# Homework #1 solutions, AST 303, Fall 2016

**100 points total**

*General grading rules: 3 points off per arithmetic or algebraic mistake. 1 point off for grossly too many significant figures. 1 point off for giving answers without units. If the first part of a calculation is wrong, full credit for later parts that depend on it that are consistent.*

**1. Different types of transitions (70 points)**

In class, we said that electronic transitions (i.e., transitions between energy states of an electron in an atom) typically have energies corresponding to a photon in the optical or ultraviolet region of the spectrum. Vibrational transitions in molecules are of lower energy, and rotational transitions are lower energy still. In this problem, we're going to understand where these statements come from, using order-of-magnitude estimates.

**a.** Let us start with the electronic transition. Go through the classic "old quantum mechanics" Bohr atom argument: consider a single electron in a (classical) circular orbit around a nucleus of charge $Ze$, where $Z$ is the atomic number. The angular momentum of the electron is quantized in units of $\hbar \equiv h/2\pi$. Derive the energy levels of the electron, and calculate the energy (in electron volts) and corresponding wavelength and frequency of the emitted photon in the transition from $n = 2$ to $n = 1$ for $Z = 1$ (hydrogen) and $Z = 26$ (hydrogenic Iron). In what region of the electromagnetic spectrum do these photons lie? *(10 points)*

**Solution:** The angular momentum of the electron, $mvr$ is quantized, $n\hbar$, thus the speed $v = n\hbar/mr$. Here $m$ is the mass of the electron (strictly speaking, I should use the *reduced* mass of the electron, but I'll ignore that tiny correction). But Newton and Coulomb tell us that:

$$m\frac{v^2}{r} = \frac{Ze^2}{r^2}.$$

(I am using cgs units; if I were to use SI units, I would substitute $e^2/4\pi\epsilon_0$ for $e^2$ here and in what follows.) Substituting for $v$, and solving for $r$:

$$\frac{n^2\hbar^2}{mr^3} = \frac{Ze^2}{r^2}$$

$$r = \frac{n^2\hbar^2}{mZe^2}$$

We have just derived what is called the *Bohr radius.* The energy of an orbit is given by the sum of potential and kinetic energy, thus:

$$-\frac{Ze^2}{r} + \frac{1}{2}mv^2 = -\frac{1}{2}\frac{Ze^2}{r} = -\frac{mZ^2e^4}{2\,n^2\hbar^2}$$

Note that the total energy is half the potential energy; this is a manifestation of the *Virial Theorem*, and holds in a variety of contexts, including any self-gravitating system, such as star clusters or entire galaxies.

For $n = 1$, plugging numbers into the above equation gives $-13.6Z^2$ electron volts (eV); 13.6 eV corresponds to a photon of wavelength 912Å, and frequency $3.29 \times 10^{15}$ Hz. For transitions from $n = 1$ to $n = 2$, the energy difference is $1 - 1/4 = 3/4$ of this. So for hydrogen, we get an energy of 10.2 eV, a wavelength of 1216Å, and a frequency of $2.47 \times 10^{15}$ Hz; this is in the ultraviolet part of the spectrum. For hydrogenic iron, these numbers scale by the factor $Z^2 = 26^2$, thus an energy of 6.9 keV, a wavelength of 1.8Å, and a frequency of a whopping $1.67 \times 10^{18}$ Hz; this is the Iron K$\alpha$ line mentioned in lecture. It is an X-ray photon. This line of hydrogenic iron is seen in the X-ray spectra of hot clusters of galaxies, and in quasars.

**b.** It turns out that there are electronic transitions of hydrogen that emit radio waves; in the very diffuse regions of interstellar space, one can have transitions from level $n$ to $n-1$, where $n$ is very large. Determine the order of magnitude of $n$ to get transitions from Hydrogen in the radio part of the spectrum (i.e., $\lambda > 1$ cm), and calculate the radius of the electron orbit. *Hint: it may be useful to use the approximation that $n \gg 1$.* These so-called 'radio recombination lines' have been seen with radio telescopes observing the interstellar medium. *(10 points)*
**Solution:** The wavelength of a transition from level $n$ to $n-1$ is:

$$\lambda = \frac{912\text{Å}}{1/(n-1)^2 - 1/n^2} \approx 912\text{Å}\,\frac{n^3}{2},$$

where the approximation is valid for large $n$; see below how this approximation was done. If we set the above expression equal to 1 cm, or $10^8$Å, we find $n^3 \approx 2 \times 10^5$, or $n \approx 60$. The Bohr ("Classical") radius of the atom, above, is $n^2\hbar^2/me^2$, or $0.5n^2$ Å, or 0.2$\mu$m. This is *huge* for a single atom! Such an atom is very fragile, and will be collisionally de-excited or ionized very quickly in any environment in which the density is high. In fact, one cannot make a vacuum on Earth good enough to allow such atoms to exist. Such atoms can only be observed in the hard vacuum of interstellar space.

Let me explain how the above approximation was done; without this approximation, solving this problem becomes much harder. The trick is to realize that this expression looks a lot like a derivative. Let's define $f(n) = 1/n^2$, so

$$\frac{1}{1/(n-1)^2 - 1/n^2} = \frac{1}{f(n-1) - f(n)} \approx \frac{-1}{\frac{df}{dn}\delta n},$$

where the last step uses the definition of the derivative, and here $\delta n = 1$. This approximation only works if $\delta n \ll n$, which indeed the case we're thinking about. Taking the derivative of $f$ is straightforward, and the expression becomes $n^3/2$.

Note that we use the phrase "order of magnitude" throughout. It refers to a calculation in which you can make rough approximations, and doesn't necessarily mean "round to the nearest factor of 10".

**c.** The case of the vibrations of a molecule cannot be done analytically, so we will use a simple order-of-magnitude argument to estimate the energies involved. Consider a simple diatomic

molecule which is vibrating (i.e., the distance between the two nuclei is oscillating); let's take CO to be definite. The overall size of the molecule (in its electronic ground state) is of the same order of magnitude as that of the Bohr atom you derived in part (a). It turns out that you can approximate the potential in which the two nuclei are sitting in terms of a harmonic oscillator, and thus will execute simple harmonic motion. For a harmonic oscillator of angular frequency $\omega$, the energy levels are quantized in steps of $\hbar\omega$. To order of magnitude, the potential in which the *nuclei* sit is similar to that of an *electron* in the vicinity of one nucleus, which you calculated in Part (a). Use this to determine the characteristic frequency, and therefore energy, of the vibration. Plug in numbers, and determine the characteristic wavelength of emitted photons in transitions between vibrational energy levels. *(10 points)*
**Solution:** In the Bohr atom, the frequencies of photons emitted in the electronic transitions are similar to the orbital frequency of the electron. Similarly, the frequencies of the photons emitted in the vibrational transitions are similar to the frequencies of the vibrations of the nuclei. From the description above, the electrostatic potential energy, and thus the kinetic energy of the nuclei of the diatomic molecule are of the same order of magnitude as that of the electron in the Bohr atom. We can think of this in terms of a classical harmonic oscillator: the spring constant $k$ of the two-nucleus oscillator is the same as that of the oscillator involving the electron, but the characteristic frequency is proportional to $\sqrt{k/m}$.

As the energy of the vibrational modes is proportional to $\omega$, the characteristic energy of transitions between quantum states of these modes is down from that of the electronic transitions by a factor $\sqrt{m_e/m_{nucleus}}$ (and the wavelength is correspondingly larger). This corresponds to photons of wavelength of order:

$$1000\text{Å} \times \sqrt{7 \times 1836} \approx 11\,\mu\text{m}.$$

This is in the mid-infrared part of the spectrum. Here, I used the fact that the reduced mass of CO $(m_C\, m_O/(m_C + m_O))$ is roughly 7 times that of the proton, and of course the ratio of proton to electron mass is 1836.

**d.** Now consider a *rotating* diatomic molecule. The angular momentum of the molecule is also quantized. Use this, and your knowledge of the characteristic size of a molecule, to determine the characteristic energy emitted in rotational transitions. In what part of the electromagnetic spectrum are these photons emitted? Look up the wavelengths of the rotational transitions of CO; are they within a few orders of magnitude of your very rough calculation? How about $H_2$? To order of magnitude, what is the energy of its rotational transitions? *Hint: There is a trick question embedded here... (10 points)*
**Solution:** The rotational angular momentum of a diatomic molecule is $m\omega r^2 = n\hbar$, where $\omega$ is the rotational frequency, $r$ is of order the Bohr radius, and $m$ is the (reduced) mass of the two nuclei. The energy in this case is $\hbar\omega$, which, in comparing with part (a), is of order the Bohr energy times the ratio of electron mass to the reduced mass of the molecule. For the CO molecule, this gives about $10^{-3}$ eV, or a wavelength of 1.3 millimeters (this latter number is 1000Å times 7 times 1836). This is within a factor of two or so of the true value of 2.6 mm for

the first rotational transition of CO. Typical transitions are from a rotational angular momentum $n\hbar\omega$ to $(n-1)\hbar\omega$, so the energy of the transition is indeed of the order of $\hbar\omega$.

For $H_2$, there are no rotational transitions. The molecule is symmetric, and has no permanent electric dipole moment. Without an electric dipole moment, a rotating molecule does not make transitions between rotational energy levels, and so there is no analogous rotational emission lines it gives off. $H_2$ is of course by far the most common molecule in the interstellar medium, but is largely invisible because of its lack of rotational transitions. (In the cold regions of the ISM where it is found, the *electronic* transitions of $H_2$, which are all in the ultraviolet, are not excited). This means that astronomers are limited to observing CO as a tracer of the molecular gas in the interstellar medium, and guessing at the ratio of $H_2$ to CO in order to determine how much mass there is in molecules.

**e.** What about transitions *inside* an atomic nucleus (i.e., a change of energy levels of a proton or neutron)? The processes that cause a nucleus to emit a photon are complex, but to get a sense of the order of magnitude, estimate the *electrostatic* potential energy of a proton on the surface of an iron nucleus. This is the order of magnitude of photons emitted from nuclei when they decay from certain excited states. Express your answer in kilo-electron volts, and convert to a wavelength. In what part of the electromagnetic spectrum is such a photon emitted? *(5 points)*
**Solution:** All we're doing here is estimating the electrostatic potential energy of a proton on the surface of the nucleus:

$$PE \approx \frac{Ze^2}{r},$$

where $r$ is the radius of the nucleus. $Z = 26$ is of course the charge of an Iron nucleus. We've done a version of this calculation already, that for the Bohr atom. There we found that for $Z = 1$, and $r$ of order 1 Ångstrom, the energy was of order 10 eV. The radius of an iron nucleus is a factor of $10^5$ smaller (roughly), and the potential energy is inversely proportional to the radius. With $Z = 26$, this gives an estimate of the potential energy of order 30 MeV (or 30,000 KeV), corresponding to a wavelength 3 million times smaller than the $\sim$ 1000Å we found for transitions in Hydrogen, or 0.0003Å. This is a gamma-ray photon.

## 2. Measuring faint objects (30 points)

Consider making observations of a star in the $u$ band, with apparent magnitude $m = 20.0$. The $u$ filter is centered at 3600Å, and has a width of 800Å. The conversion between magnitudes $m$ and flux density $f_\nu$ of a star in Janskies (1 Jy = $10^{-23}$ erg/sec/Hz/cm$^2$) is:

$$f_\nu = 3631\,\mathrm{Jy} \times 10^{-0.4\,m}.$$

**a.** Calculate approximately how many photons per second from this star hit each square centimeter of the top of the atmosphere. Explain where you are making your approximations. *(10 points)*
**Solution:** The flux density of a 20th magnitude star, by the above formula, is:

$$f_\nu = 3631 \times 10^{-23} \times 10^{-8} = 3.631 \times 10^{-28}\ \mathrm{erg/sec/Hz/cm^2}.$$

We're observing this through a filter of width 800Å, or $\frac{c}{\lambda^2}\Delta\lambda = 1.85 \times 10^{14}$ Hz, so this is a total flux of $6.7 \times 10^{-14}$ erg/sec/cm$^2$. The typical photon has an energy of $hc/\lambda \approx 4.4\,\mathrm{eV}$ (I was lazy, and calculated that by remembering that 3600Å is roughly $4 \times 912$Å, a photon whose energy we know from Problem 1 is 13.6 eV), or $7 \times 10^{-12}$ ergs. *This is the most important approximation here; we're taking all photons to have the energy of the center of the filter. We're also assuming that the flux density $f_\nu$ is constant across the 800Å of the filter. In a later homework, you will explore the limits of these approximations.* The photon flux is then the flux density, times the frequency interval, and dividing by the energy per photon. Putting this together, we get that the flux of photons from the star is a mere 0.01 photon per second per square centimeter. Yow; that's faint!

**b.** When the moon is not up, the sky at a dark site has a surface brightness of 22 magnitudes in a sky patch of area one square arcsecond (sometimes astronomers will refer to surface brightnesses in units of "magnitudes per square arcsecond", which can be quite confusing.).

First, find the magnitude of the emitted light from the sky within a circular aperture of diameter $2''$? (Note: That is 2 arcseconds, a measure of angle on the sky. It is not the size of the telescope, which is immaterial for this ratio.) The atmosphere absorbs about half the light of the star in the $u$ band at airmass 1.4. Assume that all the photons from the star fall into the two arcsecond aperture. Second, find the ratio of number of photons received per second from the star and that received from the sky in that circular aperture from a telescope on the ground? When the moon is up, the sky surface brightness is 17 magnitudes in one square arcsecond. Third, find the ratio of the number of photons per second from the star and the sky when the moon is up. *(10 points)*

**Solution:** Let's first calculate the effective magnitude of the sky in the $2''$ aperture. The area of this aperture is $\pi$ square arcseconds, thus the flux is $\pi$ times larger than the flux in a square arcsec, and thus the magnitude of the sky is: $22 - 2.5\log_{10}\pi = 20.75$ (make sure you understand where this calculation came from! It comes from solving the equation above relating fluxes and magnitudes). Thus the sky is 0.75 magnitudes fainter than the object, and therefore has a factor $10^{-0.4\times 0.75} \approx 1/2$ as many photons. But wait; we lost half the photons from the star itself from absorption in the atmosphere, so the number of photons from object and sky are essentially the same. With the moon up, the sky is 5 magnitudes, or a factor of 100, brighter, so there are 100 times more sky photons than object photons.

**c.** Now consider the SDSS telescope, which has a primary mirror diameter of 2.5 meters. In practice, only about 20% of the $u$-band photons that hit the primary mirror are detected; the remainder are lost due to the imperfect reflectivity of various optical surfaces, and the fact that the detectors are not perfect. How many photons are detected from the star per second? *(10 points)*
**Solution:** Here we multiply our photon flux in part (a) (units of photons per sec per cm$^2$) by the area of the telescope, and the 20% efficiency factor, to find a flux of 60 photons per second.