A ONE-ZONE MODEL FOR SHELL FLASHES

We shall consider a model of a compact star, a white dwarf or a neutron star, accreting matter at a constant rate $\dot{M}_a$. We assume spherical symmetry, and we ignore all dynamical effects of matter falling onto the surface. The infalling matter is rich in nuclear fuel, hydrogen and/or helium. When enough matter accumulates the nuclear fuel ignites. We shall consider a steady-state nuclear burning, its stability, and the time variations in the unstable models. This type of scenario is relevant for novae (Gallagher, J. S., and Starrfield, S. 1978, *Annual Review of Astronomy and Astrophysics*, **16**, 171), X-ray bursters (Joss, P. C., and Rappaport, S. A. 1984, *Annual Review of Astronomy and Astrophysics*, **22**, 537), some symbiotic stars, and also for single stars burning hydrogen and helium in two thin shell sources surrounding a degenerate carbon-oxygen core of stars with masses between $0.7M_\odot$ and $8M_\odot$ (Iben, I. Jr., and Renzini, A. 1983, *Annual Review of Astronomy and Astrophysics*, **21**, 271). Full scale model computations require numerical integration of partial differential equations of stellar evolution. This requires a lot of programming time, computers time, and the results are difficult to interpret in simple terms. For this reason it is useful to simplify the problem as much as possible, while retaining the most fundamental physics that is necessary to have models that have properties qualitatively, and even quantitatively similar to full-scale computations. This is possible with a one-zone model (Paczynski, B. 1983, *Astrophysical Journal*, **264**, 282) .

Nuclear burning shells are geometrically thin, and it is reasonable to approximate them by thin, plane-parallel zones. We shall be interested in the region between the base of the nuclear burning shell, below which the hydrogen content $X = 0$, and the surface, onto which fresh matter is falling. The radii of the two boundaries are R_b and R_s, respectively, and $\Delta R \equiv R_s - R_b \ll R_s$, i.e. we may define $R \approx R_s \approx R_b$. It is convenient to introduce a new space-like variable, the surface mass density Σ, with $d\Sigma = \rho dr$. The stellar structure equations may be written as:

$$\frac{\partial P}{\partial \Sigma} = -g \equiv \frac{GM}{R^2}, \qquad \text{(hydrostatic equilibrium)} \tag{oz.1a}$$

$$\frac{\partial P_r}{\partial \Sigma} = -\frac{\kappa F}{c} \equiv \frac{\kappa L}{4\pi R^2 c}, \qquad \text{(radiative equilibrium),} \tag{oz.1b}$$

$$\frac{\partial F}{\partial \Sigma} = \epsilon - T\frac{\partial S}{\partial t}, \qquad \text{(heat balance),} \tag{oz.1c}$$

$$\frac{\partial X}{\partial t} = -\frac{\epsilon}{E^*}, \qquad \text{(hydrogen depletion),} \tag{oz.1d}$$

where P is total pressure, P_r is radiation pressure, ρ is density, T is temperature, S is entropy, Σ is column mass density, t is time, F is radiative heat flux, ϵ is the nuclear energy generation rate ($\text{erg g}^{-1} sec^{-1}$) , κ is opacity ($\text{cm}^2\,\text{g}^{-1}$, c is the speed of light, and E^* is the energy released by burning 1 g of hydrogen.

The boundary conditions at the top of the zone, i.e. at the stellar surface, are

$$\Sigma = \Sigma_s(t), \qquad P \approx 0, \qquad P_r \approx 0, \qquad X = X_s, \tag{oz.2a}$$

and the boundary conditions at the bottom of the zone are:

$$\Sigma = \Sigma_b(t), \qquad F = F_b, \qquad X = 0, \tag{oz.2b}$$

where F_b is the heat flux from the stellar core. The accretion rate is related to the outer boundary with

$$\frac{d\Sigma_s}{dt} = \dot{\Sigma}_a \equiv \frac{\dot{M}_a}{4\pi R^2}, \tag{oz.3}$$

and the column density of the zone is given as

$$\Delta\Sigma = \Sigma_s - \Sigma_b. \tag{oz.4}$$

The differential equations (oz.1) may be integrated over the whole hydrogen rich zone to obtain

$$P = g \int d\Sigma = g\Delta\Sigma, \tag{oz.5a}$$

$$P_r = \frac{1}{c} \int \kappa F d\Sigma \approx \frac{\kappa F}{c} \Delta\Sigma, \tag{oz.5b}$$

$$F = \int \left(\epsilon - T \frac{\partial S}{\partial t} \right) d\Sigma \approx F_b + \left(\epsilon - T \frac{dS}{dt} \right) \Delta\Sigma, \tag{oz.5c}$$

$$\int \frac{\partial X}{\partial t} d\Sigma \approx - \frac{\epsilon \Delta\Sigma}{E^*}. \tag{oz.5d}$$

In all these equations P, P_r, T, ρ, S, κ, ϵ, refer to the values of corresponding physical quantities at the bottom of the shell, and X, and F refer to the values at the surface. This looks like a very crude approximation, but it is surprisingly accurate because all the quantities vary monotonically within the shell.

Equation (oz.5d) gives the rate at which hydrogen is depleted throughout the whole zone. We shall adopt the approximation that this is also the rate at which matter processed through nuclear burning leaves the zone through the bottom of the shell and into the core. Combining this approximation with eqs. (oz.3) and (oz.4) we obtain

$$X \frac{d\left(\Delta\Sigma\right)}{dt} = X\dot{\Sigma}_a - \frac{\epsilon \Delta\Sigma}{E^*}. \tag{oz.5e}$$

The set of equations (oz.5) may be written as two ordinary, first order differential equations

$$T \frac{dS}{dt} = \epsilon - \left(F - F_b\right) \frac{g}{P}, \tag{oz.6a}$$

$$\frac{dP}{dt} = g\dot{\Sigma}_a - \frac{\epsilon P}{X E^*}, \tag{oz.6b}$$

supplemented with a number of algebraic equations:

$$\Delta\Sigma = \frac{P}{g}, \tag{oz.7a}$$

$$1 - \beta \equiv \frac{P_r}{P} = \frac{\kappa F}{cg}, \tag{oz.7b}$$

$$\kappa = \kappa_{el} = 0.2\left(1 + X\right), \tag{oz.7c}$$

and the differential relations:

$$T \frac{dS}{dt} = \frac{P}{\rho\beta} \left[\left(16 - 12\beta - 1.5\beta^2\right) \frac{d\ln T}{dt} - \left(4 - 3\beta\right) \frac{d\ln P}{dt} \right], \tag{oz.8a}$$

$$d\ln P = \left(4 - 3\beta\right) d\ln T + \beta d\ln\rho, \tag{oz.8b}$$

$$d\ln\epsilon = \nu d\ln T + d\ln\rho, \tag{oz.8c}$$

with $\nu = \mathrm{const} \gg 1$.

It is convenient to introduce the Eddington heat flux defined as

$$F_{Edd} = \frac{cg}{\kappa} = \frac{L_{Edd}}{4\pi R^2}. \tag{oz.9}$$

Combining equations (oz.7b) and (oz.9) we obtain

$$1 - \beta = \frac{F}{F_{Edd}} < 1, \tag{oz.10}$$

i.e. the surface heat flux cannot exceed the Eddington limit, just as expected.

Steady-state models

We shall look first into steady state models, i.e. with the time derivatives equal zero. Equations (oz.6) may be combined to obtain

$$\dot{\Sigma}_a = \frac{F - F_b}{XE^*} = \frac{\epsilon P}{gXE^*}. \tag{oz.11}$$

It is convenient to express the heat fluxes and the accretion rate in the Eddington units:

$$f_b \equiv \frac{F_b}{F_{Edd}} = \frac{\kappa F_b}{cg}, \tag{oz.12a}$$

$$f \equiv \frac{F}{F_{Edd}} = \frac{\kappa F}{cg}, \tag{oz.12b}$$

$$\dot{a} \equiv \frac{\dot{\Sigma}_a XE^*}{F_{Edd}} = \left[\frac{\kappa XE^*}{cg} \right] \dot{\Sigma}_a = \frac{\kappa}{cg^2} \epsilon P. \tag{oz.12c}$$

Now, we combine equations (oz.7b) and (oz.11) to write

$$1 - \beta = f = f_b + \dot{a}, \qquad \beta = \dot{a}_{max} - \dot{a}, \qquad 0 < \dot{a} < \dot{a}_{max} = 1 - f_b. \tag{oz.13}$$

The temperature, density, and nuclear burning rate may be expressed as

$$T = \left[\frac{3}{a} P_r \right]^{1/4} = \left[\left(\frac{3}{a} \right) (1 - \beta) P \right]^{1/4} = \left[\left(\frac{3}{a} \right)^{1/4} \right] (f_b + \dot{a})^{1/4} P^{1/4}, \tag{oz.14a}$$

$$\rho = \frac{\mu H}{k} \frac{\beta P}{T} = \left[\frac{\mu H}{k} \left(\frac{a}{3} \right)^{1/4} \right] \frac{(\dot{a}_{max} - \dot{a})}{(f_b + \dot{a})^{1/4}} P^{3/4}, \tag{oz.14b}$$

$$\epsilon = \epsilon_o \rho T^\nu = \left[\epsilon_o \frac{\mu H}{k} \left(\frac{3}{a} \right)^{\frac{\nu - 1}{4}} \right] (\dot{a}_{max} - \dot{a}) (f_b + \dot{a})^{\frac{\nu - 1}{4}} P^{\frac{\nu + 3}{4}}. \tag{oz.14c}$$

Combining equations (oz.14) with (oz.12c) and (oz.7a) we obtain

$$\frac{P}{P^*} = \frac{\Sigma}{\Sigma^*} = \frac{\dot{a}^{\frac{4}{\nu+7}}}{(\dot{a}_{max} - \dot{a})^{\frac{4}{\nu+7}} (f_b + \dot{a})^{\frac{\nu-1}{\nu+7}}}, \tag{oz.15a}$$

$$P^* = \left[\frac{cg^2}{\epsilon_o \kappa} \frac{k}{\mu H} \left(\frac{a}{3} \right)^{\frac{\nu-1}{4}} \right]^{\frac{4}{\nu+7}}, \qquad \Sigma^* = \frac{P^*}{g}, \tag{oz.15b}$$

Differentiating equation (oz.15a) we obtain

$$\frac{d \ln \Sigma}{d \ln \dot{\Sigma}_a} = \frac{d \ln P}{d \ln \dot{a}} = \frac{4}{\nu + 7} \left[1 + \frac{\dot{a}}{(\dot{a}_{max} - \dot{a})} - \frac{(\nu - 1) \dot{a}}{4 (f_b + \dot{a})} \right] = \tag{oz.16}$$

$$\frac{4-\beta\left(\nu-1-\alpha\right)}{\beta\left(1+\alpha\right)\left(\nu+7\right)}, \qquad\qquad \alpha\equiv\frac{f_b}{\dot{a}}. \tag{oz.16}$$

When the accretion rate is very high, then $\dot{a}\approx\dot{a}_{max}=1-f_b$, and the derivative $d\ln P/d\ln\dot{a}$ rises up to $+\infty$. When $\dot{a}\ll f_b\ll 1$, then the last two terms on the right hand side of the equation (oz.16) are negligible, and $d\ln P/d\ln\dot{a}\approx 4/(\nu+7)>0$. Therefore, for vary high, and for very low accretion rates the pressure in the one zone model, and hence the column mass density, increase with the increasing accretion rate. However, for intermediate accretion rates, $f_b\ll\dot{a}\ll 1$, the last term on the right hand side of equation (oz.16) dominates, and we have $d\ln P/d\ln\dot{a}\approx-(\nu-1)/(\nu+7)>0$, i.e. the pressure is a decreasing function of the accretion rate.

Thermal stability

We shall consider now thermal stability of a one zone model with a given surface mass density Σ, and a given heat flux through its bottom F_b. First, we take a model in a thermal equilibrium, i.e. with $dS/dt=0$ and $dP/dt=gd\Sigma/dt=0$. It satisfies the equations

$$\epsilon=\left(F-F_b\right)\frac{g}{P}, \qquad \frac{P_r}{P}=\frac{\kappa}{cg}F. \tag{oz.17}$$

Next, we make a thermal perturbation of the model, but we neglect nuclear evolution within the model, i.e. we take $dS/dt\neq 0$, but we keep $dP/dt=gd\Sigma/dt=0$. We express all perturbations of all variables, δS, $\delta\rho$, $\delta\epsilon$, and δF in terms of temperature perturbation δT, while keeping $\delta P=0$. We obtain

$$T\frac{dS}{dt}=\frac{P}{\rho\beta}\left(16-12\beta-1.5\beta^2\right)\frac{d\left(\delta\ln T\right)}{dt}, \tag{oz.18a}$$

$$\delta\ln\rho=-\frac{4-3\beta}{\beta}\delta\ln T, \tag{oz.18b}$$

$$\delta\ln\epsilon=\left(\nu-\frac{4-3\beta}{\beta}\right)\delta\ln T, \tag{oz.18c}$$

$$\delta\ln F=4\delta\ln T. \tag{oz.18d}$$

Putting all these perturbations into the heat balance equation (oz.6a) , and taking into account the relation between the unperturbed quantities, i.e. (oz.17) , we obtain

$$T\frac{dS}{dt}=\frac{P}{\rho\beta}\left(16-12\beta-1.5\beta^2\right)\frac{d\left(\delta\ln T\right)}{dt}= \tag{oz.19}$$

$$=\delta\epsilon-\delta F\frac{g}{P}=\epsilon\left(\nu-\frac{4-3\beta}{\beta}\right)\delta\ln T-\frac{gF}{P}4\delta\ln T=$$

$$=\epsilon\left[\nu-\frac{4-3\beta}{\beta}-4\frac{F}{F-F_b}\right]\delta\ln T=$$

$$=\epsilon\left[\nu-\frac{4}{\beta}+3-4\frac{f}{f-f_b}\right]\delta\ln T=$$

$$=\epsilon\left[\nu+3-\frac{4}{\left(\dot{a}_{max}-\dot{a}\right)}-4\frac{\dot{a}+f_b}{\dot{a}}\right]\delta\ln T=$$

$$= \epsilon \left[\nu - 1 - \frac{4}{(\dot{a}_{max} - \dot{a})} - 4\frac{f_b}{\dot{a}} \right] \delta \ln T =$$

$$= 4\epsilon \left[\frac{(\nu - 1)}{4} - \frac{1}{(\dot{a}_{max} - \dot{a})} + 1 - 1 - \frac{f_b}{\dot{a}} \right] \delta \ln T =$$

$$= 4\epsilon \left[\frac{(\nu - 1)}{4} - \frac{(1 - \dot{a}_{max} + \dot{a})}{(\dot{a}_{max} - \dot{a})} - \frac{(f_b + \dot{a})}{\dot{a}} \right] \delta \ln T =$$

$$= 4\epsilon \left[\frac{(\nu - 1)}{4} - \frac{(f_b + \dot{a})}{(\dot{a}_{max} - \dot{a})} - \frac{(f_b + \dot{a})}{\dot{a}} \right] \delta \ln T =$$

$$= 4\epsilon \frac{(f_b + \dot{a})}{\dot{a}} \left[\frac{(\nu - 1)\,\dot{a}}{4\,(\dot{a} + f_b)} - \frac{\dot{a}}{(\dot{a}_{max} - \dot{a})} - 1 \right] \delta \ln T. \tag{oz.19}$$

The differential equation (oz.19) may be solved to obtain

$$\delta \ln T = (\delta \ln T)_{t=0}\, e^{\sigma t}, \tag{oz.20a}$$

$$\sigma = \frac{1}{\tau_{th}} \frac{4\beta\,(f_b + \dot{a})}{(16 - 12\beta - 1.5\beta^2)\,\dot{a}} \left[\frac{(\nu - 1)\,\dot{a}}{4\,(\dot{a} + f_b)} - \frac{\dot{a}}{(\dot{a}_{max} - \dot{a})} - 1 \right] = \tag{oz.20b}$$

$$-\frac{1}{\tau_{th}} \frac{4 - \beta(\nu - 1 - \alpha)}{(16 - 12\beta - 1.5\beta^2)}, \qquad \alpha \equiv \frac{f_b}{\dot{a}},$$

$$\tau_{th} \equiv \frac{P}{\rho\epsilon} = \text{thermal time scale } \approx \frac{\text{heat content}}{\text{heat generation rate}}. \tag{oz.20c}$$

The solution (oz.20a) grows exponentially if $\sigma > 0$, i.e. when the square bracket in the equation (oz.20b) is positive. Notice, that this square bracket is equal to the square bracket in the equation (oz.16) , with the opposite sign. This means that the one zone model is unstable when, in a linear series of such models, the column mass density is a decreasing function of the accretion rate. Therefore, models with intermediate accretion rate are unstable, while those with very high, or very low accretion rate, are stable.

It is easy to understand why models with very low accretion rate, $\dot{a} \ll f_b$, are stable. In those models almost all energy radiated from the surface comes from the stellar core, with energy input from nuclear burning being insignificant.

Models with intermediate accretion rate, $f_b \ll \dot{a} \ll 1$, are unstable because of very high temperature sensitivity of nuclear reaction rate, i.e. $\nu \gg 1$. According to the equations (oz.17) the heat generation rate is proportional to ϵ, i.e. to T^ν, while the heat losses are proportional to T^4. Therefore, a slight increase of temperature over the equilibrium value will result in the net heating, and farther increase of the temperature, and so on. We have a thermal runaway, i.e. thermal instability.

Models with very high accretion rates, $\dot{a} \approx \dot{a}_{max} = 1 - f_b$, are stable again because nuclear reaction rate is sensitive to the density too, $\epsilon \sim \rho$. When the accretion rate is close to the maximum rate, the luminosity is close to the Eddington limit, and radiation pressure dominates, with $P \approx P_r = \frac{a}{3}T^4$. As pressure within the one zone model remains constant during a thermal perturbation, the slightest increase of temperature is accompanied with a very large decrease of density, and as a result the nuclear energy generation rate decreases, making the model stable.

Notice, that the transition from stable to unstable models is at the turning points of the linear series of one zone models, at which the column mass density Σ has a local maximum or a local minimum, i.e $d \ln \Sigma / d \ln \dot{\Sigma}_a = 0$.