Physical Optics and Diffraction

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Outline

- Physical optics and Kirchhoff Integral
- Diffraction by an aperture
- Fraunhofer diffraction
- Fresnel diffraction
- Image formation
- Dealing with imperfections

Kirchhoff integral

- Monochromatic scalar wave $\Psi = \psi(\mathbf{x})e^{-i\omega t}$

  with spatial part satisfying *Helmholtz eqn*:

$$\nabla^2\psi + k^2\psi = 0$$

  for free propagation, eg: one component of **E** field in absence of polarization coupling
- Assume medium is homogeneous and non-dispersive so that k is constant
- Helmholtz eqn is a *linear, elliptic PDE*, so solution inside a volume is completely determined by value of $\psi(\mathbf{x})$ and its normal derivative on the boundary.

Kirchhoff integral (cont..)

- Nice trick available!
- Greens' theorem: for any two (reasonable) scalar functions

$$\int_{\text{boundary}} (\psi \boldsymbol{\nabla} \psi_0 - \psi_0 \boldsymbol{\nabla} \psi) \cdot d\boldsymbol{\Sigma} \;=\; -\int_{\mathcal{V}} (\psi \nabla^2 \psi_0 - \psi_0 \nabla^2 \psi) dV$$

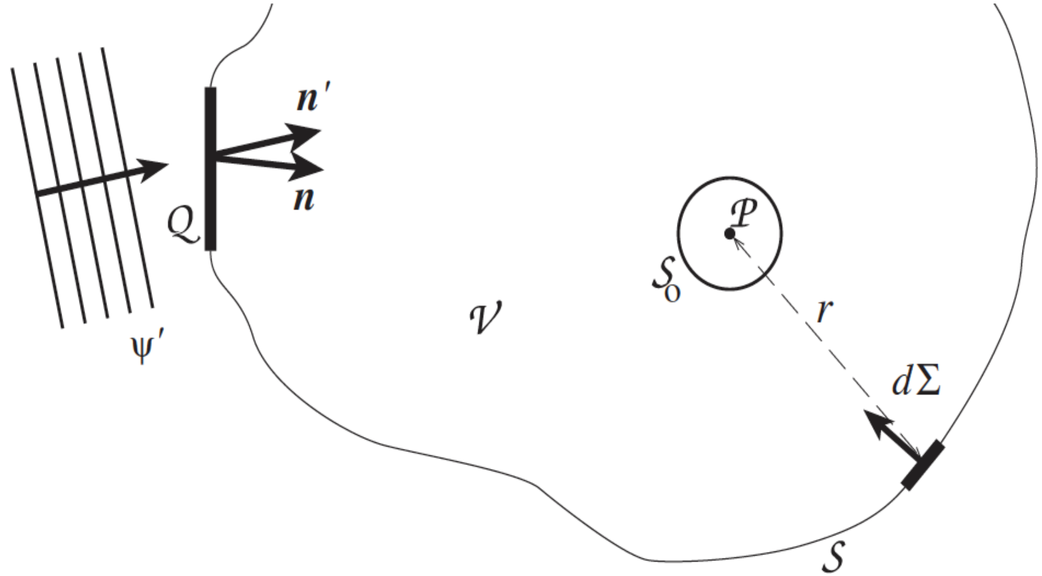
- Above intergrals **<u>= 0</u>** if both functions satisfy Helmholtz equation
- By inspection $\psi_0 = \dfrac{e^{ikr}}{r}$ is a solution

Kirchhoff integral (cont..)

$$\int_{\text{boundary}} (\psi \nabla \psi_0 - \psi_0 \nabla \psi) \cdot d\mathbf{\Sigma} = -\int_{\mathcal{V}} (\psi \nabla^2 \psi_0 - \psi_0 \nabla^2 \psi) dV = 0$$

- With $\psi_0 = \frac{e^{ikr}}{r}$, have $\psi \nabla \psi_0 - \psi_0 \nabla \psi \to -\psi(0)/r_o^2 + O(1/r_o)$
- So integral over S_0 becomes $4\pi\psi(\mathcal{P}) \equiv 4\pi\psi_{\mathcal{P}}$
- Therefore

$$\psi_{\mathcal{P}} = \frac{1}{4\pi} \int_S \left(\psi \nabla \frac{e^{ikr}}{r} - \frac{e^{ikr}}{r} \nabla \psi \right) \cdot d\mathbf{\Sigma} \, .$$



Diffraction by an aperture

- Suppose we have an aperture, which is big compared to the wavelength, but small compared to the distance to $\mathcal{P}$ .Illumination comes from a *distant* wave source.
- Characterize aperture by a complex function $\mathfrak{t}$, such that the wave just after passing through it $\psi_Q = \mathfrak{t}\,\psi'$
- On the aperture have $kr \gg 1$, so write

$\nabla(e^{ikr}/r) \simeq -ik\mathbf{n}e^{ikr}/r$, $\mathbf{n}$ pointing towards $\mathcal{P}$

$\boldsymbol{\nabla}\psi \simeq ik\mathfrak{t}\,\mathbf{n}'\psi'$, with $\mathbf{n}'$ parallel to **k**

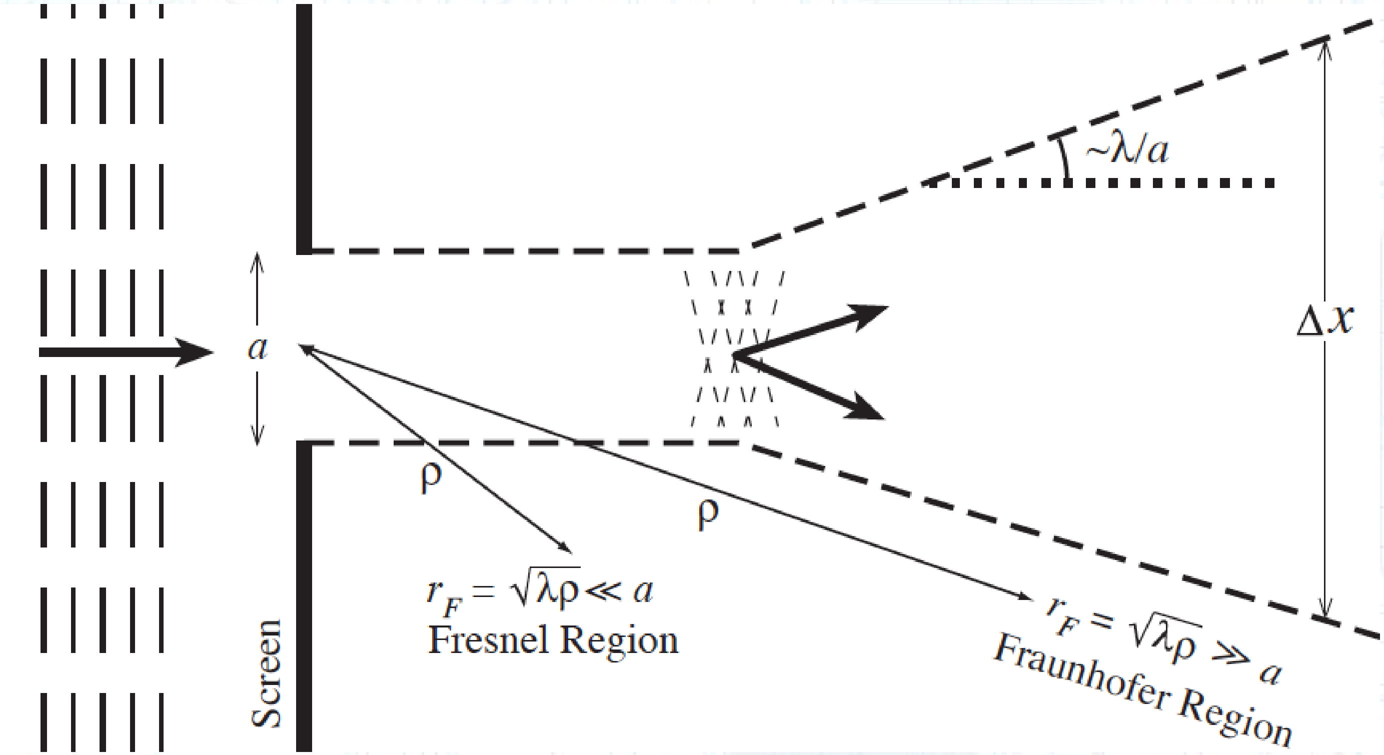
Diffraction by an aperture

Then

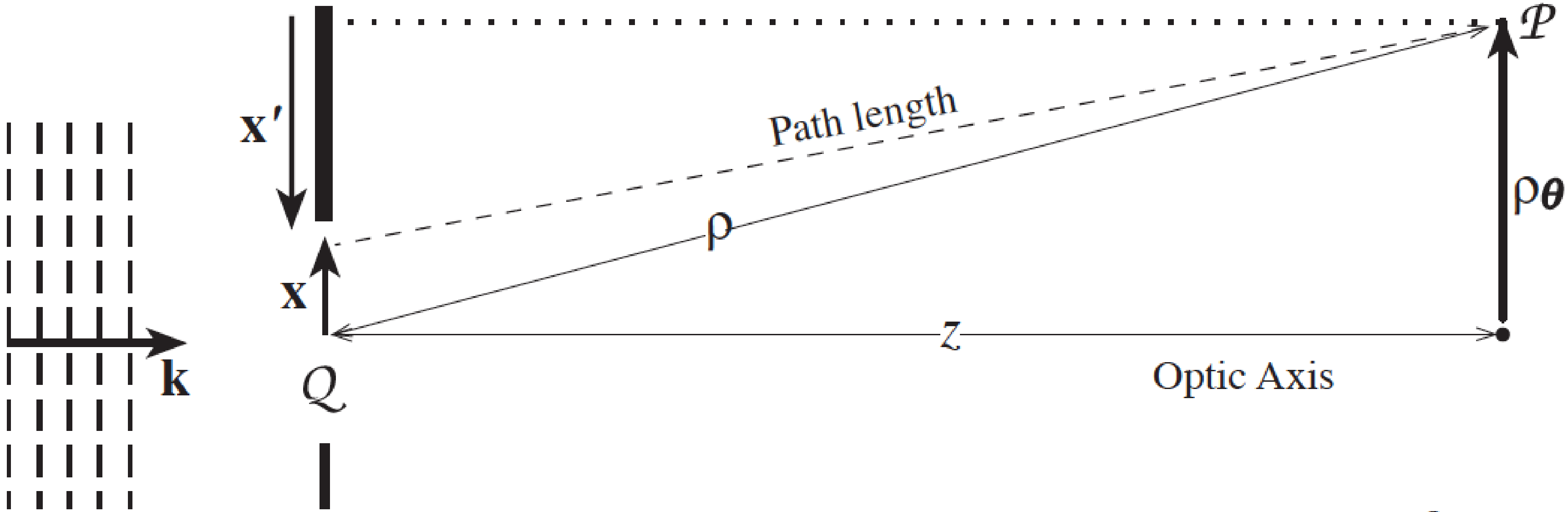
$$\psi_{\mathcal{P}} = -\frac{ik}{2\pi}\int_{\mathcal{Q}} d\mathbf{\Sigma}\cdot\left(\frac{\mathbf{n}+\mathbf{n}'}{2}\right)\frac{e^{ikr}}{r}\mathfrak{t}\,\psi'.$$

Can solve in two different regimes:

- <u>Far field</u> (*Fraunhofer*)
- <u>Near field</u> (*Fresnel*)



Fraunhofer regime (far field)

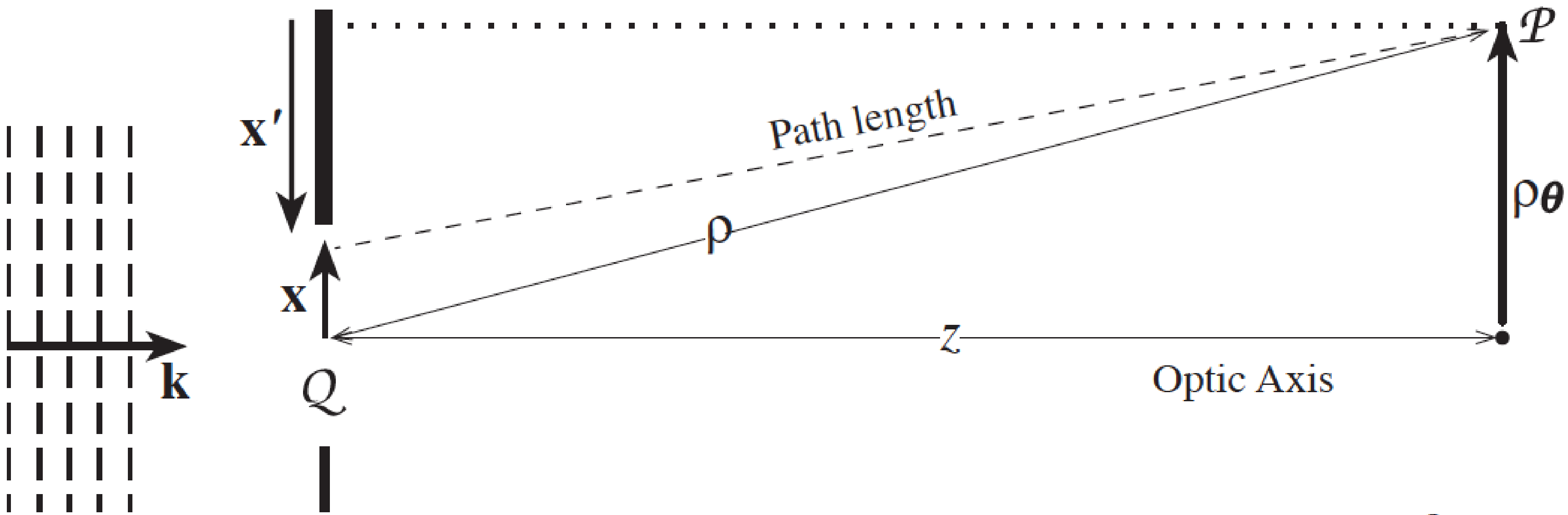


$$\text{Path length} = r = (\rho^2 - 2\rho\mathbf{x}\cdot\boldsymbol{\theta} + x^2)^{1/2} \simeq \rho - \mathbf{x}\cdot\boldsymbol{\theta} + \frac{x^2}{2\rho} + \ldots$$

Condition for Fraunhofer is that we can neglect the quadratic phase variation with position in the aperture plane

i.e. $k\frac{x^2}{2\rho} \ll 1$ for all x, or $a \ll \sqrt{\lambda\rho}$

Fraunhofer regime (far field)



$$\text{Path length } = r = (\rho^2 - 2\rho\mathbf{x}\cdot\boldsymbol{\theta} + x^2)^{1/2} \simeq \rho - \mathbf{x}\cdot\boldsymbol{\theta} + \frac{x^2}{2\rho} + \ldots$$

Neglecting the quadratic term:

$$\boxed{\psi_{\mathcal{P}}(\boldsymbol{\theta}) \propto \int e^{-ik\mathbf{x}\cdot\boldsymbol{\theta}}\mathfrak{t}\,(\mathbf{x})d\Sigma \equiv \tilde{\mathfrak{t}}(\boldsymbol{\theta})\ ,}$$

The *amplitude* of the *Fraunhofer* diffraction pattern is given by the **2D Fourier transform** of the aperture function

Fraunhofer examples

- **Diffraction by a single, finite slit**

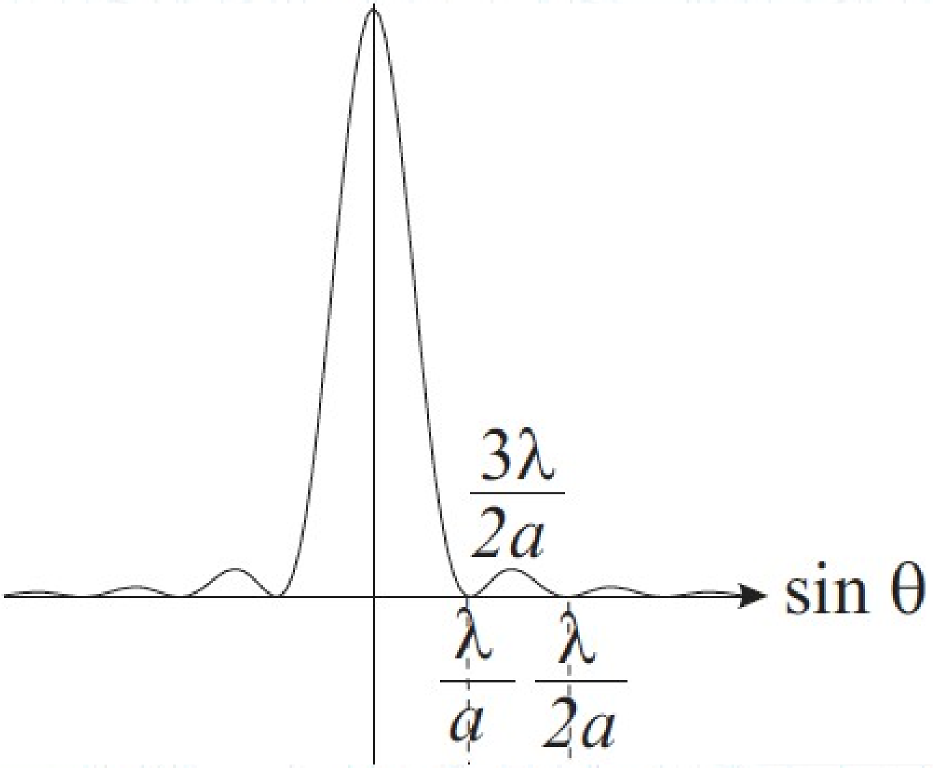
$$\mathfrak{t}(x) = H_1(x) \equiv \begin{cases} 1 & |x| < a/2 \\ 0 & |x| > a/2 \end{cases}$$

Then

$$\psi_{\mathcal{P}}(\theta) \propto \tilde{H}_1 \propto \int_{-a/2}^{a/2} e^{ikx\theta} dx \propto \text{sinc}\left(\tfrac{1}{2}ka\theta\right)$$

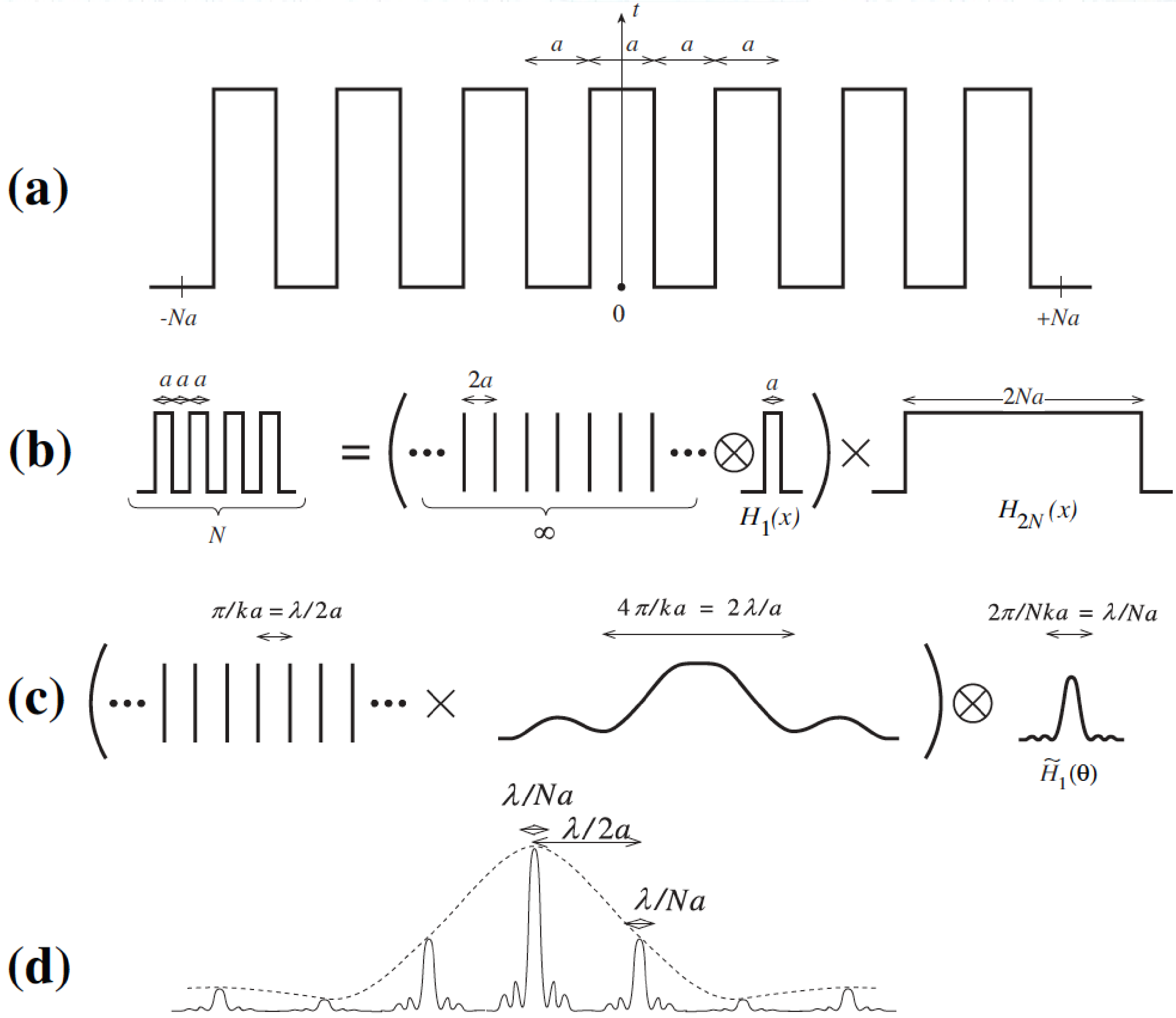
$$F(\theta) \propto \text{sinc}^2\left(\tfrac{1}{2}ka\theta\right) \ .$$

Intensity



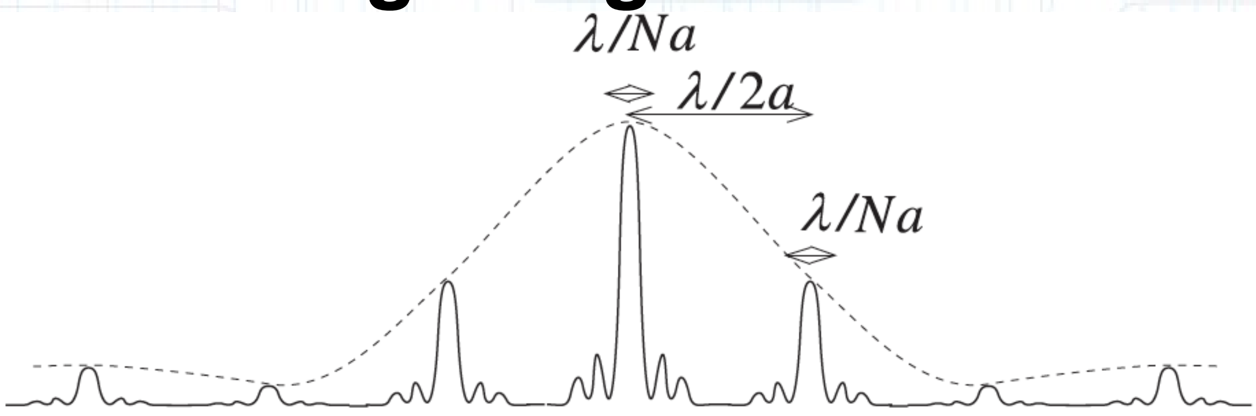
Fraunhofer examples

- **Diffraction grating**

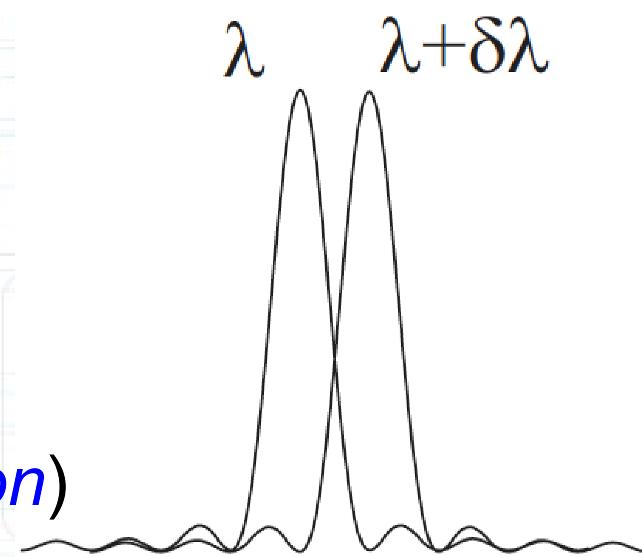


Fraunhofer examples

- **Diffraction grating**



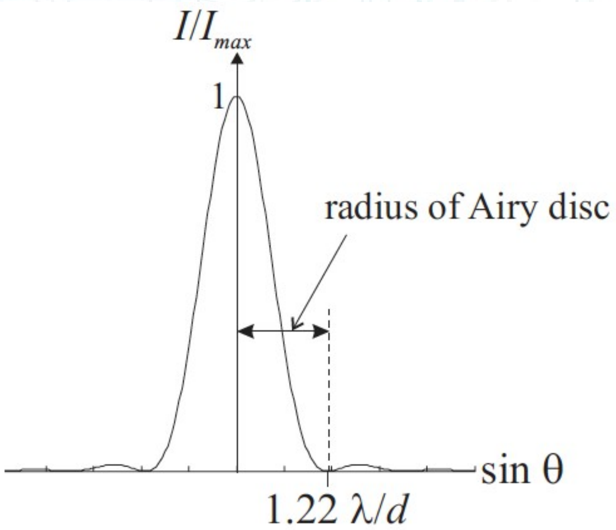
- *p*-th *order* beam deflected by $\theta = \pi p/ka = p\lambda/2a$
- Consider beams for waves at λ and $\lambda + \delta\lambda$
- Located at $\theta = p\lambda/2a$ and $p(\lambda + \delta\lambda)/2a$
- Separation $\delta\theta = p\delta\lambda/2a$
- Can distinguish them if <u>maximum of one corresponds to first minimum of the other</u>, i.e. $\delta\theta$ is at least $\lambda/2Na$ (*Rayleigh criterion*)
- Corresponds to $\frac{\lambda}{\delta\lambda} \lesssim \mathcal{R} \equiv Np$ (*chromatic resolving power*)



Fraunhofer examples

- **Diffraction by a circular aperture**

$$\psi(\theta) \propto \int_{\text{Disk with diameter } D} e^{-ik\mathbf{x}\cdot\boldsymbol{\theta}} d\Sigma \propto \frac{J_1(kD\theta/2)}{kD\theta/2}$$



- Most of the light from a distant source falls within the *Airy disc*

$$\theta_A = 1.22\lambda/D$$

- Can use to calculate the *diffraction limit* of a lens/telescope
- Two equally bright sources can be resolved only if the radius of the Airy disk is less than their separation, i.e if their angular separation is more than

$$\theta_{\min} = \theta_A = 1.22\lambda/D$$

Fraunhofer examples

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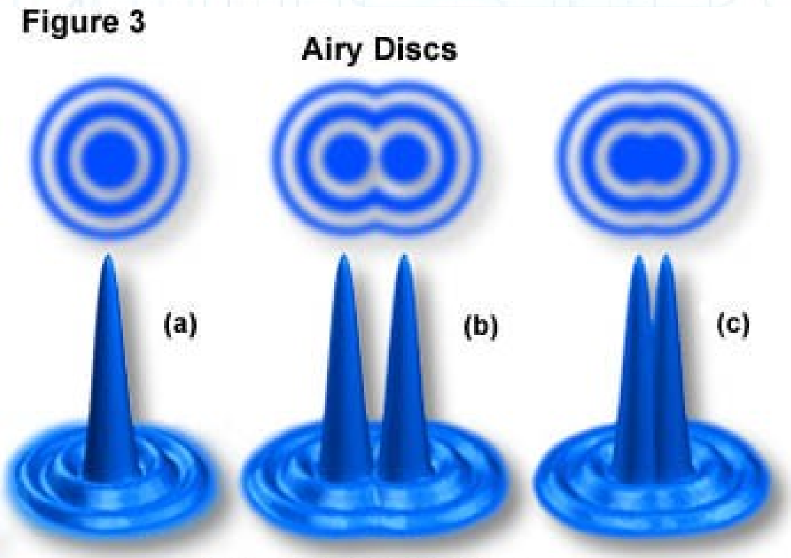
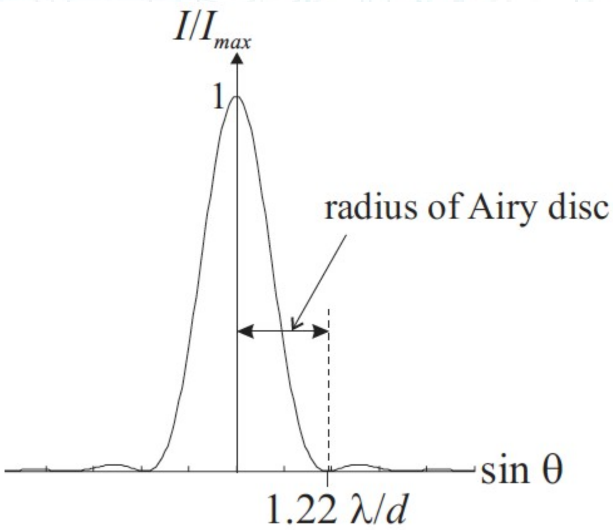


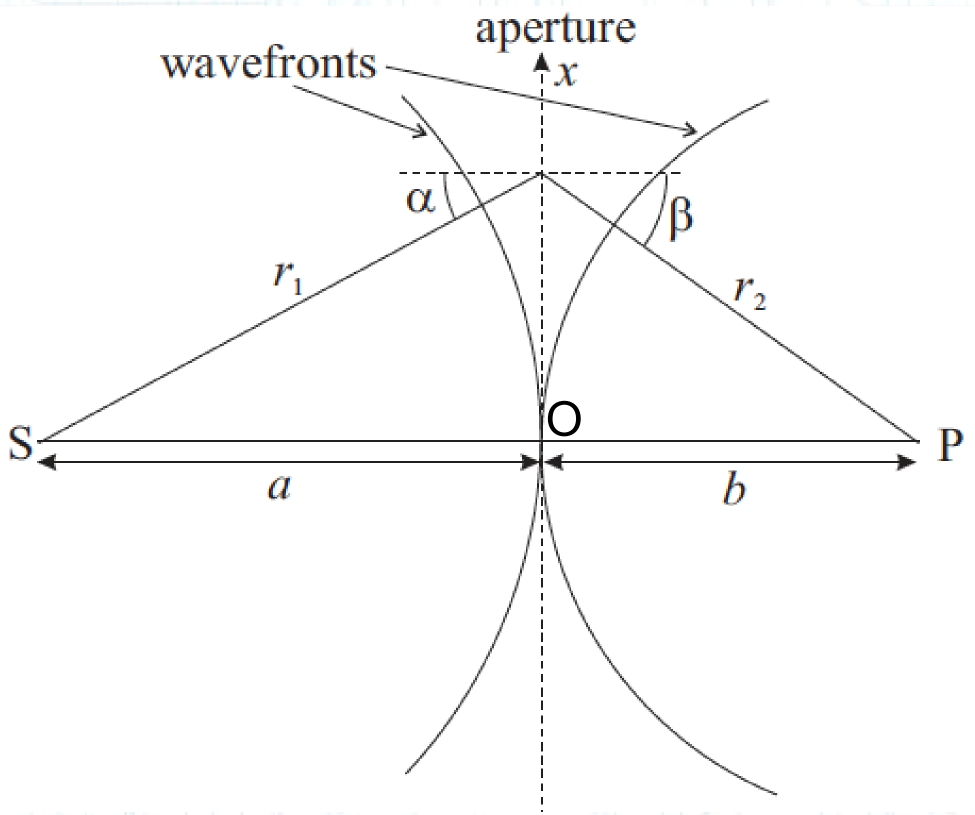


- Two equally bright sources can be resolved only if the radius of the Airy disk is less than their separation, i.e if their angular separation is more than

$$\theta_{\min} = \theta_A = 1.22\lambda/D$$

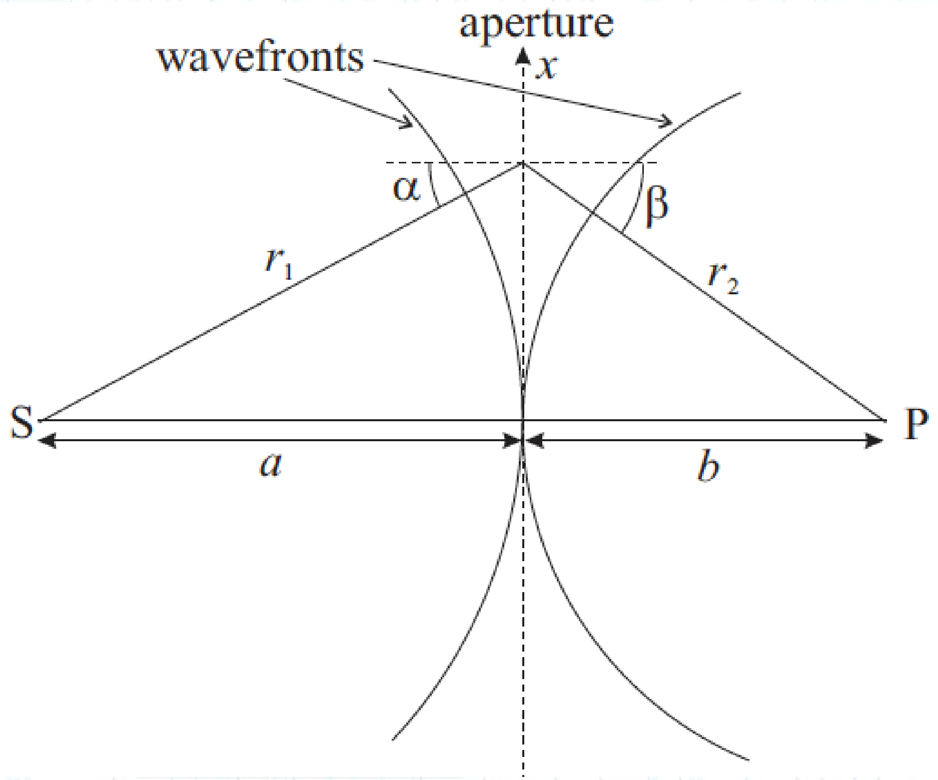
Fresnel regime (near field)

- Now can't neglect quadratic phase variation
- Problem a bit harder. Consider *special case* in which the source, the origin of coordinates and the observation point are *aligned*



Might seem uninteresting, but can make a lot of progress by moving the origin!

Fresnel regime (near field)



$$
\begin{aligned}
r_1 + r_2 &= \sqrt{a^2 + x^2 + y^2} + \sqrt{b^2 + x^2 + y^2} \\
&= a + b + \frac{x^2 + y^2}{2a} + \frac{x^2 + y^2}{2b} \\
&\quad + \text{higher order terms}
\end{aligned}
$$

Fresnel regime (near field)

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write $\frac{1}{R} = \frac{1}{a} + \frac{1}{b}$ then $\text{optical path} = \text{const.} + \frac{x^2 + y^2}{2R}$

$$\boxed{\psi_P \propto \int t(x, y) \exp\left(ik\frac{x^2 + y^2}{2R}\right) \, d\Sigma}$$

Separable aperture

If the aperture function is separable (e.g. rectangular aperture), it is convenient to rewrite in terms of *Fresnel integrals.*

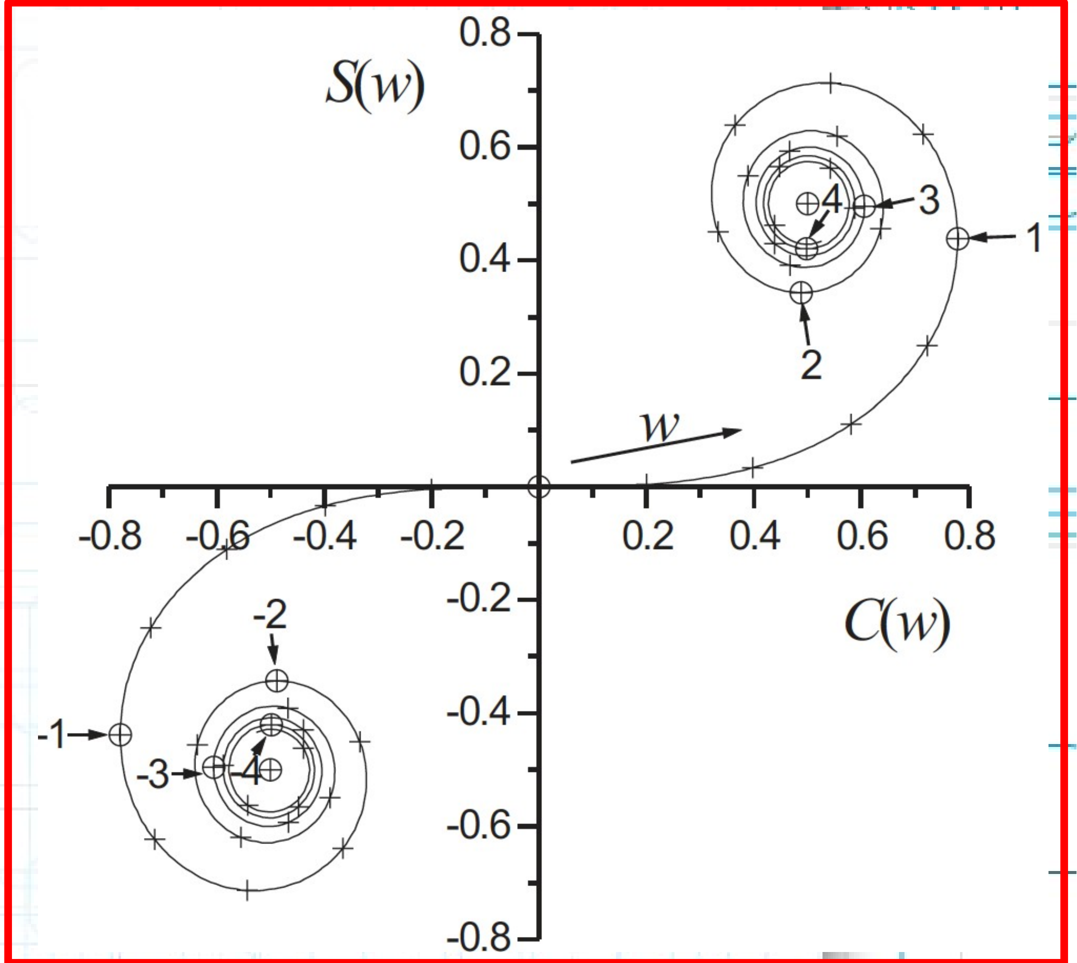
Change variables to $u = x\sqrt{\frac{2}{\lambda R}}\ ;\ v = y\sqrt{\frac{2}{\lambda R}}$

Define ***Fresnel integral***

$$\int_0^w \exp\left(\frac{i\pi u^2}{2}\right) du = C(w) + iS(w)$$

Plotting *C* vs *S*, obtain the **Cornu spiral**

$$C(\infty) = 0.5 \qquad S(\infty) = 0.5$$



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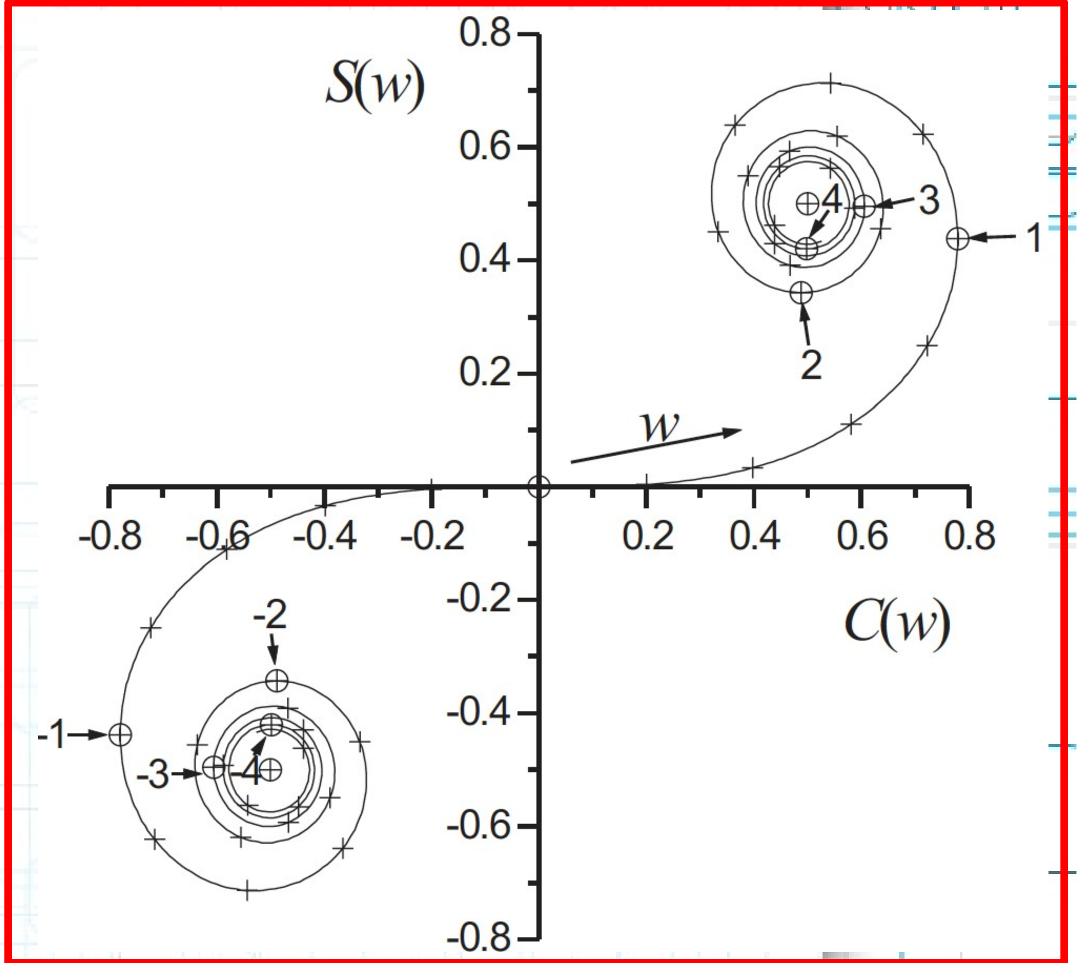

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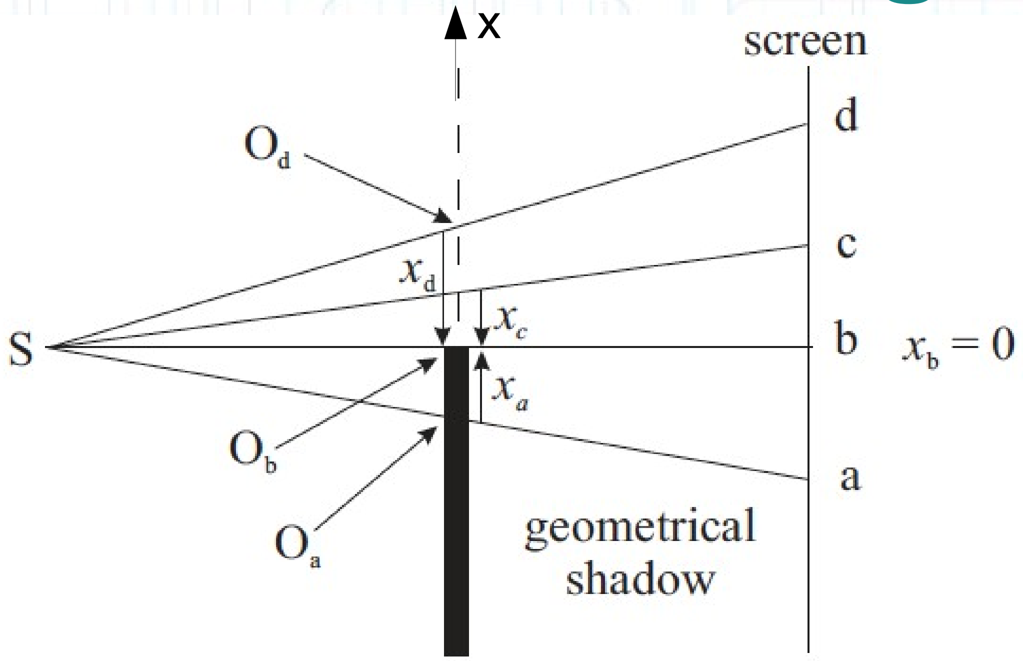
Diffraction from straight edge



$$\psi_P \propto \int_{x_1}^{x_2} \exp\left(\frac{ikx^2}{2R}\right) dx \propto \int_{w_1}^{w_2} \exp\left(\frac{i\pi u^2}{2}\right) du$$

$$= C(w_2) - C(w_1) + i(S(w_2) - S(w_1))$$

a	$x_2 = \infty$	$x_1 = x_a > 0$	
b	$x_2 = \infty$	$x_1 = x_b = 0$	the geometrical edge
c	$x_2 = \infty$	$x_1 = x_c < 0$	
d	$x_2 = \infty$	$x_1 = x_d;\ w_d \sim -1.26$	maximum $\lvert\psi_P\rvert$

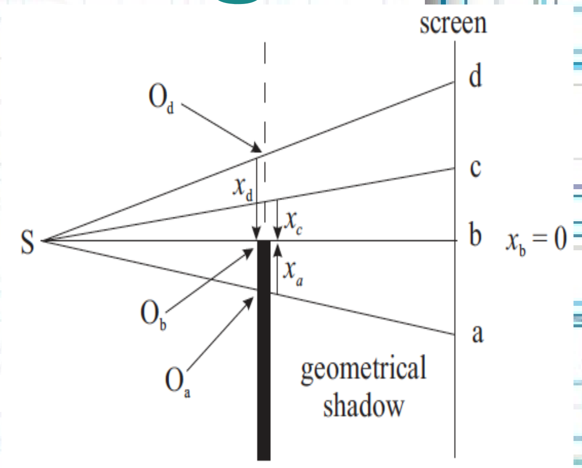
Diffraction from straight edge

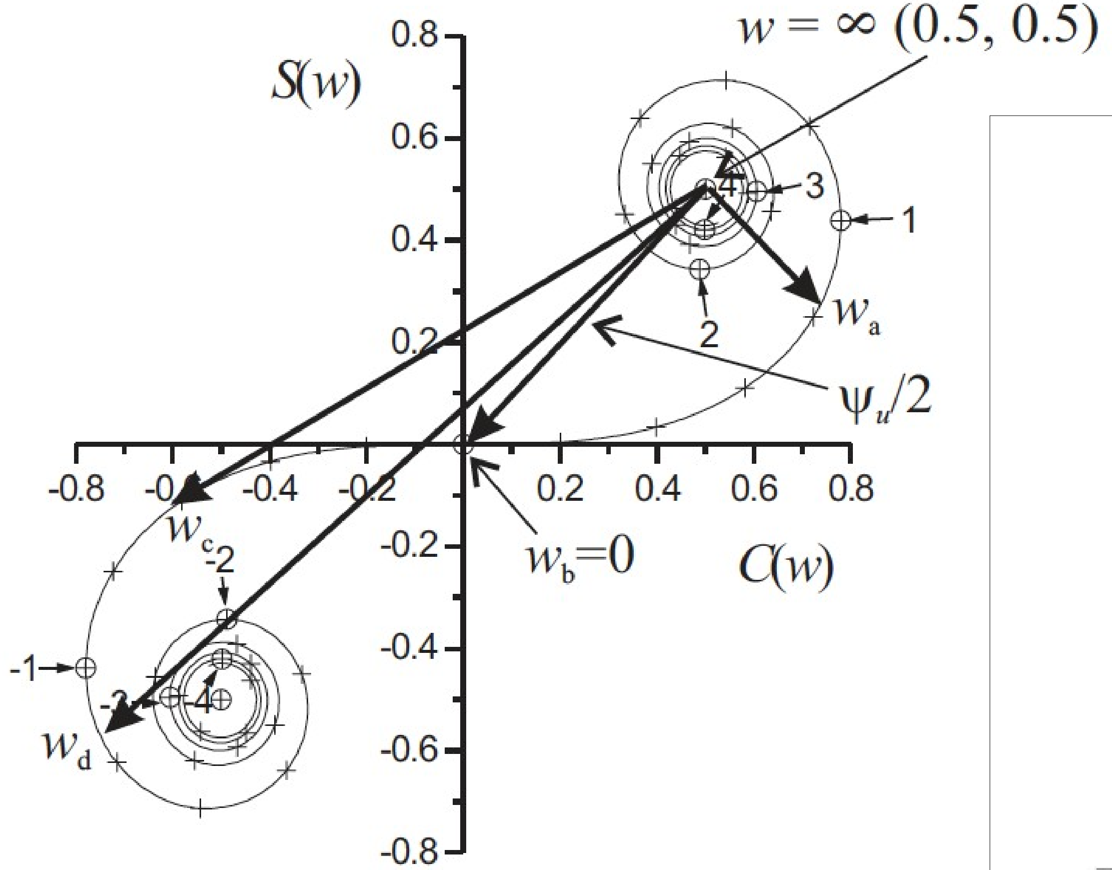
a $x_2 = \infty$ $x_1 = x_a > 0$

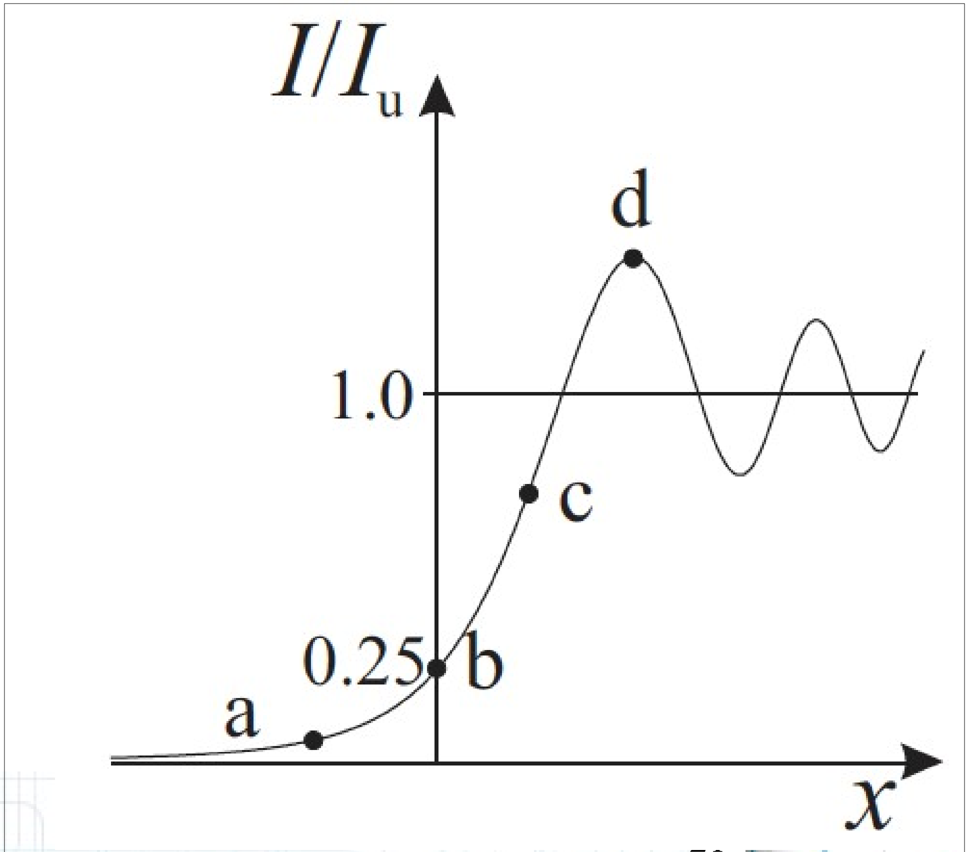
b $x_2 = \infty$ $x_1 = x_b = 0$ the geometrical edge

c $x_2 = \infty$ $x_1 = x_c < 0$

d $x_2 = \infty$ $x_1 = x_d$; $w_d \sim -1.26$ maximum $|\psi_P|$





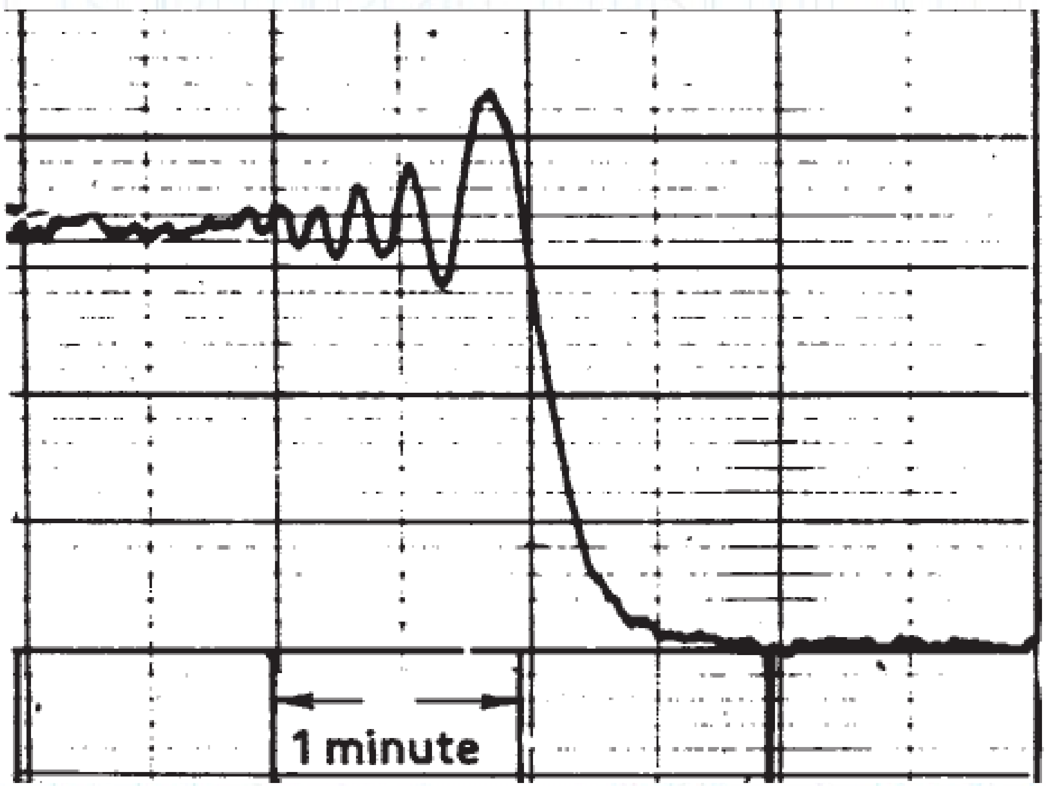


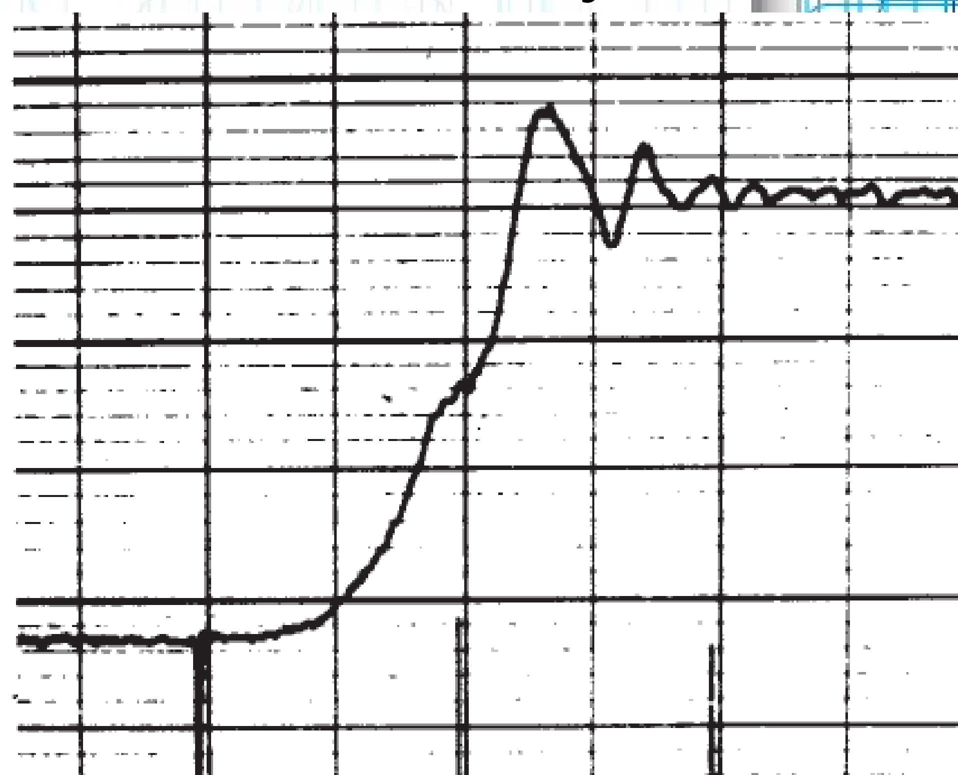
No. 4872 March 16, 1963 NATURE 1037

INVESTIGATION OF THE RADIO SOURCE 3C 273 BY THE METHOD OF LUNAR OCCULTATIONS

By C. HAZARD, M. B. MACKEY and A. J. SHIMMINS
C.S.I.R.O. Division of Radiophysics, University Grounds, Sydney

- Use the Fresnel pattern from lunar occultation of the powerful 3C 273 radio source to determine its position accurately





Diffraction by circular aperture

- Consider circular aperture w/ radius *D*
- Retain the obliquity factor *K*
- Using polar coordinates:

$$\psi_P \propto \int_{\rho=0}^{\rho=D} \frac{K}{(a^2+\rho^2)^{1/2}(b^2+\rho^2)^{1/2}} \exp\left(\frac{ik\rho^2}{2R}\right) 2\pi\rho\, d\rho.$$

- Use the substitution $\rho^2 = s\ ;\ \ 2\rho\, d\rho = ds$

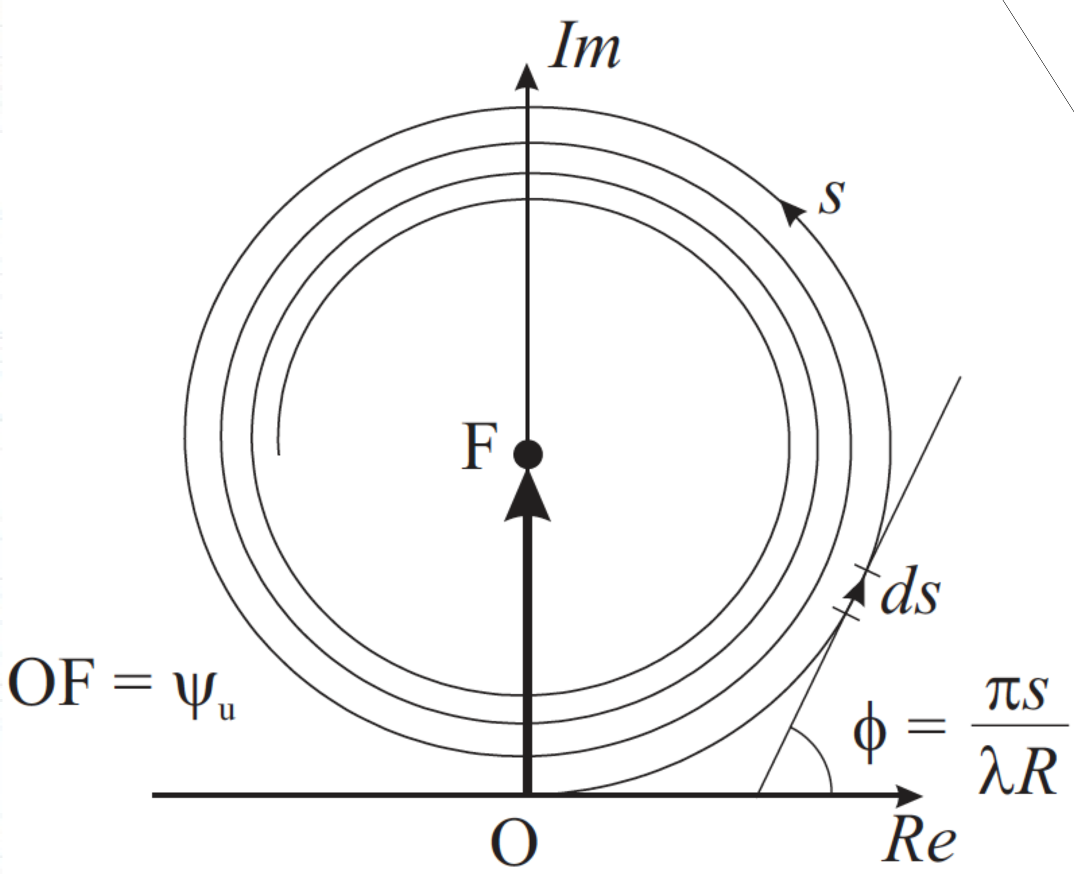
$$\psi_P \propto \int_{s=0}^{s=D^2} \frac{K}{(a^2+s)^{1/2}(b^2+s)^{1/2}} \exp\left(\frac{i\pi s}{\lambda R}\right) \pi\, ds$$

- **Note**: only valid *on axis*!

Diffraction by circular aperture

$$\psi_P \propto \int_{s=0}^{s=D^2} \frac{K}{(a^2+s)^{1/2}(b^2+s)^{1/2}} \exp\left(\frac{i\pi s}{\lambda R}\right) \pi\, ds$$

- Evaluate graphically



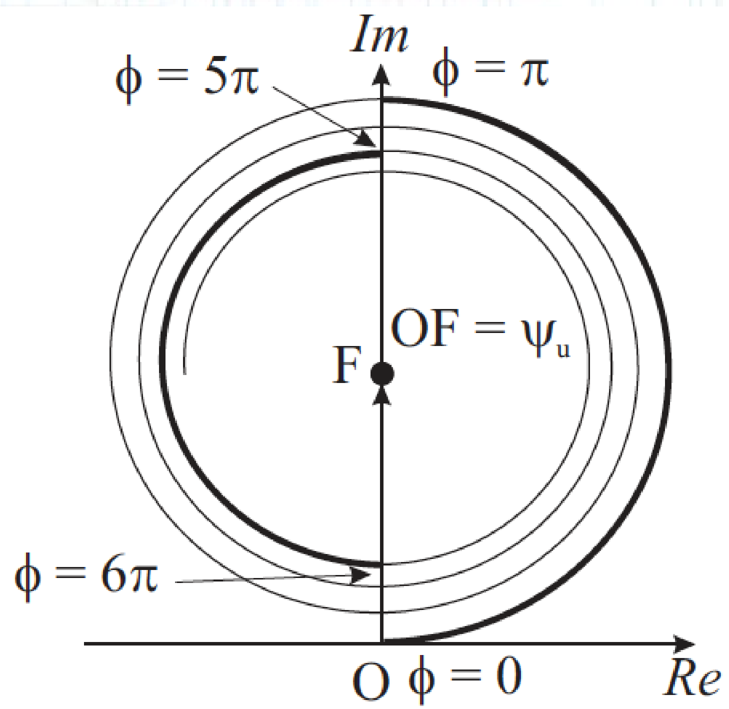
Going around in a circle

Spiralling in

Diffraction by circular aperture and Fresnel Zones

For finite apertures, the diffraction integral varies considerably

$$\phi = 2n\pi \quad \rho^2 = 2n\lambda r \quad \Rightarrow \quad \psi \approx 0$$
$$\phi = (2n+1)\pi \quad \rho^2 = (2n+1)\lambda r \quad \Rightarrow \quad \psi \approx 2\psi_u$$



Define *Fresnel zones* as concentric rings in the aperture plane over which the phase varies at the observation point varies by π

1st zone: $0 \leq \phi(\rho) \leq \pi$

$$\rho^2 \leq \lambda R.$$

Nth zone: $(n-1)\pi \leq \phi(\rho) \leq n\pi$

$$\sqrt{(n-1)\lambda R} \leq \rho \leq \sqrt{n\lambda R}.$$

Note: Area of each zone is the *same* $\pi\left(\rho_n^2 - \rho_{n-1}^2\right) = \pi\lambda R.$

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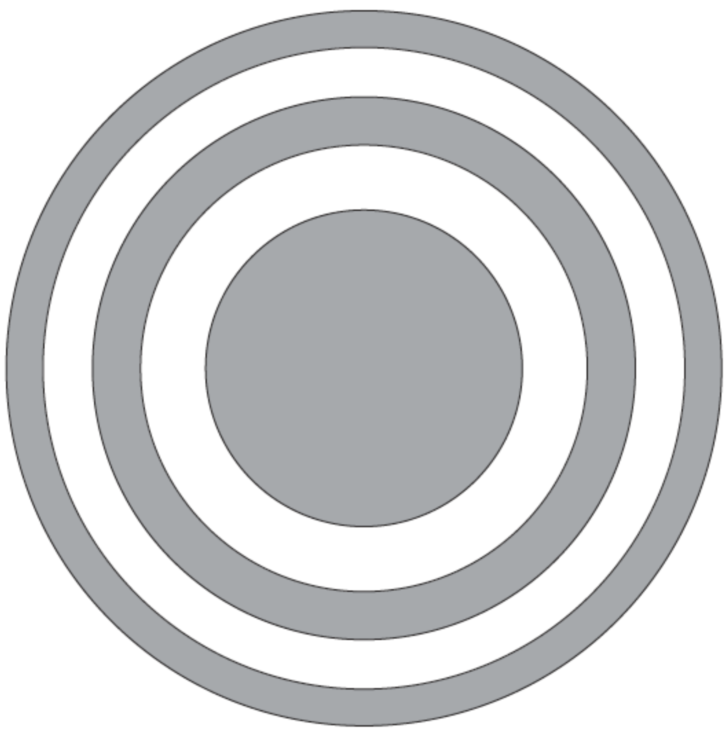

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Diffraction by circular aperture and Fresnel Zones

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$$\begin{array}{lll} \phi = 2n\pi & \rho^2 = 2n\lambda r & \Rightarrow \ \psi \approx 0 \\ \phi = (2n+1)\pi & \rho^2 = (2n+1)\lambda r & \Rightarrow \ \psi \approx 2\psi_u \end{array}$$

Odd numbered zones *add* to, and *even* numbered zones *subtract* from the overall amplitude at P.

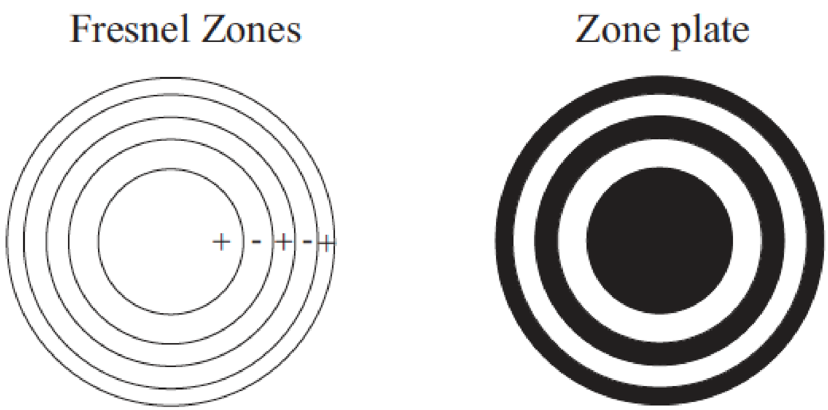
So, for an observation point P on the optic axis of a circular aperture of radius a, the aperture includes N zones, given by $a^2 = n\lambda R$.

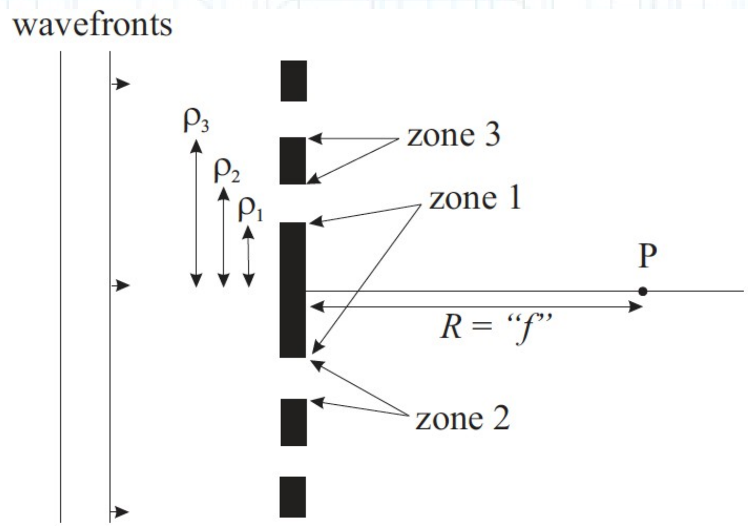
N ODD: bright spot at P; $\psi \sim 2\psi_u$; $I \sim 4I_u$

N EVEN: dark spot at P: $\psi \sim 0$; $I \sim 0$

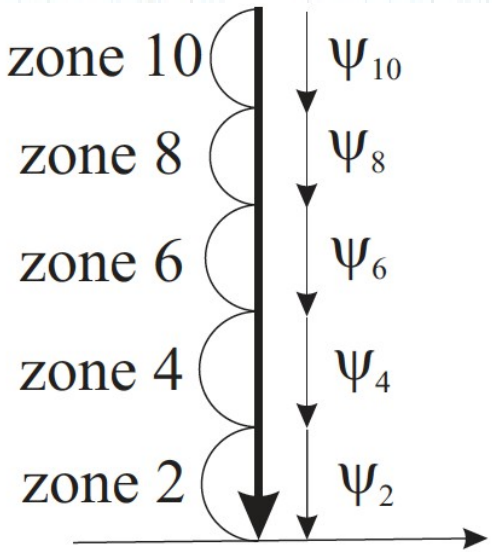
Zone plate

- A **Fresnel zone plate** is an aperture that blocks alternates half-period zones.
- Eg: block the *odd* ones





$$\rho_1 = \sqrt{\lambda R},\ \rho_2 = \sqrt{2\lambda R},\ \rho_3 = \sqrt{3\lambda R},\ \text{etc}$$



Net amplitude at *P*:

$$\psi_P = \psi_2 + \psi_4 + \psi_6 + \ldots \approx 2N\psi_u$$

Intensity $I_P \approx 4N^2 I_u$

Incident wave brought to focus at P!
Acts like a ***lens*** of focal length

$$f = \frac{\rho_1^2}{\lambda} = \frac{\rho_n^2}{n\lambda}$$

Zone plate (cont..)

- $f \propto 1/\lambda \rightarrow$ highly chromatic lens
- Focuses different wavelengths in different foci
- Works for any frequency (including X-rays)
- Maximum resolution depends on smallest zone width

Image formation (thin lens)

- Field at distance z_j along the optical axis: $\psi_j(\mathbf{x})$
- Vector **x** is perpendicular to the optic axis
- For a linear system $$\psi_2(\mathbf{x}_2) = \int P_{21}(\mathbf{x}_2, \mathbf{x}_1) d\Sigma_1 \psi_1$$
- P_{21} is the *propagator* or *Point Spread Function* (PSF)

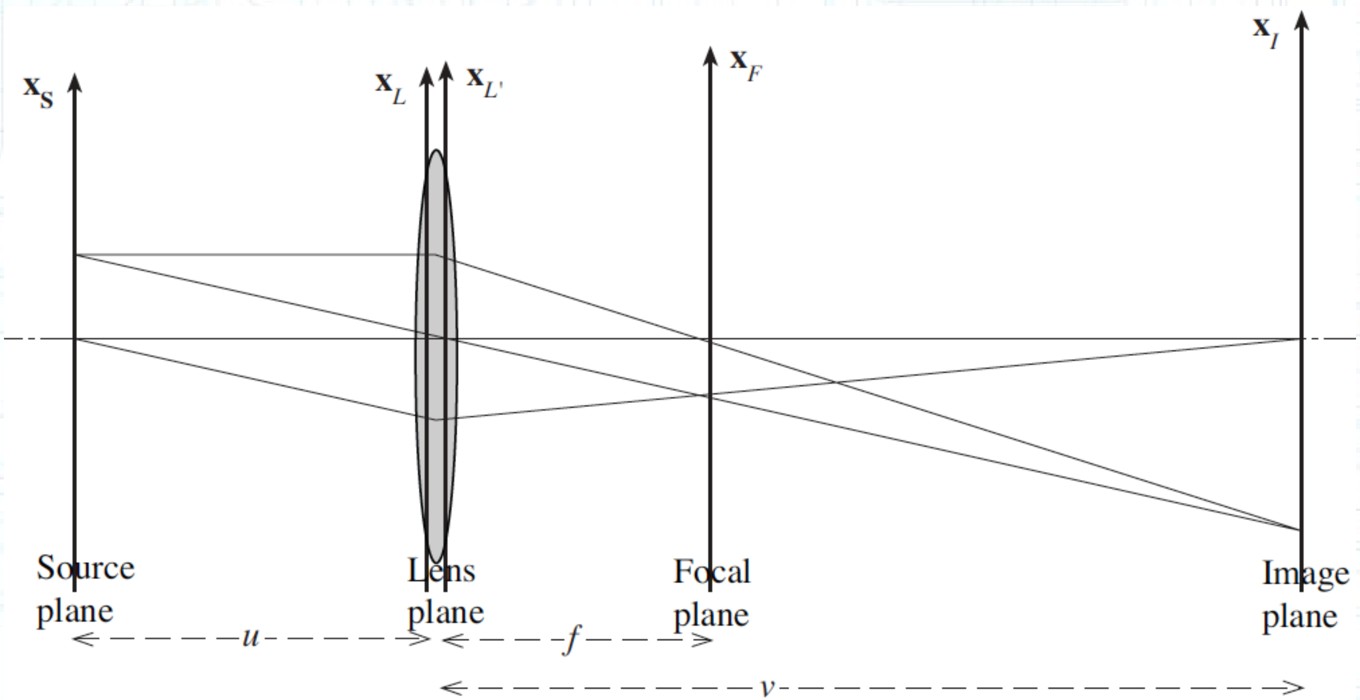


Image formation (thin lens)

- Field at distance z_j along the optical axis: $\psi_j(\mathbf{x})$
- Vector **x** is perpendicular to the optic axis
- For a linear system

$$\psi_2(\mathbf{x}_2) = \int P_{21}(\mathbf{x}_2, \mathbf{x}_1) d\Sigma_1 \psi_1$$

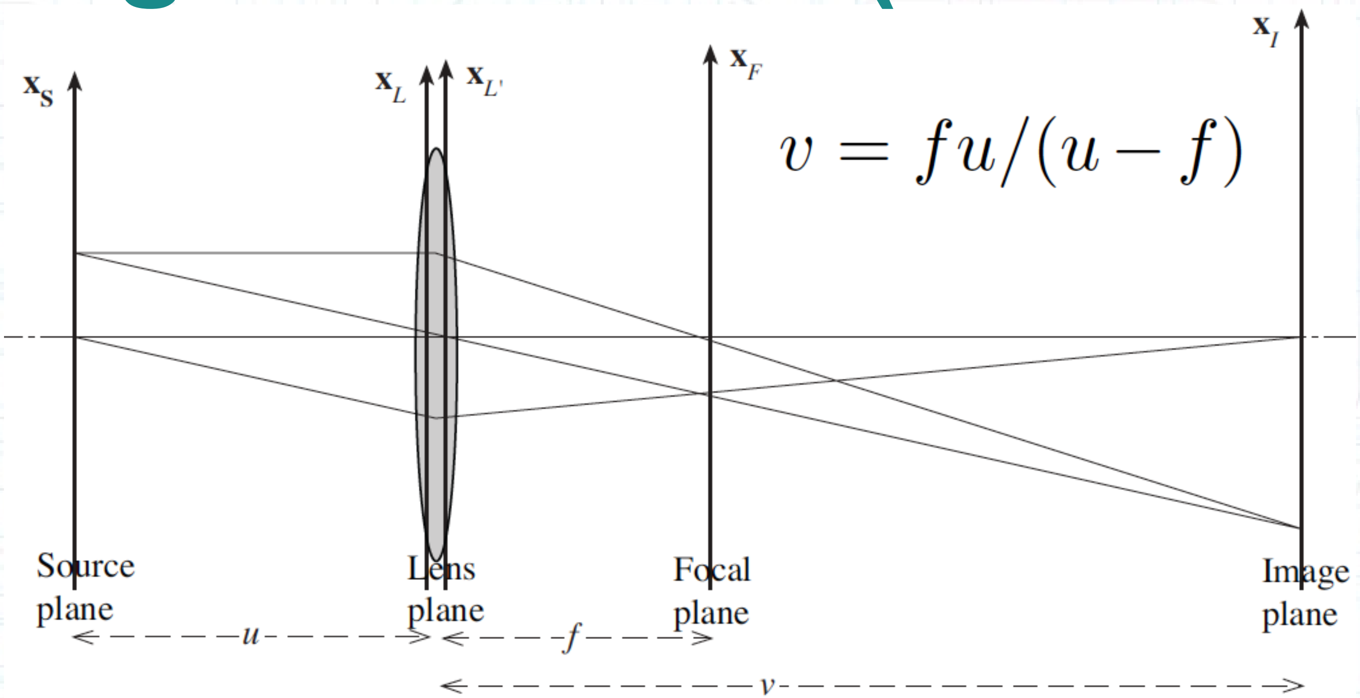
- P_{21} is the *propagator* or *Point Spread Function* (PSF)
- **Free propagation** through $d = z_2 - z_1$

$$P_{21} = \frac{-ik}{2\pi d} e^{ikd} \exp\left(\frac{ik(\mathbf{x}_1 - \mathbf{x}_2)^2}{2d}\right)$$

- **Thin lens**: phase shift depends quadratically on the distance from optic axis |**x**|

$$P_{21} = \exp\left(\frac{-ik|\mathbf{x}_1|^2}{2f}\right) \delta(\mathbf{x}_2 - \mathbf{x}_1)$$

Image formation (thin lens)



- Propagating to the *focal plane*:

$$
\begin{aligned}
P_{FS} &= \int P_{FL'} d\Sigma_{L'} P_{L'L} d\Sigma_L L P_{LS} \\
&= \int \frac{ik}{2\pi f} e^{ikf} \exp\left(\frac{ik(\mathbf{x}_F - \mathbf{x}'_L)^2}{2f}\right) d\Sigma_{L'} \delta(\mathbf{x}_{L'} - \mathbf{x}_L) \exp\left(\frac{-ik|\mathbf{x}_L|^2}{2f}\right) \\
&\quad \times d\Sigma_L \frac{-ik}{2\pi u} e^{iku} \exp\left(\frac{ik(\mathbf{x}_L - \mathbf{x}_S)^2}{2u}\right) \\
&= \frac{-ik}{2\pi f} e^{ik(f+u)} \exp\left(-\frac{ikx_F^2}{2(v-f)}\right) \exp\left(-\frac{ik\mathbf{x}_F \cdot \mathbf{x}_S}{f}\right) .
\end{aligned}
$$

Image formation (thin lens)

- Propagating from source to the *focal plane*:

$$P_{FS} = \frac{-ik}{2\pi f} e^{ik(f+u)} \exp\left(-\frac{ikx_F^2}{2(v-f)}\right) \exp\left(-\frac{ik\mathbf{x}_F \cdot \mathbf{x}_S}{f}\right)$$

- Amplitude given by $\psi_F(\mathbf{x}_F) = \int P_{FS} d\Sigma_S \psi_S(\mathbf{x}_S)$

$$\boxed{\psi_F(\mathbf{x}_F) = -\frac{ik}{2\pi f} e^{ik(f+u)} \exp\left(-\frac{ikx_F^2}{2(v-f)}\right) \tilde{\psi}_S(\mathbf{x}_F/f)\ .}$$

- Therefore the amplitude in the *focal plane* is proportional to the ***Fourier transform*** of the field in the *source plane*
- Focal plane can be used for *spatial filtering*, i.e. processing the image by altering its Fourier transform

Image formation (thin lens)

- Finally, free propagation from *focal plane* to *image plane*

$$\psi_I = \int P_{IF} d\Sigma_F \psi_F$$

$$\psi_I(\mathbf{x}_I) = -\left(\frac{u}{v}\right) e^{ik(u+v)} \exp\left(\frac{ikx_I^2}{2(v-f)}\right) \psi_S(\mathbf{x}_S = -\mathbf{x}_I u/v) \ .$$

- Apart from a phase factor, *the field in the image plane is just a magnified version of the original field*, with the correct magnification
- Theory due to E. Abbe (1873)

Dealing with imperfections...

- So far only considered perfect lenses/mirrors and uniform propagation medium
- When any of the above assumptions fail, the image quality is degraded
- Useful to define the **Strehl ratio** as the ratio between the peak amplitude of the actual PSF and the peak amplitude expected in the presence of diffraction only
- Looking through the atmosphere, the Strehl ratio will be very low, even with good optics (due to atmospheric turbulence).
- In presence of *Adaptive Optics* systems, can get very close to 1 (diffraction limited)

Dealing with imperfections...

- Can treat aberrations and imperfections in the physical optics language (but it's a hard problem, see Born & Wolf)
- Can relate rms imperfection in the optics with actual intensity in the focal plane

$$I_P = I_0 \left[1 - \left(\frac{2\pi}{\lambda} \right)^2 (\Delta\Phi)^2 \right]$$

- For example, in order to have $I_p / I_0 \approx 0.8$

we need

$$|\Delta\Phi| < \lambda/14$$

References

- Born & Wolf, *Principles of Optics*
- Hecht, *Optics*
- Blanford & Thorne, *Applications of Classical Physics lecture notes*
- Part IB *Optics lecture notes* (Cambridge)