EQUATIONS OF STELLAR STRUCTURE

General Equations

We shall consider a spherically symmetric, self-gravitating star. All the physical quantities will depend on two independent variables: **radius** and **time**, (r, t). First, we shall derive all the equations of stellar structure in a general, non spherical case, but very quickly we shall restrict ourselves to the spherically symmetric case.

The microscopic properties of matter at any given point may be described with density ρ, temperature T, and chemical composition, i.e. the abundances of various elements X_i, with $i = 1, 2, 3....$ having as many values as there are elements. All thermodynamic properties and transport coefficients are functions of (ρ, T, X_i). In particular we have: pressure $P(\rho, T, X_i)$, internal energy per unit volume $U(\rho, T, X_i)$, entropy per unit mass $S(\rho, T, X_i)$, coefficient of thermal conductivity per unit volume $\lambda(\rho, T, X_i)$, and heat source or heat sink per unit mass $\epsilon(\rho, T, X_i)$. All the partial derivatives of P, U, λ, and ϵ are also functions of ρ, T, and X_i. Using these quantities the first law of thermodynamics may be written as

$$T\,dS = d\left(\frac{U}{\rho}\right) - \frac{P}{\rho^2}d\rho, \tag{1.1}$$

If there are sources of heat, ϵ, and a non-vanishing heat flux $\vec{F}$, then the heat balance equation may be written as

$$\rho T\frac{dS}{dt} = \rho\epsilon - \mathrm{div}\,\vec{F}. \tag{1.2}$$

The heat flux is directly proportional to the temperature gradient:

$$\vec{F} = -\lambda\nabla T. \tag{1.3}$$

The equation of motion (the Navier-Stockes equation of hydrodynamics) may be written as

$$\frac{d^2\vec{r}}{dt^2} + \frac{1}{\rho}\nabla P + \nabla V = 0, \tag{1.4}$$

where the gravitational potential satisfies the Poisson equation

$$\nabla^2 V = 4\pi G\rho, \tag{1.5}$$

with $V \longrightarrow 0$ when $r \longrightarrow \infty$.

In spherical symmetry these equations may be written as

$$\frac{1}{\rho}\frac{\partial P}{\partial r} + \frac{\partial V}{\partial r} + \frac{d^2 r}{dt^2} = 0, \tag{1.6a}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) = 4\pi G\rho, \tag{1.6b}$$

$$F = -\lambda\frac{\partial T}{\partial r}, \tag{1.6c}$$

$$\frac{1}{\rho r^2}\frac{\partial\left(r^2 F\right)}{\partial r} = \epsilon - T\frac{dS}{dt}, \tag{1.6d}$$

It is convenient to introduce a new variable, M_r :

$$M_r \equiv \int_0^r 4\pi r'^2 \rho dr', \tag{1.7}$$

which is the total mass within the radius r, and another variable, L_r :

$$L_r \equiv 4\pi r^2 F, \tag{1.8}$$

which is the luminosity, i.e. the total heat flux flowing through a spherical shell with the radius r, and also

$$\kappa = \frac{4acT^3}{3\rho}\frac{1}{\lambda}, \tag{1.9}$$

where κ is the coefficient of radiative opacity (per unit mass) , c is the speed of light, and a is the radiation constant. The last equation is valid if the heat transport is due to radiation.

Using the definitions and relations (1.7-1.9) we may write the set of equations (1.6) in a more standard form:

$$\frac{1}{\rho}\frac{\partial P}{\partial r} + \frac{GM_r}{r^2} + \frac{d^2 r}{dt^2} = 0, \tag{1.10a}$$

$$\frac{\partial M_r}{\partial r} = 4\pi r^2 \rho, \tag{1.10b}$$

$$\frac{\partial T}{\partial r} = -\frac{3\kappa\rho L_r}{16\pi ac T^3 r^2}, \tag{1.10c}$$

$$\frac{\partial L_r}{\partial r} = 4\pi r^2 \rho\left(\epsilon - T\frac{dS}{dt}\right), \tag{1.10d}$$

This system of equations is written in a somewhat inconvenient way, as all the space derivatives ($\partial/\partial r$) are taken at a fixed value of time, while all the time derivatives (d/dt) are at the fixed mass zones. For this reason, and also because of the way the boundary conditions are specified (we shall see them soon) , it is convenient to use the mass M_r rather than radius r as a space-like independent variable. Therefore, we replace all derivatives $\partial/\partial r$ with $4\pi r^2 \rho \partial/\partial M_r$, and we obtain

$$\frac{\partial P}{\partial M_r} = -\frac{GM_r}{4\pi r^4} - \frac{1}{4\pi r^2}\frac{\partial^2 r}{\partial t^2}, \tag{1.11a}$$

$$\frac{\partial r}{\partial M_r} = \frac{1}{4\pi r^2 \rho}, \tag{1.11b}$$

$$\frac{\partial T}{\partial M_r} = -\frac{3\kappa L_r}{64\pi^2 ac T^3 r^4}, \tag{1.11c}$$

$$\frac{\partial L_r}{\partial M_r} = \epsilon - T\frac{\partial S}{\partial t}. \tag{1.11d}$$

The set of equations (1.11) describes the time evolution of a spherically symmetric star with a given distribution of chemical composition with mass, $X_i(M_r)$, provided the initial conditions and the boundary conditions are specified. If the time derivative in the equation (1.11a) vanishes then the star is in **hydrostatic equilibrium**. If the time derivative in equation (1.11d) vanishes then the star is in **thermal** equilibrium. Notice, that we always assume that throughout the star the matter and radiation are in **local thermodynamic equilibrium**, LTE, no matter if the star as a whole is in hydrostatic or in thermal equilibrium. From now on we shall consider stars that are in the hydrostatic equilibrium, i.e. we shall assume that the time derivative in the equation (1.11a) is negligible small.

Boundary Conditions

We shall consider now the boundary conditions. At the stellar center the mass M_r, the radius r, and the luminosity L_r, all vanish. Therefore, we have the **inner boundary conditions**

$$r = 0, \qquad L_r = 0, \qquad \text{at } M_r = 0. \tag{1.12}$$

In most cases we shall be interested in structure and evolution of a star with a fixed total mass M. At the surface, where $M_r = M$, the density falls to zero, and the temperature falls to a value that is related to the stellar radius and luminosity. The proper outer boundary conditions require rather complicated calculations of a model stellar atmosphere. We shall shall adopt a very simple model atmosphere within the Eddington approximation, which means we shall use the diffusion approximation to calculate the temperature gradient not only at large optical depth, but also at small optical depth. The Eddington approximation also means that the surface temperature is $2^{1/4} \approx 1.189$ times lower than the effective temperature. The **outer boundary conditions** are

$$\rho = 0, \qquad T = T_o = \left(\frac{L}{8\pi R^2 \sigma}\right)^{1/4}, \qquad \text{at } M_r = M, \tag{1.13}$$

where σ is the Stefan-Boltzman constant. Notice, that the so called effective temperature of a star is defined as

$$T_{eff} \equiv \left(\frac{L}{4\pi R^2 \sigma}\right)^{1/4} = 2^{1/4} T_o. \tag{1.14}$$

At the stellar center we have two adjustable parameters: the central density ρ_c, and the central temperature T_c. At the stellar surface there are other two adjustable parameters: the stellar radius R, and the stellar luminosity L. These four parameters may be calculated when the differential equations of stellar structure are solved. Notice, that only two of those parameters, R and L are directly observable. Also notice, that the equations for spherically symmetric stars (10 or 11) may be derived without considering the general case, but starting with simple geometry of thin, spherically symmetric shells, and balancing mass, momentum and energy across those shells.

Simplified Equations

Let us consider now an even simpler case: a star which may be described with the equations in which all time derivatives may be neglected, i.e. a star that is in the hydrostatic and thermal equilibria. Now, that we have no time dependence, the stellar structure depends on one space-like variable only, which we may choose to be radius r, or the mass M_r. Now, we have four ordinary differential equations:

$$\frac{dP}{dM_r} = -\frac{GM_r}{4\pi r^4}, \tag{1.15a}$$

$$\frac{dr}{dM_r} = \frac{1}{4\pi r^2 \rho}, \tag{1.15b}$$

$$\frac{dT}{dM_r} = -\frac{3\kappa L_r}{64\pi^2 acT^3 r^4}, \qquad (1.15c)$$

$$\frac{dL_r}{dM_r} = \epsilon. \qquad (1.15d)$$

These have to be supplemented with the boundary conditions (1.12) and (1.13) , as well as the total stellar mass M, and the distribution of all elements with the mass, $X_i(M_r)$. We have four ordinary differential equations, four boundary conditions, and four parameters to be found: T_c, ρ_c, R, and L.

Superficially it looks like the problem described with the eqs. (1.12) , (1.13) , (1.15) has a unique solution. In fact, the so called Vogt-Russell theorem claims just that. However, this is not true. Counter-examples were found numerically first, and only later the astronomers noticed that there is no mathematical basis for the Vogt-Russell "theorem". The initial value problems usually have a unique solution. However, if the boundary conditions are specified at the two different locations, in our case at the stellar center and at the stellar surface, then there may be no solutions, or in general there may be many solutions. We shall find some examples later on. Nevertheless, the Vogt-Russell "theorem" served a useful purpose, it "explained" the nature of the Main Sequence, a linear sequence of stellar models with the total stellar mass being the parameter that varied along the sequence. Modern numerical stellar models demonstrate that in this case there is indeed a unique solution for a large range of stellar masses, with all stars being chemically homogeneous, their luminosity generated by nuclear "burning" of hydrogen into helium.

Complications

The fact that stars are luminous, i.e. they are radiating away some energy, implies that they must change in time. Indeed, it is known now that the main energy source for the stars is nuclear, and the nuclear reactions that provide heat also change chemical composition. Therefore, our stellar structure equations are incomplete. They have to be supplemented with a set of equations describing the nuclear reaction network, i.e. providing $\partial X_i/\partial t$ as a function of ρ, T, X_j. This will introduce a new time dependence, with its accompanying nuclear time scale.

There is another problem with our equations. So far we were assuming that the heat is streaming out of the star by means of some diffusive process, as described with the original equation (1.3) . However, it may be that this process is unstable. In fact it is known to be unstable to convection whenever the temperature gradient becomes too steep. As soon as convection develops it carries some of the heat flux, and the temperature gradient is modified. It turns out that in the deep interior of a star convection, when present, brings the temperature to the adiabatic value. However, near the surface of a star convection is not very efficient in carrying heat, and there is no good theory to calculate its efficiency. For most practical purposes astronomers use the so called "mixing length theory", which parameterizes our lack of knowledge about convection with one free parameter α, which is equal to the ratio of a characteristic "mixing length" to the pressure scale height, and is usually of the order unity.

In addition to carrying heat convection mixes various stellar layers, with possibly different chemical composition. As a result chemical composition may change not only due to nuclear reactions, but also because of convective mixing. As the mixing is a non local phenomenon, the solution of full stellar structure equations becomes much more complicated.

Still another physical process which is important in some stars is a diffusion of elements with different mean molecular weight, or different ratio of electric charge to mass, or different cross section for interaction with radiation. In some cases this process may produce chemical inhomogeneity with important consequences for stellar appearance and/or evolution.

There may be some other processes which lead to some mixing that is not important as the energy transport mechanism, but which may be important for the distribution of chemical composition. This may be meridional circulation induced by very rapid rotation of a star, or some poorly understood instabilities.

Recently, a very interesting energy transfer mechanism was proposed to explain the solar neutrino problem. The process depends on the existence of weakly interacting particles with a mean free path that might be a large fraction of the solar radius. This process, if real would lead to a non-local heat transfer problem. The non-local heat transfer is known to operate in extended stellar atmospheres due to large mean free path of the photons, and in the hot neutron stars and interiors of Type II supernovae, where the mean free path of neutrinos is large. It remains to be seen if the non-local heat transfer is indeed important for the sun, or for other stars.

Equations for Numerical Integrations

In general, the temperature gradient is determined by heat diffusion and by convection, if the particular layer inside a star is convectively unstable. The standard, Schwarzschild, criterion for stability is:

$$\nabla_{rad} < \nabla_{ad} \qquad \text{stable,} \tag{1.16a}$$

$$\nabla_{rad} > \nabla_{ad} \qquad \text{unstable.} \tag{1.16b}$$

where

$$\nabla_{rad} \equiv \frac{\kappa L_r}{16\pi c G M_r} \frac{3P}{aT^4}, \tag{1.17a}$$

$$\nabla_{ad} \equiv \left(\frac{\partial \ln T}{\partial \ln P} \right)_S \tag{1.17b}$$

The temperature gradient may be calculated as

$$\nabla_T = \nabla_{rad} \quad \text{when} \quad \nabla_{rad} < \nabla_{ad}, \tag{1.18a}$$

$$\nabla_T = \nabla_{conv} \quad \text{when} \quad \nabla_{rad} > \nabla_{ad}, \tag{1.18b}$$

$$\nabla_{rad} > \nabla_{conv} > \nabla_{ad}. \tag{1.18c}$$

The usual way to calculate the convective temperature gradient is to use the so called mixing length theory, which gives a prescription how to interpolate ∇_{conv} between the ∇_{rad} and ∇_{ad}.

As many physical quantities are expressed in terms of density and temperature, it is convenient to use the temperature and the density as the two quantities to be integrated. All the equations written so far have explicitly derivatives of pressure and temperature. We have to change this to the derivatives of temperature and density. We may always write:

$$d \ln P = \left(\frac{\partial \ln P}{\partial \ln \rho} \right)_T d \ln \rho + \left(\frac{\partial \ln P}{\partial \ln T} \right)_\rho d \ln T, \tag{1.19a}$$

$$\nabla_\rho \equiv \frac{d \ln \rho}{d \ln P} = \left[1 - \left(\frac{\partial \ln P}{\partial \ln T} \right)_\rho \nabla_T \right] \Big/ \left(\frac{\partial \ln P}{\partial \ln \rho} \right)_T . \tag{1.19b}$$

Now we may write our four equations as

$$\frac{dT}{dM_r} = \frac{T}{P}\nabla_T \frac{dP}{dM_r}, \tag{1.20a}$$

$$\frac{d\rho}{dM_r} = \frac{\rho}{P}\nabla_\rho \frac{dP}{dM_r}, \tag{1.20b}$$

$$\frac{dr}{dM_r} = \frac{1}{4\pi r^2 \rho}, \tag{1.20c}$$

$$\frac{dL_r}{dM_r} = \epsilon, \tag{1.20d}$$

where

$$\frac{dP}{dM_r} = -\frac{GM_r}{4\pi r^4}. \tag{1.20e}$$

For actual numerical integrations we make one more approximation and one more change in the equations. First, to simplify the program we shall assume that the convective gradient is always equal to the adiabatic gradient. This is a very good approximation for the dwarf stars, it is not very good for the Sun, and it is very bad for the red giants. It is also convenient to use solar units for the mass, luminosity and radius. Let us define:

$$M_r^* \equiv M_r/M_\odot, \tag{1.21a}$$

$$L_r^* \equiv L_r/L_\odot, \tag{1.21b}$$

$$r^* \equiv r/R_\odot. \tag{1.21c}$$

The final equations become:

$$\frac{dT}{dM_r^*} = \frac{T}{P}\nabla_T \frac{dP}{dM_r^*}, \tag{1.22a}$$

$$\frac{d\rho}{dM_r^*} \equiv \frac{\rho}{P}\nabla_\rho \frac{dP}{dM_r^*}, \tag{1.22b}$$

$$\frac{dr^*}{dM_r^*} = \left(\frac{M_\odot}{4\pi R_\odot^3}\right)\frac{1}{{r^*}^2 \rho}, \tag{1.22c}$$

$$\frac{dL_r^*}{dM_r^*} = \frac{M_\odot}{L_\odot}\epsilon, \tag{1.22d}$$

where

$$\frac{dP}{dM_r^*} = -\left(\frac{GM_\odot^2}{4\pi R_\odot^4}\right)\frac{M_r^*}{{r^*}^4}, \tag{1.23a}$$

$$\nabla_T = \min\left(\nabla_{rad}, \nabla_{ad}\right), \tag{1.23b}$$

$$\nabla_{rad} = \left(\frac{L_\odot}{16\pi c G M_\odot}\right)\frac{\kappa L_r^*}{M_r^*}\frac{P}{P_{rad}}, \tag{1.23c}$$

$$P_{rad} = \frac{a}{3}T^4, \tag{1.23d}$$

$$\nabla_{ad} = \left(\frac{d\ln T}{d\ln P}\right)_S, \tag{1.23e}$$

$$\nabla_\rho = \left[1 - \left(\frac{\partial \ln P}{\partial \ln T}\right)_\rho \nabla_T\right] \Big/ \left(\frac{\partial \ln P}{\partial \ln \rho}\right)_T , \tag{1.23f}$$

and all the quantities: P, $(\partial \ln P/\partial \ln T)_\rho$, $(\partial \ln P/\partial \ln \rho)_T$, ∇_{ad}, κ, ϵ are the known functions of temperature, density and chemical composition. In the computer programs these are calculated by the subroutines "state", "opact", and "nburn", respectively.

The boundary conditions are:

at the surface, where $M_r^* = M^*$ we have:

$$\rho = 10^{-12}, \qquad T = 2^{-1/4} T_{eff}, \qquad R^* = \left(T^2_{\odot\, eff}\right) L^{*1/2} T_{eff}^{-2}, \tag{1.24a}$$

and at the center, where $M_r^* = 0$ we have:

$$r^* = 0, \qquad L_r^* = 0. \tag{1.24b}$$

The parameters that may be adjusted are the effective temperature, the surface luminosity, the central temperature and the central density, or the logarithms of these quantities.