GENERAL THERMODYNAMIC CONSIDERATIONS

The first law of thermodynamics may be written as a heat balance within one gram of matter

$$dQ = T\,dS = du + P\,dV, \tag{gt.1a}$$

where dQ is the heat input or the heat loss, T is temperature, S is entropy per gram, u is internal energy density per gram, P is pressure, and $V = 1/\rho$ is specific volume, i.e. the inverse of density. The equation (gt.1a) gives a balance between the heat input, the change in gas energy, and the mechanical work. In this equation entropy, energy density, and pressure are all functions of density and temperature, $S(\rho, T), u(\rho, T), P(\rho, T)$, but the heat Q is not a unique function of (ρ, T) . We shall frequently write the equation (gt.1a) in different forms:

$$T\,dS = d\left(\frac{U}{\rho}\right) - \frac{P}{\rho^2}\,d\rho, \tag{gt.1b}$$

where U is energy density per unit volume, or

$$dS = \frac{1}{T}\,du - \frac{P}{T\rho^2}\,d\rho = \left[\frac{1}{T}\left(\frac{\partial u}{\partial T}\right)_\rho\right]dT + \left[\frac{1}{T}\left(\frac{\partial u}{\partial \rho}\right)_T - \frac{P}{T\rho^2}\right]d\rho = \tag{gt.1c}$$

$$\left(\frac{\partial S}{\partial T}\right)_\rho dT + \left(\frac{\partial S}{\partial \rho}\right)_T d\rho.$$

From the last equation it follows that

$$\left(\frac{\partial S}{\partial T}\right)_\rho = \frac{1}{T}\left(\frac{\partial u}{\partial T}\right)_\rho, \tag{gt.2a}$$

$$\left(\frac{\partial S}{\partial \rho}\right)_T = \frac{1}{T}\left(\frac{\partial u}{\partial \rho}\right)_T - \frac{P}{T\rho^2}. \tag{gt.2b}$$

The second derivatives of a well behaved function satisfy the relation

$$\left[\frac{\partial}{\partial T}\left(\frac{\partial S}{\partial \rho}\right)_T\right]_\rho = \left[\frac{\partial}{\partial \rho}\left(\frac{\partial S}{\partial T}\right)_\rho\right]_T, \tag{gt.3}$$

i.e. it does not matter in which order we differentiate. Therefore, when we take a derivative of equation (gt.2a) with respect to density, and a derivative of equation (gt.2b) with respect to temperature, we should get the same result, i.e.

$$\frac{1}{T}\frac{\partial^2 u}{\partial \rho \partial T} = -\frac{1}{T^2}\frac{\partial u}{\partial \rho} + \frac{1}{T}\frac{\partial^2 u}{\partial T \partial \rho} + \frac{P}{T^2\rho^2} - \frac{1}{T\rho^2}\frac{\partial P}{\partial T}. \tag{gt.4}$$

The term on the left hand side of (gt.4) cancels with the corresponding term on the right hand side. Rearranging the three remaining terms we obtain

$$\left(\frac{\partial u}{\partial \rho}\right)_T = \frac{P}{\rho^2} - \frac{T}{\rho^2}\left(\frac{\partial P}{\partial T}\right)_\rho = \frac{P}{\rho^2}\left[1 - \left(\frac{\partial \ln P}{\partial \ln T}\right)_\rho\right]. \tag{gt.5}$$

The relation (gt.5) is a thermodynamic identity which must be satisfied by any equation of state. It is useful for either quick finding of the derivative $(\partial u/\partial \rho)_T$, or as a check on the results.

We shall need various thermodynamic functions. Among those are the adiabatic exponent γ and the adiabatic temperature gradient ∇_{ad} defined as

$$\gamma \equiv \left(\frac{\partial \ln P}{\partial \ln \rho}\right)_S, \qquad \nabla_{ad} \equiv \left(\frac{\partial \ln T}{\partial \ln P}\right)_S. \tag{gt.6a}$$

We have, of course

$$\left(\frac{\partial \ln T}{\partial \ln \rho}\right)_S = \gamma \nabla_{ad}. \tag{gt.6b}$$

We may also need specific heats, c_V and c_P, i.e. the amount of heat needed to rise a temperature of 1 gram of matter by 1 K under constant density, or under constant pressure, respectively. We have

$$c_V = T\left(\frac{\partial S}{\partial T}\right)_\rho = \left(\frac{\partial u}{\partial T}\right)_\rho, \quad \left[\text{erg g}^{-1}\,\text{K}^{-1}\right], \tag{gt.7a}$$

$$c_P = T\left(\frac{\partial S}{\partial T}\right)_P = \left(\frac{\partial u}{\partial T}\right)_P - \frac{P}{\rho^2}\left(\frac{\partial \rho}{\partial T}\right)_P \quad \left[\text{erg g}^{-1}\text{K}^{-1}\right], \tag{gt.7b}$$

We shall express all these thermodynamic quantities in terms of ρ, T, P, $(\partial P/\partial T)_\rho$, $(\partial P/\partial \rho)_T$, and $(\partial u/\partial T)_\rho$. For any equation of state we may write:

$$dP = \left(\frac{\partial P}{\partial T}\right)_\rho dT + \left(\frac{\partial P}{\partial \rho}\right)_T d\rho, \tag{gt.8}$$

which allows to calculate

$$\left(\frac{\partial \rho}{\partial T}\right)_P = -\left(\frac{\partial P}{\partial T}\right)_\rho \bigg/ \left(\frac{\partial P}{\partial \rho}\right)_T. \tag{gt.9}$$

In order to be able to express c_P in a desirable form we shall write

$$du = \left(\frac{\partial u}{\partial T}\right)_P dT + \left(\frac{\partial u}{\partial P}\right)_T dP = \tag{gt.10}$$

$$\left[\left(\frac{\partial u}{\partial T}\right)_P + \left(\frac{\partial u}{\partial P}\right)_T \left(\frac{\partial P}{\partial T}\right)_\rho\right] dT + \left[\left(\frac{\partial u}{\partial P}\right)_T \left(\frac{\partial P}{\partial \rho}\right)_T\right] d\rho.$$

The last equation gives

$$\left(\frac{\partial u}{\partial T}\right)_P + \left(\frac{\partial u}{\partial P}\right)_T \left(\frac{\partial P}{\partial T}\right)_\rho = \left(\frac{\partial u}{\partial T}\right)_\rho = c_V, \tag{gt.11a}$$

$$\left(\frac{\partial u}{\partial P}\right)_T \left(\frac{\partial P}{\partial \rho}\right)_T = \left(\frac{\partial u}{\partial \rho}\right)_T = \frac{P}{\rho^2} - \frac{T}{\rho^2}\left(\frac{\partial P}{\partial T}\right)_\rho. \tag{gt.11b}$$

Combining equations (gt.7b), (gt.9), (gt.11a), and (gt.11b) we obtain

$$c_P = c_V + \frac{T}{\rho^2}\left(\frac{\partial P}{\partial T}\right)_\rho^2 \bigg/ \left(\frac{\partial P}{\partial \rho}\right)_T. \tag{gt.12}$$

Combining equations (gt.2a), (gt.2b), (gt.5), and (gt.7a) we have

$$\left(\frac{\partial S}{\partial T}\right)_\rho = \frac{c_V}{T}, \qquad \left(\frac{\partial S}{\partial \rho}\right)_T = -\frac{1}{\rho^2}\left(\frac{\partial P}{\partial T}\right)_\rho. \tag{gt.13}$$

Setting $dS = 0$ in the equation (gt.1c) we obtain

$$\left(\frac{\partial T}{\partial \rho}\right)_S = -\left(\frac{\partial S}{\partial \rho}\right)_T \bigg/ \left(\frac{\partial S}{\partial T}\right)_\rho. \tag{gt.14}$$

Combining equations (gt.6b), (gt.13), and (gt.14) we have

$$\left(\frac{\partial \ln T}{\partial \ln \rho}\right)_S = \frac{1}{\rho c_V}\left(\frac{\partial P}{\partial T}\right)_\rho = \gamma \nabla_{ad}. \tag{gt.15}$$

Now, we shall express ∇_{ad} in terms of easy to calculate quantities. We may write

$$dS = \left(\frac{\partial S}{\partial T}\right)_P dT + \left(\frac{\partial S}{\partial P}\right)_T dP = \tag{gt.16}$$

$$\left[\left(\frac{\partial S}{\partial T}\right)_P + \left(\frac{\partial S}{\partial P}\right)_T \left(\frac{\partial P}{\partial T}\right)_\rho\right] dT + \left[\left(\frac{\partial S}{\partial P}\right)_T \left(\frac{\partial P}{\partial \rho}\right)_T\right] d\rho.$$

The second term in square brackets in the last equation may be written as (cf. equation gt.13)

$$\left[\left(\frac{\partial S}{\partial P}\right)_T \left(\frac{\partial P}{\partial \rho}\right)_T\right] = \left(\frac{\partial S}{\partial \rho}\right)_T = -\frac{1}{\rho^2}\left(\frac{\partial P}{\partial T}\right)_\rho, \tag{gt.17}$$

and therefore

$$\left(\frac{\partial S}{\partial P}\right)_T = -\frac{1}{\rho^2}\left(\frac{\partial P}{\partial T}\right)_\rho \Big/ \left(\frac{\partial P}{\partial \rho}\right)_T, \tag{gt.18a}$$

while (cf. gt.7b) :

$$\left(\frac{\partial S}{\partial T}\right)_P = \frac{c_P}{T}. \tag{gt.18b}$$

Taking $dS = 0$ in the equation (gt.16) , and using equations (gt.18) we obtain

$$\nabla_{ad} \equiv \frac{P}{T}\left(\frac{\partial T}{\partial P}\right)_S = -\frac{P}{T}\left(\frac{\partial S}{\partial P}\right)_T \Big/ \left(\frac{\partial S}{\partial T}\right)_P = \frac{P}{\rho^2 c_P}\left(\frac{\partial P}{\partial T}\right)_\rho \Big/ \left(\frac{\partial P}{\partial \rho}\right)_T. \tag{gt.19}$$

Finally, combining equations (gt.15) and (gt.19) we have

$$\gamma \equiv \left(\frac{\partial \ln P}{\partial \ln \rho}\right)_S = \frac{c_P}{c_V}\frac{\rho}{P}\left(\frac{\partial P}{\partial \rho}\right)_T = \frac{c_P}{c_V}\left(\frac{\partial \ln P}{\partial \ln \rho}\right)_T. \tag{gt.20}$$

In general we may have a gas made of a number of components, like ions, electrons, photons. We assume that gas particles are weakly interacting with each other. There is enough interaction that a Local Thermodynamic Equilibrium (LTE) is established, with the same temperature for all components. The interaction is weak enough that we may consider all particles to be free, and we may neglect energy of interaction in evaluating pressure and energy density. Therefore, if there are many components, each with its density ρ_k , pressure P_k , and energy density u_k or U_k , then gas as a whole has density, pressure and energy density given as

$$\rho = \sum_k \rho_k, \qquad P = \sum_k P_k, \qquad U = \sum_k U_k, \tag{gt.21}$$

where the summation extends over all components. As energy of interaction is ignored, we may evaluate the derivatives: $(\partial P/\partial T)_\rho$, $(\partial P/\partial \rho)_T$, $(\partial u/\partial T)_T$, for all the components, and then add them up to obtain

$$\left(\frac{\partial P}{\partial T}\right)_\rho = \sum_k \left(\frac{\partial P_k}{\partial T}\right)_\rho, \quad \left(\frac{\partial P}{\partial \rho}\right)_T = \sum_k \left(\frac{\partial P_k}{\partial \rho}\right)_T, \quad \left(\frac{\partial u}{\partial T}\right)_\rho = \sum_k \left(\frac{\partial u_k}{\partial T}\right)_\rho, \tag{gt.22}$$

Given these, all other thermodynamic quantities may be calculated according to the equations that are rewritten here together:

$$\left(\frac{\partial u}{\partial \rho}\right)_T = \frac{P}{\rho^2}\left[1 - \left(\frac{\partial \ln P}{\partial \ln T}\right)_\rho\right], \tag{gt.23a}$$

$$c_V = \left(\frac{\partial u}{\partial T}\right)_\rho, \quad \left[\mathrm{erg\ g^{-1}K^{-1}}\right], \tag{gt.23b}$$

$$c_P = c_V + \frac{T}{\rho^2}\left(\frac{\partial P}{\partial T}\right)_\rho^2 \bigg/ \left(\frac{\partial P}{\partial \rho}\right)_T, \tag{gt.23c}$$

$$\nabla_{ad} \equiv \frac{P}{T}\left(\frac{\partial T}{\partial P}\right)_S = \frac{P}{\rho^2 c_P}\left(\frac{\partial P}{\partial T}\right)_\rho \bigg/ \left(\frac{\partial P}{\partial \rho}\right)_T = \frac{c_P - c_V}{c_P}\left(\frac{\partial \ln T}{\partial \ln P}\right)_\rho, \tag{gt.23d}$$

$$\gamma \equiv \left(\frac{\partial \ln P}{\partial \ln \rho}\right)_S = \frac{c_P}{c_V}\left(\frac{\partial \ln P}{\partial \ln \rho}\right)_T, \tag{gt.23e}$$

$$\left(\frac{\partial \ln T}{\partial \ln \rho}\right)_S = \frac{1}{\rho c_V}\left(\frac{\partial P}{\partial T}\right)_\rho = \gamma \nabla_{ad}. \tag{gt.23f}$$

We shall consider now two limiting cases, gas that is non-relativistic (N-R) , and gas that is ultra-relativistic (U-R) , and in both cases the gas energy density is equal to its kinetic energy density. According to the equations (st.15) and (st.16) in the chapter: EQUATION OF STATE we have $U = 1.5P$, and $U = 3P$, in the two limiting cases, respectively. Combining these two relations with the heat balance equation in the form given by the equation (gt.1b) at the beginning of this chapter, and taking $dS = 0$, we find the following adiabatic relation for a non-relativistic gas:

$$0 = d\left(\frac{U}{\rho}\right) - \frac{P}{\rho^2}\,d\rho = 1.5\,d\left(\frac{P}{\rho}\right) - \frac{P}{\rho^2}\,d\rho = \frac{1.5}{\rho}\,dP - \frac{2.5P}{\rho^2}\,d\rho, \tag{gt.24}$$

and therefore

$$\gamma \equiv \left(\frac{\partial \ln P}{\partial \ln \rho}\right)_S = \frac{5}{3}, \qquad \text{(non-relativistic)}, \tag{gt.25}$$

and the following adiabatic relation for an ultra-relativistic gas:

$$0 = d\left(\frac{U}{\rho}\right) - \frac{P}{\rho^2}\,d\rho = 3d\left(\frac{P}{\rho}\right) - \frac{P}{\rho^2}\,d\rho = \frac{3}{\rho}\,dP - \frac{4P}{\rho^2}\,d\rho, \tag{gt.26}$$

and therefore

$$\gamma \equiv \left(\frac{\partial \ln P}{\partial \ln \rho}\right)_S = \frac{4}{3}, \qquad \text{(ultra-relativistic)}. \tag{gt.27}$$

These formulae for the adiabatic exponent γ hold for any gas which has pressure and energy density dominated by particles with either $p \ll mc$ (N-R) , or $p \gg mc$ (U-R) , no matter what are the details of their distribution functions. Therefore, stars that have adiabatic structure, for example stars that are fully convective, have a power law relation between pressure and density, $P \sim \rho^\gamma$, with $\gamma = 5/3$ if gas is non-relativistic, or $\gamma = 4/3$ if gas is ultra-relativistic. Such power law equation of state is called **polytropic**, and self-gravitating spheres with such equation of state are called **polytropes**. Many different types of stars are, to a very high precision, polytropes. These include very massive, and very low mass main sequence and pre-main sequence stars, very low mass and very large mass white dwarfs. In addition, convective cores or convective envelopes of stars can also be described as partial polytropes. Historically, polytropes were very important in the development of a theory of stellar structure. A very detailed account of that is given by S. Chandrasekhar in his book: *An Introduction to the Study of Stellar Structure.*