

SPHERICAL WINDS – SPHERICAL ACCRETION

Spherical winds.

Many stars are known to loose mass. The solar wind carries away about $10^{-14}M_{\odot}yr^{-1}$ of very hot plasma. This rate is insignificant. In fact, solar radiation carries away $4 \times 10^{33} \text{ erg s}^{-1}$, which reduces solar mass by about $10^{-13}M_{\odot}yr^{-1}$. However, very luminous, supergiant stars are losing matter at a very high rate, frequently in the range $10^{-6}M_{\odot}yr^{-1}$ to $10^{-4}M_{\odot}yr^{-1}$ (cf. *Late Stages of Stellar Evolution*, Proceedings of the Workshop held in Calgary, Canada, from 2–5 June, 1986, Editors: S. Kwok and S. R. Pottasch, 1987, D. Reidel Publ. Co., Volume 132 of Astrophysics and Space Science Library). A supergiant with a luminosity 10^4L_{\odot} must burn about $10^{-7}M_{\odot}yr^{-1}$ to compensate for the radiative energy losses. In this case a mass loss rate at the level exceeding $10^{-7}M_{\odot}yr^{-1}$ is more important for the mass balance then nuclear burning is.

In spite of common presence of mass loss from bright stars there is no good quantitative theory that could explain it. It is believed that radiation pressure in atmospheric lines, i.e. the bound – bound transitions are responsible for mass loss from blue supergiants, while radiation pressure on dust is responsible for mass loss in red supergiants. The solar wind is due to very high temperature of the solar corona, above 10^6 K . We shall consider here two very simple models of mass outflow, none of which is directly related to any of the real, and complicated objects. In spite of their simplicity the models offer a good qualitative picture of the character of a steady – state outflow.

We shall consider spherically symmetric mass outflow, with the radial velocity of gas $v \equiv (\partial r / \partial t)_{M_r}$ being a function of radius, but not of time. That is we shall consider a case of a steady – state, time independent outflow. This is possible when the amount of matter within the dynamically important flow is much smaller than the amount of matter within the *container* from which the mass is flowing out. Typically, the *container* may be the stellar envelope, i.e. this part of a star which may be lost in the particular mass loss process. The assumption of a steady – state condition has a very well defined mathematical meaning. Let us consider any physical quantity, say q , which in general may be a function of time and radius. We may always write:

$$\left(\frac{\partial q}{\partial t}\right)_{M_r} = \left(\frac{\partial q}{\partial t}\right)_r + \left(\frac{\partial r}{\partial t}\right)_{M_r} \times \left(\frac{\partial q}{\partial r}\right)_t = \left(\frac{\partial q}{\partial t}\right)_r + v \left(\frac{\partial q}{\partial r}\right)_t. \quad (\text{w1.1})$$

In a strict steady – state we have $(\partial q / \partial t)_r = 0$. In practice, we shall assume that there is a steady state when $|(\partial q / \partial t)_r| \ll |(\partial q / \partial t)_{M_r}|$. In this case we may write the relation (w1.1) as

$$\left(\frac{\partial q}{\partial t}\right)_{M_r} = v \left(\frac{\partial q}{\partial r}\right)_t = \dot{M} \left(\frac{\partial q}{\partial M_r}\right)_t, \quad (\text{w1.2})$$

where

$$\dot{M} \equiv 4\pi r^2 \rho v = \text{const}, \quad (\text{w1.3})$$

is the rate of mass flow across a spherical surface with a radius r . In a steady – state model this rate is constant in space and in time.

There are two equations of stellar structure which contain time derivatives: the equation of motion, and the equation of heat balance:

$$\left(\frac{\partial P}{\partial M_r}\right)_t = -\frac{GM_r}{4\pi r^4} - \frac{1}{4\pi r^2} \left(\frac{\partial^2 r}{\partial t^2}\right)_{M_r}, \quad (\text{w1.4})$$

$$\left(\frac{\partial L_r}{\partial M_r}\right)_t = \epsilon - T \left(\frac{\partial S}{\partial t}\right)_{M_r}. \quad (\text{w1.5})$$

These equations may be simplified in case of a steady – state outflow by noticing that the total amount of mass within the steady – state part of the flow must be much smaller than the total stellar mass, otherwise the flow would not be stationary. Therefore, once we decided to study a steady – state outflow we should adopt $M_r = M = \text{const}$ in the term describing gravitational acceleration. Also, in the outer parts of a star there are no nuclear energy sources, and we have $\epsilon = 0$. The two time derivatives may be written as follows:

$$\frac{1}{4\pi r^2} \left(\frac{\partial^2 r}{\partial t^2} \right)_{M_r} = \frac{1}{4\pi r^2} \left(\frac{\partial v}{\partial t} \right)_{M_r} = \frac{1}{4\pi r^2} \dot{M} \left(\frac{\partial v}{\partial M_r} \right)_t = \frac{v}{4\pi r^2} \left(\frac{\partial v}{\partial r} \right)_t, \quad (\text{w1.6})$$

$$-T \left(\frac{\partial S}{\partial t} \right)_{M_r} = -T \dot{M} \left(\frac{\partial S}{\partial M_r} \right)_t = -\frac{T}{4\pi r^2 \rho} \dot{M} \left(\frac{\partial S}{\partial r} \right)_t. \quad (\text{w1.7})$$

Now, that we replaced time derivatives with space derivatives, the equations of stellar structure become **ordinary differential equations**. We may write them as follows:

$$\frac{1}{\rho} \frac{dP}{dr} = -\frac{GM}{r^2} - v \frac{dv}{dr} = \frac{d}{dr} \left(\frac{GM}{r} - \frac{v^2}{2} \right), \quad (\text{w1.8})$$

$$\frac{dL_r}{dr} = -T \dot{M} \frac{dS}{dr} = -\dot{M} \left(\frac{du}{dr} - \frac{P}{\rho^2} \frac{d\rho}{dr} \right) = -\dot{M} \left[\frac{d}{dr} \left(u + \frac{P}{\rho} \right) - \frac{1}{\rho} \frac{dP}{dr} \right]. \quad (\text{w1.9})$$

The last two equations may be combined to obtain

$$\frac{d}{dr} \left[L_r + \dot{M} \left(u + \frac{P}{\rho} + \frac{v^2}{2} - \frac{GM}{r} \right) \right] = \frac{d\dot{E}}{dr} = 0, \quad (\text{w1.10})$$

where

$$\dot{E} \equiv L_r + \dot{M} \left(u + \frac{P}{\rho} + \frac{v^2}{2} - \frac{GM}{r} \right) = \text{const}, \quad (\text{w1.11})$$

is the total energy carried with the flow across a spherical surface with a radius r .

The equation of motion and the equation of heat balance gave us energy conservation in a steady state flow. Together with mass conservation as described with equation (w1.3) we have two conservation laws. Therefore, we need just two differential equations to describe the flow, these may be the equation of motion (w1.8) and the equation of radiative equilibrium, which should hold in the optically thick part of the flow:

$$\frac{d \left(\frac{a}{3} T^4 \right)}{dr} = -\frac{\kappa \rho L_r}{4\pi c r^2}. \quad (\text{w1.12})$$

These two differential equations, (w1.8) and (w1.12), together with the two conservation laws, (w1.3) and (w1.11), allow to calculate the variation of T , ρ , L_r , and v with radius r . Of course, they have to be supplemented with proper boundary conditions in the deep interior, and at the stellar surface. This general problem is difficult to solve, because the diffusion approximation for the heat transport, (w1.12), is not valid at small optical depth, above stellar photosphere. Normally, in a star that is in a hydrostatic equilibrium, an optically thin atmosphere is also geometrically thin, and Eddington approximation give reasonable results. In the wind case the optically thin atmosphere is geometrically extended, and there is nothing as simple and as good as the Eddington approximation.

There is one very general property of a steady – state wind model: in the deep stellar interior the star should be in a hydrostatic equilibrium, i.e. we should have $v \ll v_s$, where v_s is speed of sound. At very large distance from a star we expect the flow to escape from gravitational potential, and hence we expect $v \gg v_s$. Therefore, somewhere in between there should be a transition from a subsonic flow to a supersonic flow. It turns out that the point at which $v = v_s$ is very special. It is called a sonic point, or a critical point, and its existence is a common property of all wind models. We shall demonstrate its existence in the following way. The equation of motion (w1.8) may be written as

$$\frac{1}{\rho} \frac{dP}{dr} = \frac{1}{r} \frac{dP}{d\rho} \frac{d \ln \rho}{d \ln r} = -\frac{1}{r} \left(\frac{GM}{r} + v^2 \frac{d \ln v}{d \ln r} \right), \quad (\text{w1.13})$$

and this gives

$$\frac{d \ln \rho}{d \ln r} = -\frac{GM}{rv_s^2} - \frac{v^2}{v_s^2} \frac{d \ln v}{d \ln r}, \quad (\text{w1.14})$$

where the speed of sound is defined as

$$v_s \equiv \left(\frac{dP}{d\rho} \right)^{1/2}. \quad (\text{w1.15})$$

Taking a logarithmic derivative of the mass conservation equation (w1.3) we obtain

$$2 + \frac{d \ln \rho}{d \ln r} + \frac{d \ln v}{d \ln r} = 0. \quad (\text{w1.16})$$

Combining equations (w1.14) and (w1.16) we find

$$\frac{d \ln v}{d \ln r} = \frac{\frac{GM}{r} - 2v_s^2}{v_s^2 - v^2} = \frac{v_{esc}^2 - 2v_s^2}{v_s^2 - v^2}, \quad (\text{w1.17})$$

with the condition $v \ll v_s$ at small radii, and $v \gg v_s$ at large radii. Notice, that $GM/r = v_{esc}^2$, where v_{esc} is the escape velocity from the gravitational potential GM/r .

At the sonic point we have $v = v_s$. The solution of the differential equation (w1.17) may be smooth at this point only if $GM/r = 2v_s^2$ at the same point. This is a non trivial condition on the flow, and it is as important as any boundary condition in determining a unique solution of the differential equation. Of course, the right hand side of equation (w1.17) being of the 0/0 type at the critical point, cannot be calculated directly. Instead, we can use de l'Hopital's rule, according to which $f/g = df/dg$ if $f = 0$, and $g = 0$, simultaneously. Remembering, that at the critical point we have $GM/r = 2v_s^2 = 2v^2$, we obtain

$$\begin{aligned} \frac{d \ln v}{d \ln r} &= \frac{\frac{d}{dr} \left(\frac{GM}{r} - 2v_s^2 \right)}{\frac{d}{dr} (v_s^2 - v^2)} = \\ &= \frac{-\frac{GM}{r} - 4v_s^2 \frac{d \ln v_s}{d \ln r}}{2v_s^2 \left(\frac{d \ln v_s}{d \ln r} \right) - 2v^2 \left(\frac{d \ln v}{d \ln r} \right)} = \frac{-1 - 2 \left(\frac{d \ln v_s}{d \ln r} \right)}{\left(\frac{d \ln v_s}{d \ln r} \right) - \left(\frac{d \ln v}{d \ln r} \right)}. \end{aligned} \quad (\text{w1.18})$$

This may be written as a quadratic equation for $(d \ln v / d \ln r)$:

$$\left(\frac{d \ln v}{d \ln r} \right)^2 - \left(\frac{d \ln v_s}{d \ln r} \right) \left(\frac{d \ln v}{d \ln r} \right) - 1 - 2 \left(\frac{d \ln v_s}{d \ln r} \right) = 0. \quad (\text{w1.19})$$

There are two real roots of this equations, provided

$$\left(\frac{d \ln v_s}{d \ln r} \right)^2 + 8 \left(\frac{d \ln v_s}{d \ln r} \right) + 4 > 0, \quad (\text{w1.20})$$

This inequality is satisfied when

$$\left(\frac{d \ln v_s}{d \ln r} \right) > -4 + \sqrt{12} = -0.5359, \quad (\text{w1.21a})$$

or

$$\left(\frac{d \ln v_s}{d \ln r} \right) < -4 - \sqrt{12} = -7.4641, \quad (\text{w1.21b})$$

The logarithmic derivative of the speed of sound with respect to radius may be expressed as

$$\begin{aligned} \frac{d \ln v_s}{d \ln r} &= \frac{1}{2} \frac{d \ln P}{d \ln r} - \frac{1}{2} \frac{d \ln \rho}{d \ln r} = \\ &= \frac{1}{2} \left(\frac{\partial \ln P}{\partial \ln T} \right)_\rho \frac{d \ln T}{d \ln r} - \frac{1}{2} \left[\left(\frac{\partial \ln P}{\partial \ln \rho} \right)_T - 1 \right] \frac{d \ln \rho}{d \ln r}, \end{aligned} \quad (\text{w1.22})$$

The inner boundary condition requires the wind model to be in a hydrostatic equilibrium at small radii, so it could be matched with a hydrostatic equilibrium model of the whole star. There are also two outer boundary conditions. We expect the flow to expand into empty space, and therefore, density and pressure should be falling down to zero at very large radii. Also, at the photosphere a thermal boundary condition must be satisfied, i.e.

$$L_r = 4\pi r^2 \sigma T^4, \quad (\text{w1.23})$$

at optical depth $\tau \approx 2/3$. This is only approximate condition, because Eddington approximation is not good in the extended atmosphere of the wind, and the condition (w1.23) may be satisfied at some other optical depth. As our equation (w1.12) and the interaction between gas and radiation become complicated at small optical depth, there is no simple and accurate way to formulate the thermal outer boundary condition.

We shall consider now a very simple, **isothermal wind model**. The model is so simple that we shall find analytical solution for the flow. At the same time the isothermal model retains all the most important characteristics of the general, steady – state outflow. For the isothermal flow we have

$$v_s^2 = \left(\frac{dP}{d\rho} \right)_T = \frac{kT}{\mu H} = \text{const}. \quad (\text{w1.24})$$

Let us define dimensionless radius ξ , and dimensionless velocity u :

$$\xi \equiv \frac{r}{r_c}, \quad u \equiv \frac{v}{v_s}, \quad (\text{w1.25})$$

where r_c is the critical radius, i.e. the radius where the velocity of outflow is equal to the speed of sound. According to equation (w1.17) this corresponds to

$$r_c = \frac{GM}{2v_s^2} = \frac{GM\mu H}{2kT}. \quad (\text{w1.26})$$

The equation (w1.17) may be written in dimensionless variables as

$$\frac{d \ln u}{d \ln \xi} = \frac{\frac{2}{\xi} - 2}{1 - u^2}. \quad (\text{w1.27})$$

The variables in equation (w1.27) can be separated:

$$\left(\frac{1}{u} - u \right) du = 2 \left(\frac{1}{\xi^2} - \frac{1}{\xi} \right) d\xi, \quad (\text{w1.28})$$

and the equation (w1.28) can be integrated to obtain

$$\ln u - \frac{u^2}{2} = -\frac{2}{\xi} - 2 \ln \xi + C, \quad (\text{w1.29})$$

where C is the integration constant. For the solution to pass through the critical point we need $C = 1.5$. The whole family of solutions corresponding to various values of the constant C is shown in figure 1. The two thick solid lines that cross at the critical point: $u = 1$, $\xi = 1$, are the two critical solutions. The line that satisfies the wind boundary condition is the one that goes from the lower left corner to the upper right corner, i.e. it correspond to a subsonic flow at $\xi < 1$ (i.e. at small radii), and to a supersonic flow at $\xi > 1$ (i.e. at large radii).

Spherical accretion

We may consider a problem opposite to the wind outflow, and this is accretion of matter onto a star. In this case we need some finite density medium at large radii, gradually falling onto the star. The inward flow would be subsonic at very large radii, and would become supersonic free fall at small radii. The equation are the same as for the wind outflow, just the direction of the flow is reversed. In the accretion flow there is also a critical point, where the infall velocity is equal to the speed of sound. For an isothermal accretion the relevant critical solution corresponds to a thick solid line that goes from the lower right corner of figure 1 to the upper left corner of that figure.

If the accreting object is a black hole, which does not have a hard surface, the supersonic accretion flow may go right into the black hole. Of course, the relevant equations of motion have to be relativistic, but this does not change the topology of solutions – they remain the same as those in figure 1. However, if the accreting star has a surface, the supersonic flow has to be stopped at some point, and the infalling matter must come to rest. A shock wave is formed at radius smaller than critical radius, and the infall velocity is reduced from supersonic to subsonic. It may be shown that the solution for a stationary accretion flow below the shock wave corresponds to one of the curves in figure 1 that go to the lower left corner – the inflowing matter comes to a hydrostatic equilibrium, and merges with the star. A possible solution is indicated in figure 1 with a thick broken line.