40</td>
<td>$2 \times 10^{12}$</td>
</tr>
</tbody>
</table>

From NRL Plasma Formulary (very useful)
Occasionally 2 particles will get close enough to cause a 90° scatter.

\[ b = \frac{e^2}{b} \sim T \]

\( b \) is small compared to typical inter-particle spacing,

\[ \frac{b}{n^{1/3}} \sim \frac{1}{\Lambda^{2/3}} \sim 10^{-4} \ll 1 \]

Cross-section for 90° collision = \( \pi b^2 \)

Mean-free Path for 90° collision = \( \lambda_{\text{mfp}, 90} \)

\[ n \pi b^2 \lambda_{\text{mfp}, 90} = 1 \Rightarrow \lambda_{\text{mfp}, 90} = \frac{\Lambda}{\lambda_{\text{Debye}}} \]

Area = \( \pi b^2 \)
Small-angle Collisions Dominate

However, cumulative effect of many small-angle scatters dominate over the few 90° scattering events.

This figure is not to scale: have to go ln(\lambda) further to get a single 90 degree scattering event, than the distance required for many small-angle scatters to add up to 90 degrees (rms average).

Effective \( \lambda_{mfp} \) is given by 
\[
\lambda_{mfp} \approx \frac{\lambda_{mfp,90}}{\ln \lambda} = \lambda_D \frac{\Lambda}{\ln \Lambda} \approx \lambda_D \frac{10^8}{10}
\]

Electron collision frequency,
\[
\nu_e = \frac{\nu_{te}}{\lambda_{mfp}} = 2.91 \times 10^{-6} \frac{N_e}{T_e^{3/2}} \ln \Lambda
\]

Ion collision frequency, \( \nu_{ei} \approx \frac{\nu_i}{\sqrt{m_e/m_i}} \approx \frac{\nu_i}{\sqrt{m}} \)

As \( T_e \) increases, \( \nu_e \) decreases.

\[
\frac{\nu_e}{\nu_{pe}} = \frac{\nu_{te}}{\lambda_{mfp}} \frac{\lambda_D}{\nu_{te}} \approx \frac{\ln \Lambda}{\Lambda} \approx 10^{-7}
\]
Review Fundamental Length Scales of a Plasma

\[ \Lambda \sim n \frac{1}{\lambda_D^3} \]

\[ = \text{# of particles in Debye Sphere} \]

\[ \lambda_D \sim 10^{-2} \text{ cm} \]

\[ \text{TFTR} \]

\[ \lambda_{mfp} \sim 100 \text{ m} \]

Distance of closest approach for 90° scatter

Avg. interparticle spacing

Debye shielding length

\[ \Lambda \sim \Lambda^{-1/3} : \lambda_D : \lambda_{mfp} \]

\[ b : n^{-1/3} : \lambda_D : \lambda_{mfp} \]

\[ \Lambda^{-1} : \Lambda^{-1/3} : 1 : \frac{\Lambda}{\text{mfp}} \]

\[ 10^{-6} : 10^{-2} : 1 : \frac{10^6}{15} \sim 10^5 \]
**Typical Plasma Parameters: Mean-Free Path Often Large**

<table>
<thead>
<tr>
<th></th>
<th>Magnetic Fusion</th>
<th>Solar Wind @ 1 AU (L=1 earth radius)</th>
<th>Galactic Center @ Bondi accretion radius</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_e$ (eV):</td>
<td>1.00E+04</td>
<td>10</td>
<td>2.00E+03</td>
</tr>
<tr>
<td>$n_e$ (/cm$^3$):</td>
<td>1.00E+14</td>
<td>10</td>
<td>1.00E+02</td>
</tr>
<tr>
<td>L macro scale (cm)</td>
<td>1.00E+02</td>
<td>6.40E+05</td>
<td>2.20E+17</td>
</tr>
<tr>
<td>beta</td>
<td>2.87E-02</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>$A_{amu}$</td>
<td>2.5</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$(\text{potential energy})/(\text{kinetic energy})$</td>
<td>4.5E-07</td>
<td>2.1E-08</td>
<td>2.2E-10</td>
</tr>
<tr>
<td>$B$ (g)</td>
<td>5.3E+04</td>
<td>2.8E-05</td>
<td>1.3E-03</td>
</tr>
<tr>
<td>L_Debye (cm)</td>
<td>7.4E-03</td>
<td>7.4E+02</td>
<td>3.3E+03</td>
</tr>
<tr>
<td>L_Debye/L</td>
<td>7.4E-05</td>
<td>1.2E-03</td>
<td>1.5E-14</td>
</tr>
<tr>
<td># of particles in Debye Sphere</td>
<td>1.7E+08</td>
<td>1.7E+10</td>
<td>1.5E+13</td>
</tr>
<tr>
<td>log($\Lambda$)</td>
<td>1.9E+01</td>
<td>2.4E+01</td>
<td>3.0E+01</td>
</tr>
<tr>
<td>log($\Lambda$) for collisions</td>
<td>2.0E+01</td>
<td>2.5E+01</td>
<td>3.2E+01</td>
</tr>
<tr>
<td>Plasma frequency (rad/s)</td>
<td>5.6E+11</td>
<td>1.8E+05</td>
<td>5.6E+05</td>
</tr>
<tr>
<td>ion collision frequency (/s)</td>
<td>6.2E+01</td>
<td>3.8E-07</td>
<td>1.7E-09</td>
</tr>
<tr>
<td>lambda_mfp (cm)</td>
<td>1.0E+06</td>
<td>8.2E+12</td>
<td>2.6E+16</td>
</tr>
<tr>
<td>ion Cyclotron Frequency (rad/s)</td>
<td>2.0E+08</td>
<td>2.7E-01</td>
<td>1.2E+01</td>
</tr>
<tr>
<td>rho_i ion gyroradius (cm)</td>
<td>3.0E-01</td>
<td>1.1E+07</td>
<td>3.6E+06</td>
</tr>
<tr>
<td>rho_i/L</td>
<td>3.0E-03</td>
<td>1.8E+01</td>
<td>1.6E-11</td>
</tr>
<tr>
<td>lambda_mfp/rho_i</td>
<td>3.3E+06</td>
<td>7.2E+05</td>
<td>7.1E+09</td>
</tr>
<tr>
<td>lambda_mfp/L</td>
<td>1.0E+04</td>
<td>1.3E+07</td>
<td>1.2E-01</td>
</tr>
</tbody>
</table>
Particle Motion in Uniform $\mathbf{B}$ Field

Lorentz Force Law:

$$\mathbf{F} = q(E + \mathbf{v} \times \mathbf{B})$$

$$m \frac{d\mathbf{v}}{dt} = q(E + \mathbf{v} \times \mathbf{B})$$

$$\dot{x} = \frac{d}{dt}(\frac{1}{2}mv^2) = q \mathbf{v} \cdot \mathbf{E}$$

only $\mathbf{E}$ can change particle's energy,
only $\mathbf{B}$ can only change particle's direction.

$$\mathbf{B} = B \hat{z}$$

$$\frac{d\mathbf{v}}{dt} = \Omega_c \mathbf{v} \times \hat{z}$$

$$\Omega_c = \frac{qB}{mc} = \text{cyclotron freq. or gyro freq.}$$

Gyroradius $\rho$:

$$\rho = \frac{v_i}{\Omega_c}$$

Free to move along magnetic field line $\rightarrow$
Physical Picture of $E \times B$ drift:

$E$ acceleration causes $v_\perp$ to be bigger on top half of orbit for ion
(or for electrons on bottom half of orbit. But electrons gyrate around $B$ in the opposite direction, $\Omega e < 0$, $\Omega e_i > 0$, since $q_e = -q_i$; so net $E \times B$ drift is in same direction for electrons & ions)

Gyroradius $\rho = \frac{v_\perp}{\Omega_e}$ is bigger on top half of orbit (for ions)

$\Rightarrow$ Net drift to right

$V_{E \times B} = \frac{q_e}{m_e} E \times B$ is the same for electrons + all types of ions ($v_E$ independent of $q \times m$ of particle species)

Surprise! Net particle motion is not in the direction of $E$!
(for constant $E$, $E \perp B$)
Particles gyrate rapidly around a slowly drifting "guiding center"

\[ \mathbf{x}(t) = \mathbf{R}_{\text{g.c.}}(t) + \mathbf{\rho}(t) \]

particle position = guiding center + rapidly gyrating gyrovector
  \[ \mathbf{\rho} = \frac{\mathbf{b} \times \mathbf{v}(t)}{\mathbf{\Omega}_{\text{c}}} \]

(actually, \( \mathbf{b} \times \mathbf{E} \times \mathbf{B} \) can be fast, but \( \mathbf{b} \times \mathbf{B} \) and other drifts are slow...)
\[ \mathbf{E} \times \mathbf{B} \text{ Drift} \]

\[ \mathbf{E} = \text{const} = E_0 \mathbf{y} \quad \mathbf{B} = \text{const} = B_0 \mathbf{z} \]

\[ \frac{d\mathbf{V}}{dt} = \frac{q}{m} \left[ \mathbf{E} + \frac{\mathbf{V} \times \mathbf{B}}{c} \right] \]

Substitute:

\[ \mathbf{V} = \frac{c}{B^2} \mathbf{E} \times \mathbf{B} + \mathbf{V}' \]

\[ \frac{c}{B^2} \mathbf{E} \times \mathbf{B} = \frac{1}{B^2} \left[ (\mathbf{E} \times \mathbf{B}) \times \mathbf{B} \right] = -\mathbf{E}_d \]

Leaves

\[ \frac{d\mathbf{V}'}{dt} = \frac{qB}{mc} \mathbf{V}' \times \mathbf{z} \]

same oscillatory motion as before with \( \mathbf{E}=0 \).

Equivalent to a relativistic transformation to a frame of reference where \( \mathbf{E} \) vanishes.

Fast gyration: \( \mathbf{E} \times \mathbf{B} \) drift

\[ V_x = V_\perp \sin (\Omega t + \alpha) + \frac{c}{B} E_y \]

\[ V_y = V_\perp \cos (\Omega t + \alpha) \]

\[ V_z = V_\parallel \]

More generally, guiding-center moves at the velocity:

\[ \frac{dR_{g.c.}}{dt} = V_\parallel \mathbf{B} + \frac{c}{B^2} \mathbf{E} \times \mathbf{B} + \text{higher-order drifts} \]
Particle Drift Due to $\nabla B$

\[ \nabla \times B = \frac{4\pi}{c} \frac{\dot{I}}{\hat{z}} \]
\[ \oint dl \cdot B = \frac{4\pi}{c} I \]
\[ 2\pi R B = \frac{4\pi}{c} I \]
\[ B \propto \frac{1}{R} \propto \frac{1}{x} \]

Stranger B
Smaller $\rho$

Weaker B
larger $\rho$

electrons drift up.

ions drift down

\[ \rho = \frac{V_\perp}{\Omega} \propto \frac{1}{B} \propto R \propto x \]

"$\nabla B$ drift"

\[ V_{\nabla B} = \frac{1}{2} \frac{V_\perp^2}{\Omega c} \frac{\hat{b} \times \nabla B}{B} \sim V_\perp \frac{\rho}{R} \ll 1 \]
Adiabatic Invariance of $\mu$

$B = B_0(t) \hat{z}$

$B$ is uniform, but time-varying.

(From here on, heavily borrowed from Charles Kainey's lecture notes...)

Assume variation of $B$ is slow:

$$\frac{1}{B} \frac{\partial B}{\partial t} \ll \Omega$$

Faraday's law:

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}$$

$\Rightarrow \mathbf{E}$ $\Rightarrow$ energy of particle is not conserved.

$$\frac{d}{dt} \left( \frac{1}{2} m v_z^2 \right) = m v_z \cdot \frac{d v_z}{dt} = v_z \cdot q \mathbf{E}$$

Integrate over 1 cyclotron period $\frac{2\pi}{\Omega}$

$$\Delta \mathbf{v}_z = \Delta \frac{1}{2} m v_z^2 = \oint v_z \cdot q \mathbf{E} \, dt$$

$$= q \oint \mathbf{E} \cdot d\mathbf{z}$$

$$= q \oint_A \nabla \times \mathbf{E} \cdot dA$$

Stokes theorem

$$= -q \oint_A \nabla \frac{\partial \mathbf{B}}{\partial t} \cdot dA$$

Faraday

$$\frac{\Delta \mathbf{B}}{\Delta t} = \Delta \mathbf{B} \frac{\Omega}{2\pi}$$

$$A = -\pi \rho^2 = -\pi \left( \frac{v_z}{c} \right)^2$$

Sense of area is negative for positive $\mathbf{n}$.
\[ \Delta W_\perp = +\frac{8}{e} \frac{AB \rho_t}{m^2} \frac{1}{2\pi} \frac{q^2 W_1}{\Omega^2} \]

\[ \Rightarrow \Delta W_\perp = \Delta B \frac{W_\perp}{B} \]

\[ \frac{\Delta W_\perp}{W_\perp} = -\frac{\Delta B}{B} \]

or \[ \Delta \left( \frac{W_\perp}{B} \right) = 0 \]

\[ \Delta (\ln W_\perp) = \Delta \ln B \]

\[ \Delta (\ln W_\perp - \ln B) = 0 \]

\[ \frac{W_\perp}{B} = N \quad \text{adiabatic invariants} \]

\[ N \approx \text{magnetic moment} \]

Note: \( N \approx \pi \rho^2 B = \text{flux through gyro-orbit} \)

Gyrating particle ring acts like a perfectly conducting ring, conserving flux through its orbit very accurately, if the rate of change of \( B \) is slow compared to \( \Omega \).

Requires \( \omega \approx \Omega_c \) resonant perturbations to efficiently change \( N \). If \( \frac{\omega}{\Omega_c} \approx 1 \), then \( N \) is well conserved:

\[ \frac{\Delta N}{N} \sim \varepsilon \]

Kruskal:

\[ e^{-\frac{1}{\varepsilon}} \]

\[ f(\varepsilon) = f(0) + \varepsilon \frac{\partial f}{\partial \varepsilon} + \frac{1}{2} \varepsilon^2 \frac{\partial^2 f}{\partial \varepsilon^2} + \ldots \]
Mirror Effect

\[ \frac{\partial B}{\partial z} > 0 \]

\[ E = 0 \]

\[ W = \text{constant} \]

\[ W = W_{\perp B} + W_{\parallel B} \]

But \( \mu \) conservation \( \Rightarrow \) \( W_{\perp B} \approx B \)

\[ W = NB + \frac{1}{2}mv_1^2 \]

\[ \Rightarrow W_{\parallel B} \text{ becomes 0 at some point} \]

- particle is reflected

\( z \) component of force comes from radial component of \( B \)
Mirror force reflects particles near mirror throat.

Conservation of $\mu$ and $W$:

Particle which reflects at $B_2$ has energy $W = W_{12} = \mu B_2$ \( (v_{\perp i} = 0) \)

At $B_1$, it has the same energy $W = W_{11} + W_{\perp 1}$

$$= \mu B_1 + \frac{1}{2} m v_{\perp 1}^2$$

Can show that all particles with

$$\left. \frac{v_{\perp 1}}{v} \right|_{at \, B_1} > \sqrt{\frac{B_1}{B_2}}$$

are reflected.
Guiding Center Motion Summary

\[ \Omega = \frac{q \mathbf{B}}{m c} \quad \text{gyrofrequency} \]

\[ \rho = \frac{v_\perp}{\Omega} \quad \text{gyroradius} \]

\[ \mathbf{r} = R_{gc} + \rho \mathbf{\hat{r}} \]

Particle Position \quad \text{Guiding-center position}

Rapidly varying gyro-radius vector

For static fields (\(v \mathbf{E}\) not too strong...), the dominant guiding center drifts are:

\[ \frac{dR_{gc}}{dt} = v_\parallel \mathbf{\hat{b}} + \mathbf{v}_d \]

\[ \mathbf{\hat{b}} = \frac{\mathbf{B}}{B} \]

\[ \mu = \frac{W_0}{B} = \text{constant} \]

\[ W = \mu B + qB E + \frac{1}{2} m v_\perp^2 = \text{constant} \]

\[ \mathbf{v}_d = \frac{c}{B^2} \mathbf{E} \times \mathbf{B} + \frac{1}{\Omega} \frac{v_\parallel^2}{B} \mathbf{\hat{b}} \times \nabla B + \frac{v_\parallel^2}{\Omega} \mathbf{\hat{b}} \times (\mathbf{\hat{b}} \cdot \nabla \mathbf{\hat{b}}) \]

"\(\mathbf{E} \times \mathbf{B}\)" \quad "\(\nabla B\)" \quad "Curvature drift"

polarization drift \(\propto \frac{dE}{dt}\) important for understanding instabilities & waves from particle viewpoint.

(see textbooks...)
Particle motion in dipole magnetic field, showing bounce motion along magnetic field between magnetic mirrors, and slower grad(B) drift across magnetic field.

http://www-ssc.igpp.ucla.edu/ssc/spgroup_edu.html
This method can be extended to

**Systematic Derivation of Complete Guiding Center Drifts**

For details, see:


(Most textbooks derive particle drifts piecemeal, not systematically...)


Base idea:

\[ \mathbf{r} = R_{gc} + \mathbf{f} \]

- \( R_{gc} \) : Fast gyro-motion
- \( \mathbf{f} \) : Slowly moving guiding center

Assume gyroradius \( \rho \) is small & expand: \[ \mathbf{B} = B(R_{gc}, t) + \mathbf{f} \cdot \nabla \mathbf{B} \]

Assume \( \Omega \) is very large & average equations of motion over fast gyro-motion to obtain an equation for the slower moving guiding-center...

Subtleties! Various ways to expand \( E \) & MHD ordering (macro-instabilities)

Alternative: Modern Hamiltonian methods, non-canonical Lie perturbation theory, useful for going to higher order, investigate accuracy of adiabatic invariants, etc...
Nonlinear MHD-like behavior couples many of the time- & length-scales.

Even within the context of resistive MHD modeling, there is stiffness and anisotropy in the system of equations.

Even with the most powerful computers expected in the next 20 years, there are many problems with such an extreme range of scales that they can’t be directly solved…
Vlasov-Boltzmann Kinetic Eq.

for Self-Consistent Collective Dynamics

\[ f_s(x, v, t) \] = particle distribution function in phase space, position \( x \), velocity \( v \) at time \( t \) for species \( s \).

\[ n_s(x, t) = S d^3 v f_s \]

\[ n_s u_s = S d^3 v f_s v \]

\[ \frac{\partial f_s}{\partial t} + \frac{\partial}{\partial x} \left[ v_0 f_s \right] + \frac{\partial}{\partial v} \left[ \frac{q_s}{m_s} \left( \frac{E}{m} + \frac{v \times B}{c} \right) f \right] = G[f_s] \]

Conservation law in phase-space.

\[ G[f_s] = \text{Landau collision operator} \]

- velocity diffusion + drag Fokker-Planck Eq.
- (small-angle collisions dominate)
- Causes \( f \) to relax towards Maxwellian.

(Similar operator for collisions of stars in galaxies 
gravity + electromagnetic forces)
Maxwell's Eqs, need
charge density \( \sigma = \sum s q_s n_s \)
current density \( j = \sum s q_s n_s u_s \)

Vlasov+Maxwell \( \Rightarrow \) starting point for most kinetic theory

**Conservation laws follow from \( \nu \)-integration!**

\[
\frac{\partial n_s}{\partial t} + \frac{\partial}{\partial x} \left( n_s u_s \right) = 0
\]

particle conservation

\[
\frac{\partial}{\partial t} \left( m_s n_s u_s \right) + \frac{\partial}{\partial x} \left[ m_s n_s u_s u_s \right] = -\frac{\partial}{\partial x} \cdot \mathbf{P}
\]

momentum conservation

\[
+ q_s n_s \left( E + \frac{u_s x B}{c} \right)
\]

Sum over species \( \sigma \) use quasineutrality \( (\sigma E \rightarrow 0) \)
to get standard MHD. (Need \( \sigma E \) for relativistic MHD)

\[
\mathbf{P}_{s,ij} = m_s \int d^3v f_s \left( v_i - u_{s,i} \right) \left( v_j - u_{s,j} \right)
\]
If collisions are sufficiently rapid,

\[ f_s = \text{Maxwellian} + \text{small corrections} \]

(\( \Rightarrow \) Braginskii thermal conduction + viscosity \( \cdots \))

\[ P_{s,i,j} = P_s S_{i,j} \quad \frac{\partial P_s}{\partial x} = \frac{\partial P_s}{\partial x} = \nabla P_s \]

\[ \int d^3v \frac{1}{2} m v v \star \]

\[ \leftarrow \]

energy conservation

+ infinite chain of moments hierarchy

closed if collisions are rapid.

otherwise need kinetic theory

or model thereof
Generalized Ohm's Law (not completely general-dropped $\frac{me}{m_i}$ terms-electron inertia).

\[
\begin{align*}
E + \frac{\nu \times B}{c} &= \frac{\hat{x} \times B}{ne c} - \frac{\nabla P_e}{ne} + \eta \frac{\hat{z}}{2}
\end{align*}
\]

\[\uparrow \quad \text{"Hall term"} \]

\[\text{"FLR terms"} \]

\[\text{Finite Larmor Radius (Gyro)}\]

In "MHD", this is usually simplified further to:

\[
\begin{align*}
E + \frac{\nu \times B}{c} &= \eta \frac{\hat{z}}{2} \\
&= E' \text{ in frame moving with fluid}
\end{align*}
\]

"Resistive MHD"

or

\[
E + \frac{\nu \times B}{c} = 0
\]

"Ideal MHD" perfectly conducting fluid.
Fundamental Ordering Assumptions of MHD

Magneto-Hydro-Dynamics

Look for phenomena which are:

- slow compared to cyclotron frequencies \( \Omega_{ce}, \Omega_{ci} \)
- large scale compared to gyroradii \( \rho_i, \rho_e \)

\[
\frac{\Omega}{\Omega_{ci}} \sim \frac{\rho_i}{L} \sim \varepsilon \ll 1
\]

\[
\frac{\nabla}{\nabla} \sim \frac{1}{L}
\]

Allow flow velocity of plasma \( u \sim \frac{cE}{B} \sim v_{ci} \)

+ allow \( \beta \sim \frac{P}{B^2/8\pi} \sim O(1) \)

(subsidiary orderings, \( \beta \ll 1 \), allowed later)

Later do self-consistency checks to see if these are satisfied.
\[ \frac{\partial \mathbf{B}}{\partial t} \text{ in the MHD Approximation} \]

\[ \frac{\partial \mathbf{B}}{\partial t} = -c \nabla \times \mathbf{E} = \nabla \times (\mathbf{u} \times \mathbf{B}) - c \nabla \times (\eta \mathbf{j}) \]

using Ohm's Law:

\[ \mathbf{E} + \frac{\mathbf{u} \times \mathbf{B}}{c} = \eta \mathbf{j} \]

\[ \nabla \times \mathbf{B} = \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c^2} \mathbf{j} \]

Displacement current is ignored in the MHD approximation.

\( \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} \sim \frac{1}{c} \frac{\omega \mathbf{E}}{B} \sim \frac{\omega L}{c^2} \frac{cE}{B} \sim \frac{u^2}{c^2} \ll 1 \)

So in MHD, will make the "Magnetostatic" approximation

\[ \nabla \times \mathbf{B} = \frac{4\pi}{c^2} \mathbf{j} \quad \text{zeroth order Light waves out of our Eqs.} \]

\[ \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) - \frac{c^2}{4\pi} \nabla \times (\eta \nabla \times \mathbf{B}) \]
If $\eta \approx \text{const}$, use NRL Vector ID #14:

$$\frac{dB}{dt} = \nabla \times (u \times B) + \frac{c^2 \eta}{4\pi} \left[ \nabla^2 B + \nabla (\nabla \cdot B) \right]$$

**Magnetic Diffusion Coefficient**

If $u = 0$, $B$ decays on a time scale

$$\frac{1}{T_{\text{skin}}} \approx \frac{c^2 \eta}{4\pi L^2}$$

$$\Rightarrow T_{\text{skin}} \approx 8 \times 10^{-8} L_c \text{cm} \ T_{\text{ev}} \ \text{sec}$$

Where $L$ is a "typical" length scale for $B$ gradients.

$T_{\text{skin}} \approx 1$ sec for a copper sphere of 1 cm radius

$\sim 10^4$ years for the molten core of the Earth

$\sim 10^{10}$ years for a typical magnetic field in the Sun

(J. D. Jackson, 2nd Edition, p. 473)

$\sim 25$ sec for a tokamak with $L \sim 100 \text{ cm}$ & $T \sim 1 \text{ keV}$

$\sim 10^{27}$ years for a galaxy with $L \sim 10^{21} \text{ cm}$ & $T \sim 1 \text{ eV}$

"Important problems in Astrophysics"


However, regions of sharp gradients (such as shocks, or magnetic tearing layers) can have $L \Rightarrow 0$ and undergo magnetic diffusion more quickly.
\[
\Phi_B(t) = \int_{x_0(t)}^{x_1(t)} \, B(x, t) \, dx
\]

\[
\frac{d\Phi_B}{dt} = \int_{x_0(t)}^{x_1(t)} \frac{2B}{\partial t} + \frac{dx_1}{dt} B(x_1, t) - \frac{dx_0}{dt} B(x_0, t)
\]

**Alternate Proof**

\[
\Phi_0(t) = \int_{-\infty}^{\infty} \, H(x_1(t) - x) \, H(x - x_0(t)) \, B(x, t) \, dx
\]

\[
\frac{d\Phi_B}{dt} = \int_{x_0}^{x_1} \frac{2B}{\partial t} + \int_{-\infty}^{\infty} \left\{ \frac{H'(x_1 - x)}{\partial t} \, \delta(x - x_1(t)) - \frac{H'(x - x_0)}{\partial t} \, \delta(x - x_0(t)) \right\} B(x, t)
\]

\[
Q.E.D.
\]
**Flux - conservation**

The magnetic flux through any closed contour \( C \) that moves with the fluid remains unchanged:

\[
\Phi_B = \int_C d\mathbf{S} \cdot \mathbf{B}
\]

\[
\frac{d\Phi_B}{dt} = \int_C d\mathbf{S} \cdot \frac{\partial \mathbf{B}}{\partial t} + \oint_C \frac{\partial d\mathbf{S}}{\partial t} \cdot \mathbf{B}
\]

\[
= \int_C d\mathbf{S} \cdot \nabla \times (\mathbf{u} \times \mathbf{B}) - \oint_C d\mathbf{l} \times \mathbf{u} \cdot \mathbf{B}
\]

\[
= \oint_C d\mathbf{l} \cdot \mathbf{u} \times \mathbf{B} \quad \text{(cancel)}
\]

\[
\frac{d\Phi_B}{dt} = 0
\]
Frozen-in Field Lines (for ideal MHD with \( \rho = 0 \))

(See S. von Goeler, Lecture XIV for proof)

or Goldston + Axford, Sec. 8.5 first + then Sec. 8.4.

2 parts:

(a) The magnetic flux through a closed contour \( C \) that moves with the fluid remains unchanged.

(b) Two fluid elements that initially lie on a magnetic field line continue to lie on a field line.

Theorem: Field lines are frozen into a Perfectly Conducting Fluid proposed by Alfvén.
Even without plasma compression, can get magnitude of $B$ to be enhanced:

\[ B \]

Later

\[ |B| \text{ is larger here!} \]

"Dynamo" generation of $B$: Earth? Sun?

galactic + intergalactic magnetic fields?
The Fluid Universe

Most of the “big” questions in astrophysics require studying the fluid dynamics of the visible matter.

• How do galaxies form?
• How do stars form?
• How do planets form?

This requires solving the equations of radiation magneto-hydrodynamics (MHD).
It’s a fluid!
Some of this is a fluid too.
MHD equations: conservation laws

...for mass, momentum, energy, and magnetic flux.

Mass conservation:
Rate of change of mass in a volume is divergence of fluxes through surface

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \]

\( \rho = \) mass density
\( \mathbf{v} = \) velocity
\( \frac{\partial}{\partial t} = \) Eulerian derivative (at a fixed point in space)
\( \frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla = \) Lagrangian derivative (moving with flow)
Momentum conservation:

Rate of change of momentum within a volume is divergence of stress on surface of volume (no viscous stress)

\[ \frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot [\rho \mathbf{v} \mathbf{v} - \mathbf{BB} + P^*] = 0 \]

Energy conservation:

Rate of change of total energy density \( E \) is equal to the divergence of energy flux through the surface

\[ \frac{\partial E}{\partial t} + \nabla \cdot [(E + P^*) \mathbf{v} - \mathbf{B} (\mathbf{B} \cdot \mathbf{v})] = 0 \]

\( E = \rho \, \mathbf{v}^2/2 + e + B^2/2 \) is total energy

\( P^* = P + B^2/2 \) is total pressure (gas + magnetic)
Flux conservation:

Given by Maxwell’s equations:
\[ \frac{\partial \mathbf{B}}{\partial t} = -c \nabla \times \mathbf{E} \]
\[ \nabla \cdot \mathbf{B} = 0 \quad \text{(constraint rather than evolutionary equation)} \]

From Ohm’s Law, the current and electric field are related by
\[ \mathbf{J} = \sigma \left( \mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c} \right) \]
For a fully conducting plasma, \( \sigma \to \infty \)
So \( c\mathbf{E} = -(\mathbf{v} \times \mathbf{B}) \).

\[ \frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) = 0 \]
The results are the equations of compressible inviscid ideal MHD:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot [\rho \mathbf{v} \mathbf{v} - \mathbf{B} \mathbf{B} + P^*] = 0$$

$$\frac{\partial E}{\partial t} + \nabla \cdot [(E + P^*) \mathbf{v} - \mathbf{B} (\mathbf{B} \cdot \mathbf{v})] = 0$$

$$\frac{\partial B}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) = 0$$

Where $E = \rho \frac{v^2}{2} + e + \frac{B^2}{2}$ is total energy

$P^* = P + \frac{B^2}{2}$ is total pressure (gas + magnetic)

Plus an equation of state $P = P(\rho, T)$

**Warning:** used units so that $\mu = 1$
Can also be written in non-conservative form

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \]

\[ \frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot (\rho \mathbf{v} \mathbf{v}) = -\nabla p + \frac{1}{4\pi} (\nabla \times \mathbf{B}) \times \mathbf{B} \]

\[ \frac{\partial e}{\partial t} + \nabla \cdot (e \mathbf{v}) = -\frac{p}{\rho} \nabla \cdot \mathbf{v} \]

\[ \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) \]

Plus an equation of state \( P = P(\rho, T) \)

Useful form for numerical methods based on operator splitting (lecture 2)
Equation of state

Usually adopt the ideal gas law \[ P = nkT \]

In thermal equilibrium, each internal degree of freedom has energy \((kT/2)\). Thus, internal energy density for an ideal gas with \(m\) internal degrees of freedom

\[ e = nm(kT/2). \]

Combining,

\[ P = (\gamma - 1)e \]

where \(\gamma = (m+2)/m\)

For monoatomic gas (H), \(\gamma = 5/3\) \((m = 3)\)

diatomic gas (H\(_2\)), \(\gamma = 7/5\) \((m = 5)\)

Also common to use isothermal EOS \( P = C^2 \rho \) where C=isothermal sound speed when (radiative cooling time) \(<\ll\) (dynamical time)

In some circumstances, an ideal gas law is not appropriate, and must use more complex (or tabular) EOS (e.g. for degenerate matter)
Another important characteristic of hyperbolic PDEs is they admit solutions of the form:

\[ a = a_0 + a_1 \exp(ik \cdot x - i\omega t) \]  

(WAVES)

When \( a_1/a_0 \ll 1 \); waves are small amplitude; linear
When \( a_1/a_0 > 1 \), waves are large amplitude, nonlinear (in this case, plane wave solution does not persist, for example nonlinear terms cause steepening)

Movie of density in linear sound wave

Linear waves are produced by small amplitude disturbances, with \( v < C \) (sound waves)
Dispersion relation for *hydrodynamic* waves.

Substitute solution for plane waves into hydrodynamic equations. Assume a uniform homogeneous background medium, so \( a_0 = \text{constant} \), and \( v_0 = 0 \). Keep only linear terms. Fluid equations become:

\[-i\omega \rho_1 = -i \rho_0 k \cdot v_1 \]
\[-i\omega v_1 = -i \frac{1}{\rho_0} k P_1 \]
\[-i\omega P_1 = -i\gamma P_0 k \cdot v_1 \]

Linear system with constant coefficients! Solutions require \( \text{det}(A) = 0 \), which requires

\[
\omega^3(\omega^2 - C^2 k^2) = 0
\]

where \( C^2 = \gamma P_0 / \rho_0 \) is the adiabatic sound speed

Apparently 5 modes; 3 advection modes and 2 sound waves with \( \omega / k = \pm C \)
Summary of wave modes in *hydrodynamics*:

1. Entropy waves. Advect constant density field at V.

   ![Entropy waves](image)

   e.g. advection of sinusoidal density profile

2. Sound waves. Density, velocity, and pressure fluctuations that propagate at V+C and V-C.

   ![Sound waves](image)
Dispersion relation for MHD waves.

Substitute solution for plane waves into MHD equations. Assume a uniform homogeneous background medium, so $a_0=\text{constant}$, and $v_0=0$. Get a much more complicated dispersion relation (derivation is non-trivial! see Jackson):

$$\left[\omega^2 - (\mathbf{k} \cdot \mathbf{v}_A)^2\right]\left[\omega^4 - \omega^2 k^2 (v_A^2 + C^2) + k^2 C^2 (k \cdot \mathbf{v}_A)^2\right] = 0$$

Where $\mathbf{v}_A = \frac{\mathbf{B}}{\sqrt{4\pi\rho}}$ is the Alfven speed

$$C^2 = \gamma p_0 / \rho_0$$

is the sound speed

There are three modes (only one in hydrodynamics!):

Alfven wave propagates at $V_A$

Slow and fast magnetosononic waves propagating at $C_s$ and $C_f$

(Of course, the entropy mode is also present in both cases)
MHD Wave Modes.

1. **Alfven Waves**  
   Zero-frequency when \( k \) perpendicular to \( B \) (propagate along \( B \)), incompressible. Represent propagating transverse perturbations of field.

\[
V_A = \frac{B}{\sqrt{4\pi\rho}}
\]

2. **Fast and Slow Magnetosonic Waves**  
   Compressible perturbations of both field and gas.  
   Fast mode has field and gas compression in phase  
   Slow mode has field and gas compression out of phase.

\[
C_{f,s}^2 = \frac{1}{2} \left( [C^2 + C_A^2] \pm \sqrt{[C^2 + C_A^2]^2 - 4C^2C_{Ax}^2} \right)
\]

\[
C_A^2 = \frac{(B_x^2 + B_y^2 + B_z^2)}{i(4\pi\rho)}, \quad C_{Ax}^2 = \frac{B_x^2}{(4\pi\rho)}
\]
Phase velocities of MHD waves: Friedrichs diagrams.

Note for in some cases, modes are degenerate. Eigenvalues of linearized MHD equations are not always linearly independent. MHD equations are not strictly hyperbolic.
Linear Instabilities

Going beyond the study of waves and shocks in fluids requires learning about the zoo of MHD instabilities in fluids.

Probably the most important are:

1. Gravitational instability.
2. Thermal instability.
3. Rayleigh-Taylor (RT) instability.
4. Richtmyer-Meshkov (RM) instability.
5. Kelvin-Helmholtz (KH) instability.
6. Magneto-rotational instability (MRI)
7. Kink instability (current driven)
8. Sausage/Ballooning instability (pressure-gradient driven)

See monographs by Chandrasekhar 1965
Drazin & Reid 1981
Further Plasma References

- www.pppl.gov
- many more…

- Textbooks:
  - F. F. Chen simplest introduction with many physical insights
  - Goldston & Rutherford, somewhat more advanced, but still for beginning graduate student or upper level undergraduate
  - Kulsrud, Plasma Physics for Astrophysics
  - Many others, some much more mathematical or advanced: Hazardine & Waelbroeck, Friedberg, Boyd & Sanderson, Dendy, Bittencourt, Wesson, Krall & Trivelpiece, Miyamoto, Ichimaru, Spitzer (elegantly brief), Stix, others
  - Blandford & Thorne’s draft book has chapters on plasma physics: http://www.pma.caltech.edu/Courses/ph136/yr2006/text.html