Homework 2 Solutions

General grading rules: 3 points off per arithmetic, algebraic, or conceptual mistake. 1 point off for grossly too many significant figures. 1 point off for giving answers without units. In problems 1 and 2, take off half the points in each problem if the python code is not included.

1. The Colors of Stars *50 points*
We define the color of an astronomical object as the difference in the apparent magnitude of that object as measured in two different filters. In this problem, we will use AB magnitudes. Here, you will calculate the colors of stars using SDSS filters, approximating their spectra as blackbodies. In what follows, consider a sequence of stars (and substellar objects, or brown dwarfs) with the following surface temperatures (all in degrees Kelvin): $1000, 1500, 2000, 2500, 3000, 4000, 6000, 8000, 10^4, 1.5\times10^4, 2\times10^4, 3\times10^4, 5\times10^4$, and consider their brightness as measured through the SDSS filters, whose central wavelengths and widths are as follows:

Filter	$\lambda_{\rm central}$ (Å)	$\Delta\lambda$ (Å)
u	3551	581
g	4686	1262
r	6166	1149
i	7480	1237
z	8932	994

That is, you should approximate each filter as a top-hat, centered on the central wavelength listed, and having a width given by the value of $\Delta\lambda$.

a. *30 points* Calculate the following colors for each star: $u-g, g-r, r-i$, and $i-z$, and plot them as a function of temperature. *Hint: We did not tell you the size of the star, nor its distance. Why do you not need this information? Are there any assumptions you need to make to do this calculation? Also, you will need to numerically integrate the black-body formula. This can be done by approximating the integral as a sum; explain the details of how you've done this, and describe your choice of how finely you bin the integrand.*
Solution: To calculate the effective flux density through the filter, we integrate over the spectrum and filter. We compute the mean value of f_ν (dividing by the energy of each photon, so that we count photons) over the filter,

$$\langle f_\nu \rangle = \frac{\int f_\nu(\nu) d\nu \, \frac{1}{h\nu} \, R(\nu)}{\int d\nu \, \frac{1}{h\nu} \, R(\nu)}. \tag{1}$$

Note the normalization in the denominator; this gives us a result with the units of f_ν, which we need for the magnitude formula. So we can plug the result into the formula for AB magnitude:

$$m_{\rm AB} = -2.5 \log_{10} \left(\frac{f_\nu}{\rm erg\, s^{-1}\, cm^{-2}\, Hz^{-1}} \right) - 48.60. \tag{2}$$

Remember that we must first convert all of the ν's and $d\nu$'s to λ's and $d\lambda$'s. We do this with the chain rule, so that

$$f_\nu = \frac{df}{d\nu} = \frac{df}{d\lambda}\frac{d\lambda}{d\nu} = f_\lambda \frac{\lambda^2}{c}$$

and

$$d\nu = d\lambda \frac{d\nu}{d\lambda} = \frac{c}{\lambda^2} d\lambda.$$

There is a minus sign which we have dropped; it simply tells us that as frequency *increases*, wavelength *decreases*. Thus, after a little algebra, we may rewrite Eq. (1) as

$$\langle f_\nu \rangle = \frac{\int f_\lambda(\lambda) R(\lambda) \frac{\lambda}{hc} d\lambda}{\int R(\lambda) \frac{d\lambda}{h\lambda}}, \tag{3}$$

which we have implemented as a simple sum in the accompanying Python code.

There are more sophisticated ways to do the integration, using higher order approximations. One of the more popular choices is Simpson's rule, which approximates the integrand by piecewise parabolas and gives more accurate values than the summation I have implemented, at least for smooth functions. It is given by:

$$\int_{x_0}^{x_N} f(x)dx \approx \frac{\Delta x}{3} \sum_{i=0}^{N} \left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \ldots + 4f(x_{N-1}) + f(x_N) \right]. \tag{4}$$

Formally, we say it is a fourth order method, since its error is proportional to $(\Delta x)^4$ for smooth functions and small Δx.

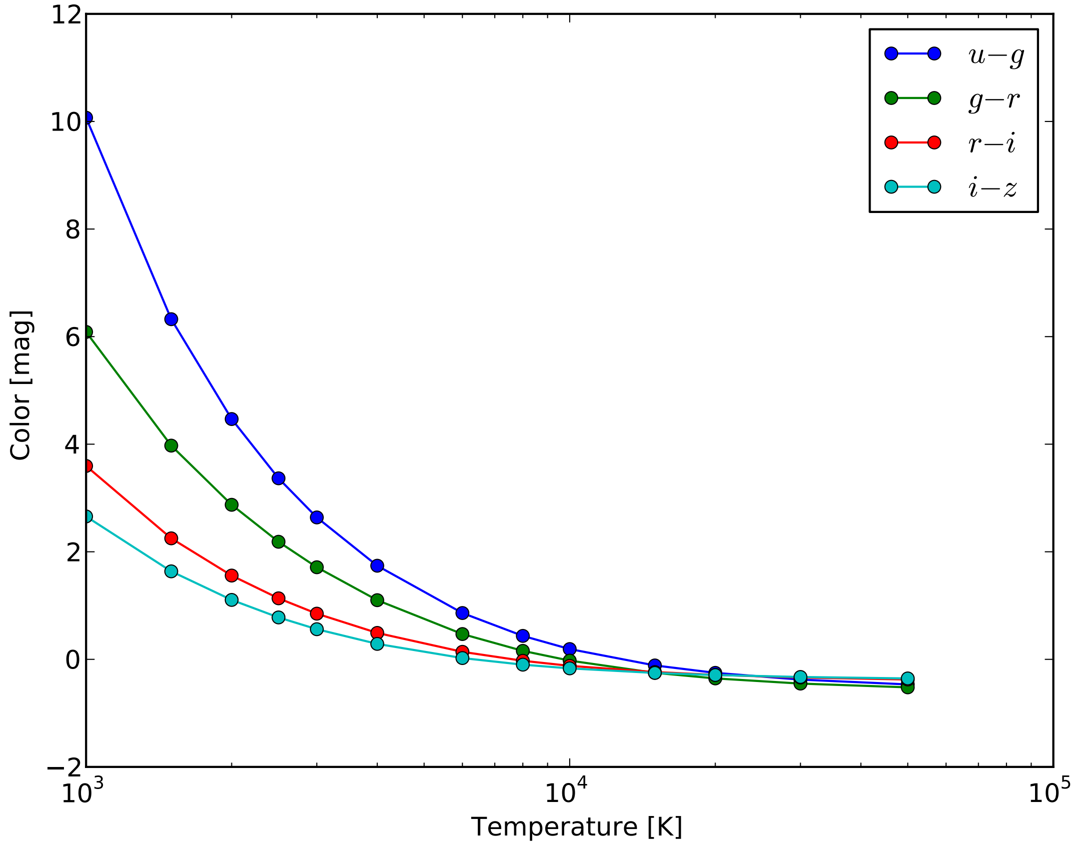
For a perfect black body, the flux density is proportional to:

$$f_\lambda \propto \frac{1}{\lambda^5} \frac{1}{e^{hc/\lambda k_B T} - 1} \tag{5}$$

Using (3) and performing the sum numerically, we find the results shown in the figure.

Note that the results do not depend on distance or the size of the star; the distance and radius factors are common to the expression for the magnitude in all filters, and thus drop out of the calculation of the color (a *difference* of magnitudes, and thus a *ratio* of flux densities). Indeed, the proportionality constant in the flux density formula above drops out as well, which is why we didn't bother including it.

Full credit for any reasonable approach to the numerical integration. 10 points off for simply using the flux density of the black body at the center of the filter, with no further discussion. 3 points off for no discussion of the reason we need not know the radius or distance of the star. 2 points off for neglecting the $1/h\nu$ term in the integral or for not normalizing. Points off if no computation or no code, dependent upon work shown. 2 points off for no labels. 2 points off for wrong values.



b. *10 points* You should have found that at high temperatures, the colors asymptote to a constant. Can you explain this behavior? In the limit of high temperature (high compared to what?), calculate the colors analytically, and compare your results with those in part (a). This insensitivity of color to temperature means that it is difficult to distinguish between stars of high temperature from their broad-band colors alone.

Solution: At high temperature ($T \gg hc/\lambda k_B$), the argument in the exponential in the blackbody formula becomes small, so we can expand it to find:

$$f_\lambda \propto \frac{1}{\lambda^5} \, \frac{1}{e^{hc/\lambda k_B T} - 1} \approx \frac{1}{\lambda^5} \frac{\lambda k_B T}{hc} \propto \frac{1}{\lambda^4} T \tag{6}$$

We see that at high T, the flux factorizes into a function of λ only and a function of T only, such that the ratio of fluxes (and hence the color) is independent of temperature. This is often referred to as the *Rayleigh-Jeans* limit. If we had kept all the constants in the above expression, we would have found that Planck's Constant drops out: this expression can be derived (and was done so by Rayleigh and Jeans) without any reference to quantum-mechanical effects or photons.

Full credit for getting this far. No points awarded if exponential was not expanded in the high T limit.

Using equation (3) for a narrow bin, we find that the color $x-y$ is given approximately by

$$x - y = -2.5 \log_{10} \left(\frac{\lambda_y^2}{\lambda_x^2} \right), \tag{7}$$

independent of temperature. It turns out that the wavelength ratios of the pairs of filters you are considering here are almost all the same, which means that all the colors asymptote to roughly the *same* constant.

c. *10 points* On the course home page, you will find the tabulated spectrum of an A0V star; the first two columns are wavelength (in Ångstroms) and flux density f_λ (in units of $10^{-17}\,\mathrm{erg\,s^{-1}\,cm^{-2}\,\AA^{-1}}$; this spectrum is from a compilation by Pickles 1998, PASP, 110, 863). Calculate the colors of this star from this spectrum, and compare with the results you found for the 10,000 K black-body (roughly the effective temperature of the star). Are they in good agreement? If not, why not? What would the colors of this star be on the Vega system? *Hint: Look up the spectral type of the star Vega.*

Solution: Using the code available on the course website, I find the following colors on the AB system for the A0V star:
$u - g = 1.07$
$g - r = -0.24$
$r - i = -0.18$
$i - z = -0.16$

The code above gave the following for the 10,000 K black-body:
$u - g = 0.44$
$g - r = 0.16$
$r - i = -0.03$
$i - z = -0.10$

Note that the biggest difference between a 10,000 K black-body and the real star occurs in the u band, where the spectrum has a huge jump (the so-called Balmer break, produced by absorption from hydrogen atoms in the first excited state to the continuum). This is where the spectrum least resembles a black-body.

Vega is an A0V star, like this one, and thus has the same colors. By definition, the magnitude of Vega is 0.00 in the Vega system, *in any filter*. Thus *all* its colors are 0.00.

Just to confuse you further, it turns out that people have found errors in the Vega system, such that the actual best calibrated magnitude of Vega itself, on the Vega system, is not exactly 0, but rather about 0.03. Yet another reason to avoid the Vega system of magnitudes!

7 points for the calculation; 3 points for any reasonable discussion of the difference of magnitude between the blackbody and the A0V star, and the colors of Vega. 2 points off if values are wrong.

2. The relationship between broad-band and monochromatic magnitudes *25 points*

One typically observes a star through a broad filter (as we'll see in the next problem), while the magnitude scale refers to a monochromatic flux. Consider a star whose spectrum is described by a power law:

$$f_\lambda = A\left(\frac{\lambda}{\lambda_0}\right)^\gamma. \tag{8}$$

You measure this star with a photon-counting detector through a filter which is a tophat: i.e., it lets 100% of the light through in the wavelength range $\lambda_0 - \frac{\Delta\lambda}{2}$ to $\lambda_0 + \frac{\Delta\lambda}{2}$, and no light through elsewhere. Calculate the average flux density of this star measured through the filter as a function of A and the power-law index γ, and expand in orders of $\Delta\lambda$, to the first non-vanishing term in $\Delta\lambda$. Make sure that your result has units of flux density! While the flux density at λ_0 is independent of γ, the flux density averaged over the filter does depend on γ. Stellar spectra of different temperatures have γ ranging from -2 to $+2$; by what factor does the flux density over the filter vary over this range? Do this calculation for $\Delta\lambda/\lambda_0 = 0.02$ (an intermediate-band filter) and $\Delta\lambda/\lambda_0 = 0.2$ (a broad-band filter).

Solution: Our instrument measures photon flux. Through a filter R, this flux (in units of, say, photons $\mathrm{cm^{-2}\,s^{-1}}$) is given by

$$f = \int f_\nu(\nu)\, R(\nu)\, \frac{1}{h\nu} d\nu = \int f_\lambda(\lambda)\, R(\lambda)\, \frac{\lambda}{hc} d\lambda. \tag{9}$$

We may convert this into a monochromatic flux by dividing by an average energy per photon and an effective wavelength or frequency. Generally, these normalizations are defined as

$$\mathcal{N}_\lambda \equiv \int R(\lambda)\, \frac{\lambda}{hc} d\lambda \quad \text{and} \quad \mathcal{N}_\nu \equiv \int R(\nu)\, \frac{1}{h\nu} d\nu.$$

Note that these are the photon fluxes for sources with f_λ = constant and f_ν = constant, respectively. Each normalization thus "privileges" a certain value of γ, 0 for $\mathcal{N}_\lambda$ and -2 for $\mathcal{N}_\nu$ (since $f_\nu = -\frac{\lambda^2}{c} f_\lambda$). Another way to think about this is to imagine Eqs. (9) as measurements of f_ν and f_λ weighted by $R(\nu)\frac{1}{h\nu}$ and $R(\lambda)\frac{\lambda}{hc}$, respectively. For example, we imagine a weighting function $W_\lambda = R(\lambda)\frac{\lambda}{hc}$ and then estimate

$$\langle f_\lambda \rangle = \frac{\int f_\lambda W_\lambda d\lambda}{\int W_\lambda d\lambda}.$$

Again, this shows that $\langle f_\lambda \rangle = f_\lambda(\lambda_0)$ for $\gamma = 0$ and $\langle f_\nu \rangle = f_\nu(\lambda_0)$ for $\gamma = -2$. For the rest of the solutions, I'll just calculate the effect of γ on the photon flux, since this is the quantity that our instrument actually measures.

Our task is then to calculate the average photon flux density

$$f = \int_{\lambda_0 - \Delta\lambda/2}^{\lambda_0 + \Delta\lambda/2} A\left(\frac{\lambda}{\lambda_0}\right)^\gamma \frac{\lambda}{hc} d\lambda,$$

where A is the normalization constant from Eq. (8). Changing variables to $x \equiv \lambda/\lambda_0$ and defining $\delta \equiv \Delta\lambda/(2\lambda_0)$, we have

$$f = \frac{A\lambda_0^2}{hc}\int_{1-\delta}^{1+\delta} x^{\gamma+1}dx = \frac{A\lambda_0^2}{hc}\frac{(1+\delta)^{\gamma+2}-(1-\delta)^{\gamma+2}}{\gamma+2}, \tag{10}$$

assuming $\gamma \neq -2$. For $\gamma = -2$, we would end up with

$$f = \frac{A\lambda_0^2}{hc}\int_{1-\delta}^{1+\delta} \frac{dx}{x} = \frac{A\lambda_0^2}{hc}\log\left(\frac{1+\delta}{1-\delta}\right). \tag{11}$$

I won't go through it explicitly, but you can expand the logarithm to get exactly the same approximation for Eq. (11) as for Eq. (10) (try it!).

Let's expand Eq. (10) in δ, which is usually much less than one (and certainly true in our case!). This expansion can be thought of as a Taylor expansion, or a binomial expansion (they are equivalent in this context). We find we have to go to third (!) order to see a non-vanishing term in δ. This makes sense if we realize that we are expanding around the midpoint of the interval. Our first term is proportional to the width of the interval, but the next term (corresponding to the function's first derivative) contributes an equal but opposite area in the $-x$ and $+x$ directions. We get, after a bit of algebra,

$$f = \frac{A\lambda_0^2}{hc}\left(2\delta + \frac{(\gamma+1)\gamma}{3}\delta^3 + O(\delta^5)\right) = \frac{2A\lambda_0^2\delta}{hc}\left(1 + \frac{(\gamma+1)\gamma}{6}\delta^2 + O(\delta^4)\right). \tag{12}$$

The first term in Eq. (12), $2A\lambda_0^2\delta/(hc)$, is equal to the measured photon flux from a source with $\gamma = 0$. The ratio of the photon flux from a source with nonzero γ to our reference object with $\gamma = 0$ is just the ratio of the second term to the first, or $(\gamma+1)\gamma\delta^2/6$. Plugging in numbers, we find that this correction ranges from a factor of $-\delta^2/24$ for $\gamma = -\frac{1}{2}$ to δ^2 for $\gamma = 2$. For an intermediate band filter, with $\delta = 0.01$, the correction is $\sim 1.7\times 10^{-5}\gamma(\gamma+1)$. That is indeed a tiny correction, less than 0.01% for most values of γ and no more than 0.04% for all values in our range. For the broad-band filter, the correction is $1.7\times 10^{-3}\gamma(\gamma+1)$, i.e., less than a 1% effect, which is difficult, albeit not impossible to measure.

15 points for the calculation of the flux algebraically, although 8 points off if there is no expansion in powers of δ. 5 points for any reasonable plugging in of numbers. 2 points off if algebraic error.

3. Local Sidereal Time and Positions on the Sky *25 points*

a. *10 points* The Local Sidereal Time is equal to the right ascension of objects on the meridian. It is therefore very useful to be able to determine the approximate Local Sidereal Time if you want to figure out which objects are available for observations at any given time. You have been scheduled telescope time on the evening of August 14. Calculate the Local Sidereal Time, accurate to the nearest hour, when your watch reads 3 AM. Make sure you've thought about Daylight Savings Time! Now repeat this exercise for a watch time of 10 PM on January 3.

Suppose we asked instead to do the calculation to the nearest *minute*. Describe qualitatively what further effects you should worry about.

Solution: We are only asked to calculate the Local Sidereal Time to the nearest hour, which will save some headaches. First, we use the fact that the Sun is at zero right ascension at solar noon on the vernal equinox. Thus the Sun is at zero hours right ascension at *midnight* on the atumnal equinox, September 21. At midnight, the Local Sidereal Time is 12 hours away from where the Sun is, namely 0 hours right ascension. But September 21 is during the period when we use Daylight Savings Time. Your watch is an hour ahead of solar time. This means that at Local Sidereal Time of 0 hours, your watch reads 1 AM. Thus at 3 AM (on your watch) on September 21, the Local Sidereal Time is 2 hours.

But we've asked about August 14. This is 1 1/4 months earlier. We know that the Local Sidereal Time advances relative to clock time by 2 hours per month, so at the same clock time August 14, we're talking about about a Local Sidereal time about 2 1/2 hours earlier, namely a sidereal time of 23:30.

Let's redo this for 10 PM on January 3. On the Vernal Equinox, the Local Sidereal Time is 12 hours at midnight (and now we don't have to adjust for daylight savings). January 3 is about 2.5 months earlier (corresponding to 5 hours difference in LST), so at midnight, the LST is roughly 7 hours. 10 PM is two hours earlier than that, or 5 hours LST.

To do this more accurately requires worrying about a number of effects. The principal one is the fact that throughout a timezone (which is roughly 15 degrees of longitude wide on the Earth), the clock time is taken to be the same, while of course the sidereal time is a continuous function of longitude at a given moment. We also would need to worry about the ellipticity of the Earth's orbit around the Sun. And we need to worry about the effects of the Earth's precession. Finally, we'd have to be much more careful than we have been here about the offset between the Autumnal Equinox and the time we're examining; the rough approximation of two hours per month is not good enough.

6 points for doing one of the two times correctly; 2 points for the other one. 3 points for a discussion of the zero point of sidereal time, and getting no further. Full credit for a more accurate treatment than we've done here, as long as it is clearly explained. 2 points for any reasonable discussion of what you'd need to do a more precise calculation. 2 points off for algebraic error.

b. *15 points* Determining the Local Sidereal Time allows you to figure out roughly the optimal time to observe any given object; all else being equal, you want to observe an object as close to the meridian as possible. First, why, all else being equal, is it optimal to observe an object at meridian?

Solution: The airmass at which any object is observed is constantly changing as the Earth rotates. The meridian is when the object is highest in the sky, at the lowest airmass, and therefore is least affected by absorption in the atmosphere.

Next, let's think about the *range* of time over which a given object can be observed. For optical astronomy, one usually wants to observe an object when it is more than 30° above the horizon (what is the airmass when it is 30° above the horizon?).

Consider an object whose right ascension is equal to the Local Sidereal Time at midnight on August 14. You are using the Apache Point Telescope in New Mexico, whose latitude on Earth is about +30°. For what period of time is your object more than 30° above the horizon, if the declination of your object is $\delta = 0°$? $+50°$? $+80°$? $-20°$? Express your results in hours. Qualitatively speaking, how do your results change for a telescope at the North Pole? On the Earth's Equator?

Solution: We need to calculate the fraction of the star's arc that is at least 30° above the horizon. If it is at exactly this altitude, the airmass will be $\sec 30° = 2$. Larger airmasses cause all sorts of trouble, from chromatic aberrations to extinction to lots of airglow to poor seeing. Since our star is on the meridian at midnight, we don't have to worry that part of this arc will overlap with daylight. This therefore becomes a geometry problem; see Figure 1 for a diagram of the setup. After a bit of analysis, we find the maximum right ascension difference from the meridian $\mathrm{RA}_{\max}$ is given by

$$\cos(\mathrm{RA}_{\max}) = \frac{(\tan 30°)(1 - \sin\delta)}{\cos\delta}. \qquad (13)$$

We then convert this right ascension into a time that the object is visible by multiplying by 2 (since it could be positive or negative), dividing by the full 2π, and multiplying by the sidereal day's 23.93447 hours. I get values of:

Declination δ	Visible Time ($z > 30°$), hours
0°	7.28
50°	10.4
80°	11.6
−20°	4.58

It is worth noting that a star at $\delta = +80°$ never sets as seen from Apache Point Observatory, although it does go to airmass greater than 2. Note that the North Pole is right at airmass 2 (independent of time), and a star at $\delta = -30°$ just hits airmass 2 when it reaches the meridian.

If the telescope is at the North Pole of the Earth, Polaris is at the zenith (independent of time), and over 24 hours, an object at any given declination lies at an altitude (i.e., 90 minus the zenith angle) equal to that declination, moving in a full circle over 24 hours. Thus its airmass never changes.

For a telescope at the equator, the North Pole appears at the Northern horizon. All declinations are above the horizon, but of course declinations North of +60° or South of −60° never get above airmass 2.

2 points for the airmass at $z = 30°$ and a discussion of why it matters, 2 points for the meridian discussion, 7 points for solving the geometry problem and numerical calculation, and 4 points for a qualitative discussion of what happens when observing from the North Pole or the Equator.

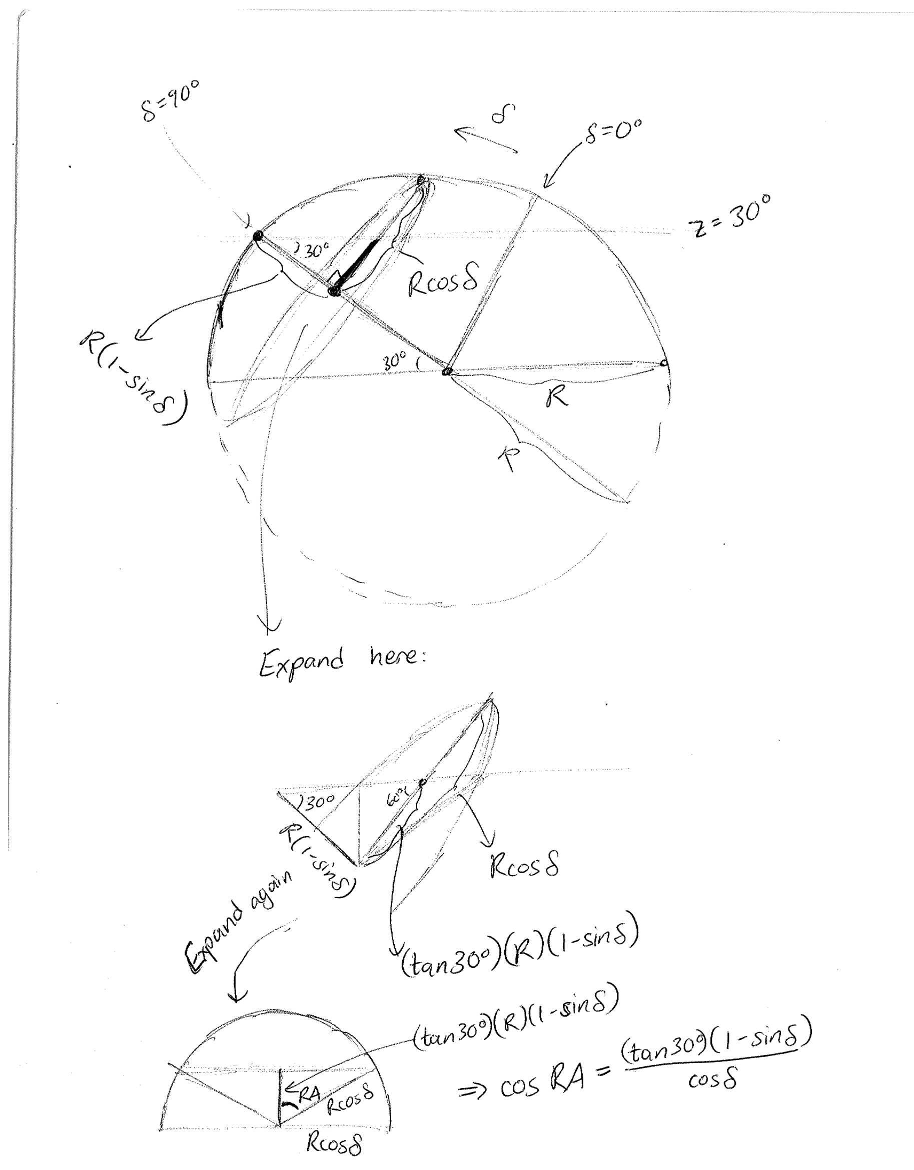


Figure 1: The geometry for Problem 2b