THERMAL STABILITY OF LOW MASS STARS

Let us consider a star in a hydrostatic equilibrium, i.e. we have $d^2r/dt^2 = 0$. Let the star be also in a thermal equilibrium, i.e. we have $dS/dt = 0$. Now we shall make a perturbation that is so slow that is does not disturb hydrostatic equilibrium but it disturbs thermal equilibrium, i.e. now $dS/dt \neq 0$. The question is: will the perturbation grow or decay? If it grows then the star is thermally unstable. If it decays then the star is thermally stable against this particular perturbation. Of course, there may be many other perturbations, and the full analysis is very complicated. However, thermal stability analysis is very simple for lower main sequence stars, because they are fully convective, i.e. the specific entropy is the same throughout the whole star. This means that the *gravitational luminosity* L_g (cf. notes on *STARS IN A HYDROSTATIC EQUILIBRIUM*) may be written as

$$L_g \equiv - \int_0^M T \left(\frac{\partial S}{\partial t} \right)_{M_r} dM_r = - \frac{dS}{dt} \int_0^M T \ dM_r. \tag{ts.1}$$

The nuclear luminosity L_n is given as

$$L_n \equiv \int_0^M \epsilon_n dM_r, \tag{ts.2}$$

and the surface luminosity is

$$L = 4\pi R^2 \sigma T_{eff}^4. \tag{ts.3}$$

A model in a thermal equilibrium satisfies the equation:

$$L = L_n + L_g, \qquad L_g = 0, \qquad \text{(thermal equilibrium)}, \tag{ts.4}$$

while in a perturbed model we have

$$\delta L = \delta L_n + \delta L_g, \qquad \text{or} \qquad \frac{\delta L}{L} = \frac{\delta L_n}{L} + \frac{\delta L_g}{L}. \tag{ts.5}$$

The important simplicity of a low mass star is the efficient convection which forces the interior to be iso-entropic. Any change in entropy in one part of a star is redistributed throughout the whole star on a time scale only slightly longer than dynamical, i.e. very much shorter than overall thermal time scale. This implies that there is only one thermal mode within fully convective star, the mode which changes specific entropy uniformly. Therefore, we have to analyze only this one mode to find if the star is thermally stable or unstable. The perturbed luminosities satisfy the equations

$$\frac{\delta L}{L} = 2 \frac{\delta R}{R} + 4 \frac{\delta T_{eff}}{T_{eff}}, \tag{ts.6}$$

$$\delta L_n = \int_0^M \delta \epsilon_n dM_r, \tag{ts.7}$$

$$\delta L_g = - \frac{d(\delta S)}{dt} \int_0^M T dM_r, \tag{ts.8}$$

A lower main sequence star may be considered to be a polytrope with an index $n = 1.5$ (cf. notes on *LOW MASS STARS*). To make our analysis as simple as possible we shall neglect electron degeneracy, and adopt the simple equation of state:

$$P = \frac{k}{\mu H}\rho T, \qquad u = 1.5\frac{k}{\mu H}T, \qquad dS = 1.5\frac{k}{\mu H}\left(d\ln T - \frac{2}{3}d\ln\rho\right), \tag{ts.9}$$

We shall also adopt a simple formula for heat generation rate in the proton-proton reaction:

$$\epsilon_n = \epsilon_0 \rho T^{\nu}, \qquad \frac{\delta\epsilon_n}{\epsilon_n} = \frac{\delta\rho}{\rho} + \nu\frac{\delta T}{T}, \qquad \nu \approx 5. \tag{ts.11}$$

Thermally perturbed polytropic star remains polytropic, just density and temperature throughout the convective interior change according to

$$\frac{\delta\rho}{\rho} = -3\frac{\delta R}{R}, \qquad \frac{\delta T}{T} = -\frac{\delta R}{R}. \tag{ts.12}$$

Notice, the perturbation of temperature described with equation (ts.12) does not apply to T_{eff}, as that is governed by outer boundary condition, i.e. model atmosphere that **is not** a part of polytropic stellar interior. Combining equations (ts.9) and (ts.12) we obtain for the variation of entropy

$$\delta S = 1.5\frac{k}{\mu H}\left(\frac{\delta T}{T} - \frac{2}{3}\frac{\delta\rho}{\rho}\right) = 1.5\frac{k}{\mu H}\frac{\delta R}{R}. \tag{ts.13}$$

For a non-degenerate polytropic ($n = 1.5$) star we also have

$$1.5\frac{k}{\mu H}\int_0^M T dM_r = \int_0^M u dM_r = E_{th} = -\frac{1}{2}\Omega = \frac{3}{7}\frac{GM^2}{R}, \tag{ts.14}$$

(cf. notes on *STARS IN A HYDROSTATIC EQUILIBRIUM* and on *POLYTROPES*). Combining equations (ts.8), (ts.13) and (ts.14) we obtain

$$\delta L_g = -\frac{3}{7}\frac{GM^2}{R^2}\frac{d(\delta R)}{dt}. \tag{ts.15}$$

Notice, that the results expressed with equation (ts.15) could be obtained directly from the equation (eql.15) of *STARS IN A HYDROSTATIC EQUILIBRIUM* , for any star supported by pressure of non-relativistic gas. In particular, the equation (ts.15) is valid even if electron gas is partially or fully degenerate, as long as it is non-relativistic, i.e. as long as the star has a small mass.

We shall look for time variability of the type $e^{\sigma t}$, where σ is the eigen-value of the problem. With $\delta R \sim e^{\sigma t}$ we have

$$\frac{d(\delta R)}{dt} = \sigma\delta R. \tag{ts.16}$$

Combining equations (ts.15) and (ts.16) we get

$$\delta L_g = -\sigma\frac{3}{7}\frac{GM^2}{R}\frac{\delta R}{R}. \tag{ts.17}$$

Combining equations (ts.7), (ts.11) and (ts.12) we find

$$\delta L_n = \int_0^M \left(\nu\frac{\delta T}{T} + \frac{\delta\rho}{\rho}\right)\epsilon_n dM_r = \tag{ts.18}$$

$$= -(\nu+3)\frac{\delta R}{R}\int_0^M \epsilon_n dM_r = -(\nu+3)L_n\frac{\delta R}{R} = -(\nu+3)L\frac{\delta R}{R},$$

because in the equilibrium model $L_n = L$ (cf. eq. ts.4).

Lower main sequence stars are on the Hayashi line (i.e. they are fully convective), and their effective temperatures are almost constant (this will be justified in a lecture on the *HAYASHI LIMIT*). For the purpose of this analysis we shall adopt $\delta T_{eff} = 0$, and this makes (cf. eq. ts.6)

$$\frac{\delta L}{L} \approx 2\frac{\delta R}{R}. \tag{ts.19}$$

This way we expressed all perturbations in terms of $\delta R/R$. Combining equations (ts.5), (ts.17), (ts.18), and (ts.19) we find

$$2\frac{\delta R}{R} = -\left(\nu + 3\right)\frac{\delta R}{R} - \sigma\frac{3}{7}\frac{GM^2}{RL}\frac{\delta R}{R}. \tag{ts.20}$$

With $\delta R/R$ in every term of equation (ts.20) we may obtain for σ

$$\sigma = -\frac{7\left(\nu + 5\right)}{3}\frac{RL}{GM^2} < 0. \tag{ts.21}$$

The eigen-value σ is negative, i.e. the model is thermally stable.

Let us consider now even lower mass stars, for which electron gas may be partly degenerate, but non-relativistic. Such stars are still fully convective, and well described by an $n = 1.5$ polytrope, but the central temperature is no longer proportional to R^{-1}. Instead, according to equation (lms.7) (cf. notes on *Low Mass Stars*) we have

$$T_c = 0.539\,\frac{\mu H}{k}\frac{GM}{R}\left[1 - \left(\frac{R_{min}}{R}\right)^2\right]^{1/2}, \qquad R_{min} = \frac{K_1}{0.4242\,GM^{1/3}}. \tag{ts.22}$$

Let us now consider the differences in density and temperature between stars with slightly different masses and radii. We have

$$\frac{\delta\rho}{\rho} = \frac{\delta\rho_c}{\rho_c} = \frac{\delta M}{M} - 3\frac{\delta R}{R}, \tag{ts.23}$$

$$\frac{\delta T}{T} = \frac{\delta T_c}{T_c} = \frac{1 - \frac{2}{3}\left(\frac{R_{min}}{R}\right)^2}{1 - \left(\frac{R_{min}}{R}\right)^2}\frac{\delta M}{M} + \frac{2\left(\frac{R_{min}}{R}\right)^2 - 1}{1 - \left(\frac{R_{min}}{R}\right)^2}\frac{\delta R}{R}. \tag{ts.24}$$

We shall consider first a sequence of models which are in a thermal equilibrium, i.e. for which $L = L_n$, and $L_g = 0$ (cf. equation ts.4). Just as before we shall adopt the following relations:

$$\frac{\delta L}{L} = 2\frac{\delta R}{R}, \qquad (\ T_{eff} = \text{const. is assumed}), \tag{ts.25}$$

$$\frac{\delta L_n}{L} = \frac{\delta\rho_c}{\rho_c} + \nu\frac{\delta T_c}{T_c}. \tag{ts.26}$$

Combining equations (ts.23) – (ts.26) we obtain

$$\frac{\delta\left(L_n - L\right)}{L} = \tag{ts.27}$$

$$= \frac{\left(\nu + 1\right) - \left(\frac{2}{3}\nu + 1\right)\left(\frac{R_{min}}{R}\right)^2}{1 - \left(\frac{R_{min}}{R}\right)^2}\frac{\delta M}{M} + \frac{\left(2\nu + 5\right)\left(\frac{R_{min}}{R}\right)^2 - \left(\nu + 5\right)}{1 - \left(\frac{R_{min}}{R}\right)^2}\frac{\delta R}{R}.$$

Along the sequence of thermal equilibrium models we require that $\delta(L_n - L) = 0$. Therefore, the equation (ts.27) gives a differential mass – radius relation which may be written in a form

$$\frac{d\ln M}{d\ln R} = \frac{\left(\nu + 5\right) - \left(2\nu + 5\right)\left(\frac{R_{min}}{R}\right)^2}{\left(\nu + 1\right) - \left(\frac{2}{3}\nu + 1\right)\left(\frac{R_{min}}{R}\right)^2}. \tag{ts.28}$$

When stars are non degenerate, i.e. when $R \gg R_{min}$, the mass – radius relation (ts.28) gives

$$\frac{d\ln M}{d\ln R} = \frac{\nu+5}{\nu+1} = \frac{5}{3}, \qquad \text{for} \qquad R \gg R_{min}, \qquad \nu = 5. \tag{ts.29}$$

This well describes the relation for low mass main sequence stars. When mass is lowered, the mass – radius relation (ts.29) implies $\rho_c \sim M/R^3 \sim M^{-4/5}$, $T_c \sim M/R \sim M^{2/3}$, and the ratio of degenerate to non – degenerate pressure $P_d/P_{nd} \sim \rho_c^{2/3}/T_c \sim M^{-14/15}$. Therefore, for stars with lower mass electron degeneracy is more important. Increase of degeneracy implies that the ratio R/R_{min} is reduced, and ultimately may approach 1, but cannot be less than 1. Therefore, the denominator of the right hand side of equation (ts.28) is always positive, but the numerator may become negative while the ratio R_{min}/R increases. In particular, the numerator vanishes, and we reach a minimum mass when

$$(\nu+5) - (2\nu+5)\left(\frac{R_{min}}{R}\right)^2 = 0, i.e. \tag{ts.30}$$

$$\frac{R}{R_{min}} = \left(\frac{2\nu+5}{\nu+5}\right)^{1/2} = 1.5^{1/2} \approx 1.225.$$

The sequence of stars in thermal equilibrium may be extended beyond the point defined with equation (ts.30). However, while the stellar radius may still decrease, the stellar mass will increase. Therefore, for masses somewhat above the minimum mass there are two different equilibrium models: one on the so called normal branch of the main sequence, along which stellar mass and radius increase together, and a second model on the so called high density branch, along which stellar radius decreases, while the stellar mass increases. This branch is only crudely described with our simplified outer boundary condition, T_{eff} = const. At some point, when the interior of the star becomes more and more degenerate, the surface temperature must drop as well. However, even this crude outer boundary condition allows to demonstrate that there is a minimum mass for hydrogen burning models in thermal equilibrium. Numerical computations show that the minimum mass is $0.08 M_\odot$.

Let us consider now thermal stability of models near the minimum mass. The relations (ts.15) and (ts.17) hold for any non-relativistic star described with an $n = 1.5$ polytrope. We have $L = L_n + L_g$, and $\delta(L_n - L) = -\delta L_g$. Combining equations (ts.17) and (ts.27), and keeping $\delta M = 0$ we obtain

$$\sigma = \frac{7}{3}\frac{RL}{GM^2}\frac{(2\nu+5)\left(\frac{R_{min}}{R}\right)^2 - (\nu+5)}{1-\left(\frac{R_{min}}{R}\right)^2}. \tag{ts.31}$$

In the limit of $R \gg R_{min}$ the equation (ts.31) gives the same result as the equation (ts.21). However, the equation (ts.31) provides the eigenvalue σ also for partly degenerate stars. It is clear that $\sigma < 0$ when $R > 1.225 R_{min}$, i.e. on the normal branch of the main sequence, $\sigma = 0$ when the minimum mass is reached, and $\sigma > 0$ for models on the high density branch of the main sequence. Therefore, the normal main sequence stars are thermally stable, while models on the high density branch are thermally unstable. The transition from stability to instability coincides with the turning point of the main sequence, where a minimum mass is reached. This is just as expected from a general relation between linear series of stellar models and stellar stability.

Similar technique can be used to analyze thermal stability of very massive stars, which are also fully convective. In that case we may use as a very good approximation the Eddington model and approximate stellar structure with $n = 3$ polytrope. In that case not only gas pressure, but also radiation pressure has to be allowed for, and a thermal perturbation does not change the surface stellar luminosity, as the luminosity depends on the stellar mass only.