TRANSFER OF RADIATION

Under LTE (Local Thermodynamic Equilibrium) condition radiation has a Planck (black body) distribution. Radiation energy density is given as

$$U_{r,\nu} d\nu = \frac{8\pi h}{c^3} \frac{\nu^3 d\nu}{e^{h\nu/kT} - 1}, \qquad \text{(LTE)}, \tag{tr.1}$$

and the intensity of radiation (measured in ergs per unit area per second per unit solid angle, i.e. per steradian) is

$$I_\nu = B_\nu(T) = \frac{c}{4\pi} U_{r,\nu}, \qquad \text{(LTE)}. \tag{tr.2}$$

The integrals of $U_{r,\nu}$ and $B_\nu(T)$ over all frequencies are given as

$$U_r = \int_0^\infty U_{r,\nu} d\nu = aT^4, \qquad \text{(LTE)}, \tag{tr.3a}$$

$$B(T) = \int_0^\infty B_\nu(T) d\nu = \frac{ac}{4\pi} T^4 = \frac{\sigma}{\pi} T^4 = \frac{c}{4\pi} U_r = \frac{3c}{4\pi} P_r, \qquad \text{(LTE)}, \tag{tr.3b}$$

where $P_r = aT^4/3$ is the radiation pressure.

Inside a star conditions are very close to LTE, but there must be some anisotropy of the radiation field if there is a net flow of radiation from the deep interior towards the surface. We shall consider intensity of radiation as a function of radiation frequency, position inside a star, and a direction in which the photons are moving. We shall consider a spherical star only, so the dependence on the position is just a dependence on the radius r, i.e. the distance from the center. The angular dependence is reduced to the dependence on the angle between the light ray and the outward radial direction, which we shall call the polar angle θ. The intensity becomes $I_\nu(r, \theta)$.

Let us consider a change in the intensity of radiation in the direction θ at the radial distance r when we move along the beam by a small distance $dl = dr/\cos\theta$. The intensity will be reduced by the amount proportional to the opacity per unit volume, $\kappa_\nu \rho$ multiplied by dl, where ρ is density of matter and κ_ν is monochromatic opacity in units $\text{cm}^2\,\text{g}^{-1}$. Also, the intensity will increase by the amount proportional to the emissivity of gas. Under nearly LTE condition the emissivity per unit volume is given given as a product $\kappa_\nu \rho B_\nu(T)$. The equation of monochromatic radiation transfer is written as

$$\frac{\partial I_\nu(\theta, r)}{\partial l} = \cos\theta \frac{\partial I_\nu(\theta, r)}{\partial r} = -\kappa_\nu(\rho, T, X)\rho I_\nu(\theta, r) + \kappa_\nu(\rho, T, X)\rho B_\nu(T), \tag{tr.4}$$

where the dependence of the coefficient of opacity on photon frequency, as well as on the density, temperature, and chemical composition of gas has been written as $\kappa_\nu(\rho, T, X)$. From now on we shall write it as κ_ν.

Monochromatic radiation energy density may be calculated as

$$U_{r,\nu} = \frac{1}{c} \int_{4\pi} I_\nu(\theta, r)\, d\omega, \tag{tr.5}$$

where the integration is extended over the whole 4π solid angle. Because of azimuthal symmetry this integral may be written as

$$U_{r,\nu} = \frac{2\pi}{c} \int_0^{\pi} I_\nu(\theta, r) \sin\theta d\theta. \qquad \text{(tr.6)}$$

The total radiation energy density is given as

$$U_r = \int_0^{\infty} U_{r,\nu} d\nu. \qquad \text{(tr.7)}$$

Monochromatic flux of radiation in the direction r may be calculated as

$$F_\nu = \int_{4\pi} I_\nu(\theta, r) \cos\theta d\omega = 2\pi \int_0^{\pi} I_\nu(\theta, r) \cos\theta \sin\theta d\theta, \qquad \text{(tr.8)}$$

and the total flux of radiation, measured in $\mathrm{erg\ cm^{-2}\ s^{-1}}$, is given as

$$F = \int_0^{\infty} F_\nu d\nu. \qquad \text{(tr.9)}$$

We shall look for a solution in a form of a power series

$$I_\nu(\theta, r) = \sum_{n=0}^{\infty} I_{\nu,n}(r) \cos^n\theta. \qquad \text{(tr.10)}$$

Now we insert this expansion into the equation of radiation transfer (tr.4) , integrate all terms over all angles, and we compare the terms with the same power of $\cos\theta$. This allows us to replace the partial differential equation (tr.4) with an infinite number of ordinary differential equations

$$I_{\nu,0}(r) = B_\nu(T), \qquad I_{\nu,n}(r) = -\frac{1}{\kappa_\nu\rho} \frac{\partial I_{\nu,n-1}(r)}{\partial r}, \qquad \text{(tr.11)}$$

in which the coefficients $I_{\nu,n}$ do not depend on the angle θ. In a typical stellar interior we may have $\rho \approx 1\ \mathrm{g\ cm^{-3}}$, $\kappa_\nu \approx 1\ \mathrm{cm^2\ g^{-1}}$, and $r \approx 10^{11}\ \mathrm{cm}$. Therefore $I_{\nu,n}(r) \approx -I_{\nu,n-1}(r)/\kappa_\nu\rho \approx -I_{\nu,n-1}(r)/10^{11}$, and the series rapidly converges. The physical interpretation is simple: deep inside a star the radiation field is almost isotropic.

The first of the equations (tr.11) tells us that the intensity of radiation averaged over all angles is equal to the Planck function. The second of those equations, combined with equation (tr.8) gives

$$F_\nu = 2\pi \int_0^{\pi} I_\nu(\theta, r) \cos\theta \sin\theta d\theta = 2\pi \int_0^{\pi} I_{\nu,1}(r) \cos^2\theta \sin\theta d\theta = \frac{4\pi}{3} I_{\nu,1} = \qquad \text{(tr.12)}$$

$$-\frac{4\pi}{3\kappa_\nu\rho} \frac{\partial I_{\nu,0}(r)}{\partial r} = -\frac{4\pi}{3\kappa_\nu\rho} \frac{\partial B_\nu(T)}{\partial r} = -\frac{4\pi}{3\kappa_\nu\rho} \frac{\partial B_\nu(T)}{\partial T} \frac{dT}{dr}.$$

The total radiative energy flux is an integral of F_ν over all frequencies (cf. eq. 9) , i.e

$$F = -\frac{4\pi}{3\rho} \frac{dT}{dr} \int_0^{\infty} \frac{1}{\kappa_\nu} \frac{\partial B_\nu(T)}{\partial T} d\nu = -\frac{4\pi}{3\kappa\rho} \frac{dB(T)}{dr} = -\frac{c}{\kappa\rho} \frac{dP_r}{dr}, \qquad \text{(tr.13)}$$

where the Rosseland mean opacity is defined as

$$\frac{1}{\kappa} \equiv \frac{\int_0^{\infty} \frac{1}{\kappa_\nu} \frac{\partial B_\nu(T)}{\partial T} d\nu}{\int_0^{\infty} \frac{\partial B_\nu(T)}{\partial T} d\nu} = \frac{\int_0^{\infty} \frac{1}{\kappa_\nu} \frac{\partial B_\nu(T)}{\partial T} d\nu}{\frac{ac}{\pi} T^3}. \qquad \text{(tr.14)}$$

Of course, the Rosseland mean opacity is a function of density, temperature and chemical composition, $\kappa(\rho, T, X)$.

The equation (tr.13) may be written as

$$\frac{L_r}{4\pi r^2} = F = -\frac{c}{3\kappa\rho}\frac{dU_r}{dr} = -\left(\frac{4acT^3}{3\kappa\rho}\right)\frac{dT}{dr} = -\lambda\frac{dT}{dr}, \tag{tr.15}$$

where $L_r = 4\pi r^2 F$ is stellar luminosity at a radius r, i.e. the total amount of radiation energy flowing across a spherical surface with a radius r, and U_r is the radiation energy density. The last equation looks just like the equation for heat diffusion, with the coefficient of thermal conductivity λ related to the coefficient of opacity with a relation

$$\lambda = \frac{4acT^3}{3\kappa\rho}. \tag{tr.16}$$

As heat may be transferred not only by photons, but also by electrons, it may be safer to write the last equation as

$$\lambda_{rad} = \frac{4acT^3}{3\kappa_{rad}\rho}, \tag{tr.17}$$

where λ_{rad} and κ_{rad} are explicitly related to radiation. We may write a similar relation for electrons:

$$\lambda_{el} = \frac{4acT^3}{3\kappa_{el}\rho}, \tag{tr.18}$$

where λ_{el} and κ_{el} are the coefficients of thermal conductivity and "opacity" for the electrons. While it is reasonable to think of the coefficient of thermal conductivity for photons, it is somewhat funny to use a term "opacity" in reference to the heat transferred by electrons. Nevertheless, both relations (tr.17) and (tr.18) may be treated as a definition of one quantity (e.g opacity) when the other quantity (e.g. coefficient of thermal conductivity) is given.

In general, we may have some heat transferred by photons, and some by electrons. As the two means of heat transport are additive, the combined coefficient of thermal conductivity may by calculated as

$$\lambda = \lambda_{rad} + \lambda_{el}, \tag{tr.19}$$

or equivalently, we may write a formula for the combined coefficient of opacity as

$$\frac{1}{\kappa} = \frac{1}{\kappa_{rad}} + \frac{1}{\kappa_{el}}. \tag{tr.20}$$

Notice, that if there are two independent carriers of a heat flow, e.g. photons and electrons, then the combined coefficient of thermal conductivity is larger, while the combined coefficient of opacity is smaller than the corresponding coefficients for either of the carriers. In most cases there is no need to be very careful with the subscripts "rad" and "el", because the heat transport is dominated by photons when gas is not degenerate, and it is dominated by electrons when electron gas is degenerate. The transition between the two regimes is very rapid.

We shall consider now radiation transport in stellar envelopes and atmospheres, where electrons do not contribute to heat conduction. Therefore, we shall use the coefficient of opacity without any subscript, with an understanding that it refers to the Rosseland mean opacity as defined by equation (tr.14) . It is perhaps surprising, that the equation (tr.15) , which was derived under the assumption that radiation field is almost isotropic, holds very well all the way to the stellar surface, where radiation pressure is no longer well defined as the radiation field becomes highly anisotropic. At the stellar surface itself one hemisphere, towards the star is bright, while the other hemisphere, facing the outer space, is dark. Under these conditions radiation energy density may still be calculated according to equation (tr.7) , and we may estimate temperature from the radiation energy density using the LTE relation (tr.3a) .

Near the stellar surface the luminosity and radius can be taken as L and R, and the radiation energy flux is $F = L/4\pi R^2$. Let us define optical depth τ as

$$d\tau \equiv -\kappa \rho dr, \qquad \tau = 0 \quad \text{at} \quad r = R. \tag{tr.21}$$

Now, we may write the equation (tr.15) as

$$\frac{dT^4}{d\tau} = \frac{3F}{ac} \approx \text{const}, \tag{tr.22}$$

and therefore

$$T^4 = T_0^4 + \frac{3F}{ac}\tau, \tag{tr.23}$$

where T_0 is the temperature at the stellar surface.

Consider now a surface radiating as a black body with a temperature T. At a point just above the surface the radiation comes from one hemisphere only, and we may use equations (tr.7), (tr.6), and (tr.2) to calculate

$$U_r = \int_0^\infty \left[\frac{2\pi}{c} \int_0^\pi I_\nu(\theta) \sin\theta d\theta \right] d\nu = \tag{tr.24}$$

$$\int_0^\infty \left[\frac{2\pi}{c} \int_0^{\pi/2} B_\nu(T) \sin\theta d\theta \right] d\nu = \frac{2\pi}{c} B(T) = \frac{1}{2} a T^4.$$

We obtained only one half of the radiation energy density expected under LTE conditions for the temperature T, because radiation was comming from one hemisphere only. The radiative energy flux may be calculated for our case using equations (tr.9) and (tr.8)

$$F = \int_0^\infty \left[2\pi \int_0^\pi I_\nu(\theta) \sin\theta \cos\theta d\theta \right] d\nu = \int_0^\infty \left[2\pi \int_0^{\pi/2} B_\nu(T) \sin\theta \cos\theta d\theta \right] d\nu = \tag{tr.25}$$

$$\pi B(T) = \frac{ac}{4} T^4 = \sigma T^4.$$

We shall define the effective temperature of a star with a relation

$$\frac{L}{4\pi R^2} = F \equiv \sigma T_{eff}^4. \tag{tr.26}$$

This is a temperature that a black body would have if it radiated just as much energy per unit area as the star does. The radiation energy density at the surface of a black body is half of the LTE energy density corresponding to the temperature T. We shall adopt an approximation that at the stellar surface, i.e. at $\tau = 0$ the radiation energy density is $aT_{eff}^4/2$, by analogy with a black body case. Combining this with the equation (tr.23) we find that

$$T_0^4 = \frac{1}{2} T_{eff}^4, \tag{tr.27}$$

and the temperature distribution close to the stellar surface is given as

$$T^4 = \frac{1}{2} T_{eff}^4 + \frac{3F}{ac}\tau = T_{eff}^4 \left(\frac{1}{2} + \frac{3}{4}\tau \right). \tag{tr.28}$$

Therefore, we have $T = T_{eff}$ at $\tau = 2/3$. The optical depth 2/3 corresponds to a **photosphere**, which is defined as

The diffusion approximation to the transfer of radiation near the stellar surface, and the approximation according to which at the very surface the temperature is $2^{1/4}$ times lower than the effective temperature (cf. equations tr.27 and tr.28) is known as the **Eddington approximation**. This equation (tr.28) may be written as (cf. equation tr.13)

$$\frac{dP_r}{dr} = -\frac{\kappa\rho}{c}F = -\frac{\kappa\rho}{c}\frac{L_r}{4\pi r^2}. \tag{tr.29}$$

This may be combined with the equation of hydrostatic equilibrium to obtain

$$\frac{dP_g}{dr} = \frac{dP}{dr} - \frac{dP_r}{dr} = -\frac{GM_r}{r^2}\rho + \frac{\kappa\rho}{4\pi c}\frac{L_r}{r^2} = \tag{tr.30}$$

$$-\frac{GM_r}{r^2}\rho\left(1 - \frac{\kappa L_r}{4\pi c G M_r}\right).$$

Near the stellar surface we have $L_r = L$, and $M_r = M$. We shall find later on that when luminosity is very high then density in a stellar atmosphere is very low, and the opacity is dominated by scattering of photons on free electrons. For a fully ionized gas the electron scattering opacity is given as

$$\kappa_e = \frac{n_e}{\rho}\sigma_e = 0.2\,(1+X)\,, \qquad [\mathrm{cm}^2\,\mathrm{g}^{-1}], \tag{tr.31}$$

where X is hydrogen content by mass fraction, n_e is number of electrons per cubic centimeter, and σ_e is equal to the Thompson scattering cross-section for scattering photons on electrons

$$\sigma_e = \frac{8\pi}{3}r_e^2 = \frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2 = 0.665 \times 10^{-24}\,\mathrm{cm}^2. \tag{tr.32}$$

Putting the electron scattering opacity into the equation (tr.30) we obtain near the stellar surface

$$\frac{dP_g}{dr} = -\frac{GM}{r^2}\rho\left(1 - \frac{\kappa_e L}{4\pi c G M}\right), \tag{tr.33}$$

while the gradient of the total pressure P is given as

$$\frac{dP}{dr} = -\frac{GM}{r^2}\rho. \tag{tr.34}$$

Dividing the last two equations side by side we obtain

$$\frac{dP_g}{dP} = 1 - \frac{\kappa_e L}{4\pi c G M} = \text{const.} \tag{tr.35}$$

This may be integrated to obtain

$$P_g = (P - P_0)\left(1 - \frac{\kappa_e L}{4\pi c G M}\right), \tag{tr.36}$$

where $P_0 = P_{r,(\tau=0)} = 2F/3c$ is a very small radiation pressure at the stellar surface. It is clear, that at a modest depth below the stellar surface the pressure is very much larger than at the surface, and therefore, the equation (tr.36) gives

$$\beta \equiv \frac{P_g}{P} = \left(1 - \frac{\kappa_e L}{4\pi c G M}\right). \tag{tr.37}$$

It is obvious that $0 < \beta < 1$, and therefore

$$0 < L < L_{Edd} \equiv \frac{4\pi c G M}{\kappa_e} = \frac{4\pi c G}{0.2\,(1+X)}M = \frac{1.256 \times 10^5\,\mathrm{erg\ s}^{-1}}{1+X}M = \tag{tr.38}$$

$$\frac{2.50 \times 10^{38}\,\mathrm{erg\ s}^{-1}}{1+X}\frac{M}{M_\odot} = \frac{65300 L_\odot}{1+X}\frac{M}{M_\odot},$$

where L_{Edd} is the **Eddington luminosity**. For a normal hydrogen abundance, $X = 0.7$ we have $L_{Edd}/L_\odot = 4 \times 10^4\ M/M_\odot$.