WHITE DWARFS (DEGENERATE DWARFS)

White dwarfs are stars supported by pressure of degenerate electron gas, i.e. in their interiors thermal energy kT is much smaller then Fermi energy E_F. We shall derive the equations of structure of white dwarfs, sometimes called degenerate dwarfs, in the limiting case when their thermal pressure may be neglected, but the degenerate electron gas may be either non-relativistic, somewhat relativistic, or ultra-relativistic.

We shall introduce variable x defined as a dimensionless electron momentum:

$$x \equiv p/mc, \qquad x_F \equiv p_F/mc. \tag{wd.1}$$

Following the derivations in the chapter "Equation of state" we may write density and pressure as a function of dimensionless Fermi momentum x_F

$$\rho = A\mu_e x_F^3, \qquad \mu_e = \frac{2}{1+X}, \tag{wd.2a}$$

$$P = B \int_0^{x_F} \frac{x^4 dx}{(1+x^2)^{1/2}}, \tag{wd.2b}$$

where X is hydrogen abundance by mass fraction, μ_e is the mean number of nucleons per electron, and

$$A \equiv \frac{8\pi}{3}\left(\frac{mc}{h}\right)^3 H = 0.981 \times 10^6 \qquad [\text{g cm}^{-3}], \tag{wd.3a}$$

$$B \equiv \frac{8\pi}{3}\left(\frac{mc}{h}\right)^3 mc^2 = 4.80 \times 10^{23} \qquad [\text{erg cm}^{-3}]. \tag{wd.3b}$$

The equation of hydrostatic equilibrium may be written as

$$\frac{dP}{dr} = \frac{dP}{dx}\frac{dx}{dr} = B\frac{x^4}{(1+x^2)^{1/2}}\frac{dx}{dr} = \tag{wd.4}$$

$$-\frac{GM_r}{r^2}\rho = -\frac{GM_r}{r^2}A\mu_e x^3,$$

where we wrote x instead of x_F, for simplicity. The equation of mass conservation may be written as

$$\frac{dM_r}{dr} = 4\pi r^2 \rho = 4\pi r^2 A\mu_e x^3. \tag{wd.5}$$

We shall introduce dimensionless variables x_1, x_2, and x_3, defined with:

$$r \equiv \alpha_r x_1, \qquad M_r \equiv \alpha_m x_2, \qquad 1 + x^2 \equiv x_3. \tag{wd.6}$$

Combining equations (wd.4) , (wd.5) , and (wd.6) we obtain equations in dimensionless variables

$$\frac{dx_3}{dx_1} = -\left[\frac{2GA}{B}\frac{\alpha_m}{\alpha_r}\right] \frac{x_2}{x_1^2}\, x_3^{1/2}, \tag{wd.7}$$

$$\frac{dx_2}{dx_1} = \left[4\pi A\mu_e \frac{\alpha_r^3}{\alpha_m}\right] x_1^2 \, (x_3 - 1)^{1.5}. \tag{wd.8}$$

The two scaling parameters α_r and α_m may be adjusted so as to make the constants in the square brackets in the equations (wd.7) and (wd.8) equal to one. This is accomplished by setting

$$\alpha_r = \left(\frac{B}{8\pi G}\right)^{1/2} \frac{1}{A\mu_e} = 5.455 \times 10^8 \ \mu_e^{-1} \ (\,\mathrm{cm}\,) = 0.00784 \ R_\odot \mu_e^{-1}, \tag{wd.9}$$

$$\alpha_m = \frac{1}{(2\pi)^{1/2}} \left(\frac{B}{G}\right)^{1.5} \frac{1}{(2A\mu_e)^2} = 2.00 \times 10^{33} \ \mu_e^{-2} \ (\,\mathrm{g}\,) = 1.005 \ M_\odot \mu_e^{-2}. \tag{wd.10}$$

With these scaling parameters the two equations (wd.7) and (wd.8) may by written as

$$\frac{dx_3}{dx_1} = -\frac{x_2}{x_1^2} \ x_3^{1/2}, \tag{wd.11}$$

$$\frac{dx_2}{dx_1} = x_1^2 \left(x_3 - 1\right)^{1.5}. \tag{wd.12}$$

The dimensionless equations (wd.11) and wd.12) have to be supplemented with the boundary conditions. These are:

$$x_3 = x_{3,c}, \qquad x_2 = 0, \qquad \text{at} \ \ x_1 = 0, \qquad \text{(inner boundary conditions)}, \tag{wd.13}$$

$$x_3 = 1, \qquad x_2 = x_{2,s}, \qquad \text{at} \ \ x_1 = x_{1,s}, \qquad \text{(outer boundary condition)}, \tag{wd.14}$$

where $x_{3,c}$ is the central value of the variables x_3, and $x_{2,s}$ and $x_{1,s}$ are the surface values of dimensionless mass and radius, respectively. The total mass and radius of a white dwarf are given as

$$M = \alpha_m x_{2,s}, \qquad R = \alpha_r x_{1,s}. \tag{wd.15}$$

As we have two inner boundary conditions (wd.13) , and there are two ordinary differential equations (wd.11) and (wd.12) , we may treat (wd.13) as the initial conditions for the integrations, with $x_{3,c}$ being a free parameter. For a given value of $x_{3,c}$ we may calculate central density using equations (wd.2a) and (wd.6) , integrate numerically equations (wd.11) and (wd.12) , calculate $x_{1,s}$ and $x_{2,s}$ corresponding to $x_3 = 1$, i.e. $\rho = 0$ at the white dwarf surface, and finally calculate the total mass and radius with equations (wd.15) . In this way we may obtain the mass - radius relation for white dwarfs.

The mass - radius relation for white dwarfs may be estimated using the usual algebraic approximation to the differential equations of stellar structure and an analytical approximation to the equation of state for degenerate electron gas. The equations of stellar structure may be approximated with:

$$\frac{M}{R} \approx R^2 \rho, \qquad \text{i.e.} \qquad \rho \approx \frac{M}{R^3}, \tag{wd.16}$$

$$\frac{P}{R} \approx \frac{GM}{R^2} \rho \approx \frac{GM^2}{R^5}, \qquad \text{i.e.} \qquad P \approx \frac{GM^2}{R^4}. \tag{wd.17}$$

The equation of state may be approximated as

$$P \approx \left[\left(K_1 \rho^{5/3}\right)^{-2} + \left(K_2 \rho^{4/3}\right)^{-2} \right]^{-1/2}. \tag{wd.18}$$

Equations (wd.16) , (wd.17) , and (wd.18) may be combined to obtain

$$P^{-2} \approx \frac{R^8}{G^2 M^4} \approx \frac{R^{10}}{K_1^2 M^{10/3}} + \frac{R^8}{K_2^2 M^{8/3}}. \tag{wd.19}$$

This may be rearranged to have

$$R \approx \frac{K_1}{GM^{1/3}} \left[1 - \frac{G^2 M^{4/3}}{K_2^2} \right]^{1/2} . \qquad \text{(wd.20)}$$

The last equation should have the correct asymptotic form, but there may be dimensionless coefficients of the order unity that our approximate analysis cannot provide. However, we may recover the coefficients noticing that in the two limiting cases, $\rho \ll 10^6$ g cm $^{-3}$, and $\rho \gg 10^6$ g cm $^{-3}$, the equation of state (wd.18) is very well approximated with a polytrope with index $n = 1.5$ and $n = 3$, respectively. In these two limiting cases we have exact mass - radius relations:

$$R = \frac{K_1}{0.4242 \ GM^{1/3}} \qquad \text{for} \quad n = 1.5, \qquad \text{(wd.21)}$$

$$M = \left(\frac{K_2}{0.3639 \ G} \right)^{1.5} = 1.142 \times 10^{34} \ \mu_e^{-2} \ \ (\,\text{g}\,) = 5.74 \ M_\odot \mu_e^{-2}, \qquad \text{for} \quad n = 3, \qquad \text{(wd.22)}$$

Combining equations (wd.21) and (wd.22) we may write (wd.20) as

$$R \approx \frac{K_1}{0.4242 \ GM^{1/3}} \left[1 - \left(\frac{M}{M_{Ch}} \right)^{4/3} \right]^{1/2} = \qquad \text{(wd.23)}$$

$$0.0126 \ R_\odot \left(\frac{2}{\mu_e} \right)^{5/3} \left(\frac{M}{M_\odot} \right)^{-1/3} \left[1 - \left(\frac{M}{M_{Ch}} \right)^{4/3} \right]^{1/2} ,$$

with the Chandrasekhar's mass

$$M_{Ch} = 1.435 \ M_\odot \left(\frac{2}{\mu_e} \right)^2 . \qquad \text{(wd.24)}$$

The analytical formula (wd.23) approximates the exact numerical mass - radius relation for white dwarfs with an error smaller than 15% for masses near Chandrasekhar limit, and a much better accuracy at lower masses.

The following table gives a comparison between the numerical and analytical values of white dwarf radii for $\mu_e = 2$. The first column gives the logarithm of central density, the second white dwarf mass in units of $M_\odot$, the third and fourth give numerical and analytical white dwarf radii, respectively, in units of $R_\odot$, and the fifth column gives the fractional error of the analytical radii.

$\log \rho_c$	$M/M_\odot$	$R/R_\odot$		error
		numerical	analytical	
4	.04811	.03448	.03446	.0008
5	.14600	.02339	.02335	.0015
6	.39366	.01566	.01558	.0048
7	.80146	.01013	.00997	.0158
8	1.16176	.00619	.00593	.0411
9	1.34619	.00353	.00325	.0803
10	1.41096	.00188	.00165	.1230