STARS IN HYDROSTATIC EQUILIBRIUM

Gravitational energy and hydrostatic equilibrium

We shall consider stars in a hydrostatic equilibrium, but not necessarily in a thermal equilibrium. Let us define some terms:

$$U = \text{kinetic, or in general internal energy density} \qquad [\text{erg cm}^{-3}], \tag{eql.1a}$$

$$u \equiv \frac{U}{\rho} \qquad \left[\text{erg g}^{-1}\right], \tag{eql.1b}$$

$$E_{th} \equiv \int_0^R U 4\pi r^2 dr = \int_0^M u \, dM_r = \text{thermal energy of a star}, \qquad [\text{erg}], \tag{eql.1c}$$

$$\Omega = -\int_0^M \frac{GM_r dM_r}{r} = \text{gravitational energy of a star}, \qquad [\text{erg}], \tag{eql.1d}$$

$$E_{tot} = E_{th} + \Omega = \text{total energy of a star} , \qquad [\text{erg}]. \tag{eql.1e}$$

We shall use the equation of hydrostatic equilibrium

$$\frac{dP}{dr} = -\frac{GM_r}{r}\rho, \tag{eql.2}$$

and the relation between the mass and radius

$$\frac{dM_r}{dr} = 4\pi r^2 \rho, \tag{eql.3}$$

to find a relations between thermal and gravitational energy of a star. As we shall be changing variables many times we shall adopt a convention of using "c" as a symbol of a stellar center and the lower limit of an integral, and "s" as a symbol of a stellar surface and the upper limit of an integral. We shall be transforming an integral formula (eql.1d) so, as to relate it to (eql.1c) :

$$\Omega = -\int_c^s \frac{GM_r dM_r}{r} = -\int_c^s \frac{GM_r}{r} 4\pi r^2 \rho dr = -\int_c^s \frac{GM_r \rho}{r^2} 4\pi r^3 dr = \tag{eql.4}$$

$$\int_c^s \frac{dP}{dr} 4\pi r^3 dr = \int_c^s 4\pi r^3 dP = 4\pi r^3 P \Big|_c^s - \int_c^s 12\pi r^2 P dr =$$

$$-3 \int_c^s P 4\pi r^2 dr = \Omega.$$

Our final result: gravitational energy of a star in a hydrostatic equilibrium is equal to three times the integral of pressure within the star over its entire volume.

Now, we shall use a relation between pressure and energy density in two limits. First, in a non-relativistic limit (NR) we have $U = 1.5P$, and hence:

$$\Omega = -2 \int_c^s U 4\pi r^2 dr = -2E_{th}, \qquad \text{(NR)}, \qquad \text{(eql.5a)}$$

and in the ultra-relativistic limit (UR) we have $U = 3P$, and

$$\Omega = -\int_c^s U 4\pi r^2 dr = -E_{th}, \qquad \text{(UR)}, \qquad \text{(eql.5b)}$$

These equations also give

$$E_{tot} = \Omega + E_{th} = \frac{1}{2}\Omega < 0 \qquad \text{(NR)}, \qquad \text{(eql.6a)}$$

$$E_{tot} = 0 \qquad \text{(UR)}. \qquad \text{(eql.6b)}$$

The non-relativistic case is equivalent to the well known virial theorem. The ultra-relativistic results is somewhat paradoxical, with the total energy of a star being zero. This result is only approximate, as it relates to the case when all particles within the star are moving with the speed of light. This limit is never quite reached, and our result just indicates that close to that limit the total stellar energy is close to zero.

Heat balance in a star

Let us consider now the equation of heat balance for a star. It may be written as

$$\left(\frac{\partial L_r}{\partial M_r}\right)_t = \epsilon_n - \epsilon_\nu - T\left(\frac{\partial S}{\partial t}\right)_{M_r}, \qquad \text{(eql.7)}$$

where ϵ_n and ϵ_ν are the heat generation and heat loss rates in nuclear reactions and in thermal neutrino emission, respectively [erg g $^{-1}$ s $^{-1}$], and S is entropy per gram. We shall define nuclear, neutrino, and "gravitational" luminosities of a star as

$$L_n = \int_c^s \epsilon_n dM_r, \qquad \text{(eql.8a)}$$

$$L_\nu = \int_c^s \epsilon_\nu dM_r, \qquad \text{(eql.8b)}$$

$$L_g = -\int_c^s T\left(\frac{\partial S}{\partial t}\right)_{M_r} dM_r, \qquad \text{(eql.8c)}$$

and the total stellar luminosity is given as

$$L = L_n - L_\nu + L_g. \qquad \text{(eql.9)}$$

According to the first law of thermodynamics we have

$$TdS = d\left(\frac{U}{\rho}\right) - \frac{P}{\rho^2}d\rho = du + Pd\left(\frac{1}{\rho}\right). \qquad \text{(eql.10)}$$

It is convenient to write "gravitational" luminosity as a sum of two terms, $L_g = L_{g1} + L_{g2}$, with

$$L_{g1} = -\int_c^s \left(\frac{\partial u}{\partial t}\right)_{M_r} dM_r = -\frac{d}{dt}\left[\int_c^s u\ dM_r\right] = -\frac{dE_{th}}{dt}, \qquad \text{(eql.11)}$$

$$L_{g2} = \int_c^s \frac{P}{\rho^2}\left(\frac{\partial \rho}{\partial t}\right)_{M_r} dM_r = -\int_c^s P\left[\frac{\partial\,(1/\rho)}{\partial t}\right]_{M_r} dM_r. \qquad \text{(eql.12)}$$

In order to modify the last integral we should note the relation

$$\frac{1}{\rho} = \frac{4\pi}{3}\left(\frac{\partial r^3}{\partial M_r}\right)_t. \qquad \text{(eql.13)}$$

Combining equations (eql.12) and (eql.13) we obtain

$$L_{g2} = -\frac{4\pi}{3}\int_c^s P\frac{\partial}{\partial t}\left(\frac{\partial r^3}{\partial M_r}\right) dM_r = -\frac{4\pi}{3}\int_c^s P\frac{\partial}{\partial M_r}\left(\frac{\partial r^3}{\partial t}\right) dM_r = \qquad \text{(eql.14)}$$

$$\left[-\frac{4\pi}{3}P\frac{\partial r^3}{\partial t}\right]_c^s + \frac{4\pi}{3}\int_c^s \frac{\partial P}{\partial M_r}\frac{\partial r^3}{\partial t} dM_r =$$

$$-\frac{4\pi}{3}\int_c^s \frac{GM_r}{4\pi r^4}3r^2\frac{\partial r}{\partial t} dM_r = -\int_c^s \frac{GM_r}{r^2}\frac{\partial r}{\partial t} dM_r =$$

$$\frac{d}{dt}\left[\int_c^s \frac{GM_r dM_r}{r}\right] = -\frac{d\Omega}{dt}.$$

Combining equations (eql.11) and (eql.14) we obtain

$$L_g = -\frac{dE_{th}}{dt} - \frac{d\Omega}{dt} = -\frac{dE_{tot}}{dt}. \qquad \text{(eql.15)}$$

Thermal stability of the Eddington model

Let us apply the results of the last two sections to the Eddington model of a massive star. Throughout the stellar model we assume $\beta \equiv P_g/P = const.$, and we can combine the eqs. (eql.1c) and (eql.4) to obtain

$$E_{th} \equiv \int_0^R U 4\pi r^2 dr = \int_0^R \left(3P_r + 1.5P_g\right) 4\pi r^2 dr = (3 - 1.5\beta)\int_0^R P 4\pi r^2 dr =$$

$$-(1 - 0.5\beta)\Omega. \qquad \text{(eql.16)}$$

The Eddington model is a polytrope with the index $n = 3$, for which there is a simple analytical formula for the gravitational potential energy (cf. eqs. [poly.18] and [eql.1d])

$$\Omega = -1.5\ \frac{GM^2}{R}, \qquad \text{(eql.17)}$$

and the total energy of the star may be calculated according to

$$E_{tot} = E_{th} + \Omega = 0.5\beta\ \Omega = -0.75\ \beta\ \frac{GM^2}{R}. \tag{eql.18}$$

Combining the eqs. (eql.15) and (eql.18) the "gravitational" luminosity of the stars is expressed as

$$L_g = -\frac{dE_{tot}}{dt} = -0.75\beta\ \frac{GM^2}{R^2}\frac{dR}{dt}. \tag{eql.19}$$

We know that the luminosity radiated away from the stellar surface depends on its mass only (as long as the Eddington model applies), and it is given with the eq. (s2.7) as

$$\frac{M}{M_\odot} = \frac{18.1}{\mu^2}\frac{(L/L_{Edd})^{1/2}}{(1-L/L_{Edd})^2}, \qquad L_{Edd} \equiv \frac{4\pi cGM}{\kappa_e}. \tag{eql.20}$$

The nuclear luminosity is given with the eq. (eql.8a). Adopting $\epsilon = \epsilon_0 \rho T^\nu$, with $\nu = 16$ we obtain

$$L_n = \int_0^R \epsilon_0 \rho T^\nu dM_r \sim R^{-(\nu+3)}, \tag{eql.21}$$

for a star with a fixed mass. The star is said to be on the Main Sequence when its luminosity (radiative energy losses) is balanced by the heat source due to hydrogen burning. Therefore, we may write

$$L_n = L\ \left(\frac{R_{MS}}{R}\right)^{\nu+3}, \tag{eql.22}$$

where R_{MS} is the main sequence radius of the star (it may be calculated numerically evaluating the integral given with the eq. [eql.8a]).

Combining the equations (eql.9, 19, 22) (and neglecting neutrino luminosity which is never important for main sequence stars) we obtain the differential equation for the time variation of the stellar radius:

$$\frac{dx}{dt} = C\left(1 - x^{\nu+3}\right), \qquad x \equiv \frac{R_{MS}}{R}. \qquad C \equiv \left(\frac{4LR_{MS}}{3GM^2\beta}\right), \tag{eql.23}$$

where the constant C depends on the stellar mass and chemical composition, but not on stellar radius. The constant C has a dimension of $[s^{-1}]$. It is customary to define the Kelvin - Helmholtz (thermal) time scale of a star as

$$\tau_{K-H} \equiv \frac{GM^2}{RL}. \tag{eql.24}$$

Apart from dimensionless factor of the order unity the constant C is of the order of τ_{K-H}^{-1}.

Clearly, there is an asymptotic solution of the differential equation (eql.23): $x = 1$, i.e. $R = R_{MS}$, i.e. the stellar radius is equal to its main sequence value. Now we may ask a question: is the main sequence star thermally stable? If we make a small perturbation, making a star slightly smaller or slightly larger, will this perturbation grow, or will it decrease with time? Let at some time t_0 the dimensionless stellar radius be $x_0 = 1 + \Delta x_0$, with $|\Delta x_0| \ll 1$. The eq. (eql.23) may be written as

$$\frac{dx}{dt} = \frac{d\Delta x}{dt} = \frac{4}{3\tau_{K-H}\beta}\left[1 - (1+\Delta x)^{\nu+3}\right] \approx -\frac{4}{3\tau_{K-H}\beta}(\nu+3)\Delta x. \tag{eql.25}$$

This equation has the solution

$$\frac{R_{MS}}{R} = 1 + \Delta x = 1 + \Delta x_0 exp\left[-\left(\frac{4(\nu+3)}{3\tau_{K-H}\beta}\right)(t - t_0)\right]. \tag{eql.26}$$

We find that the initial disturbance Δx_0 decreases exponentially with time, i.e. the Eddington model of a main sequence star is thermally stable, and the characteristic time scale on which the thermal equilibrium is restored is the Kelvin Helmholtz time scale.