Neutron stars and black holes

Like white dwarfs, neutron stars are strongly degenerate compact objects of roughly one solar mass. But whereas white dwarfs are supported against gravity by the zero-point motion of electrons (electron degeneracy pressure), neutron stars, as the name suggests, consist mostly of neutrons and are supported by the zero-point motion and interactions among the latter particles. The existence of neutron stars was hypothesized by Lev Landau and by Walter Baade and Fritz Zwicky shortly after the discovery of the neutron by Chadwick in 1932—Baade and Zwicky even proposed that neutron stars might form in supernovae, as is now believed. However, the observational history of neutron stars began only in 1967 with the discovery of radio pulsars by Jocelyn Bell and her Ph.D. advisor A. Hewish. In the meantime, the properties of such stars had been elucidated by a few important theorists, including J. R. Oppenheimer and E. Salpeter. For more historical and physical detail than can be fit into this lecture, see the book by Shapiro and Teukolsky.

We have seen that white dwarfs have a maximum mass

$$M_{\rm Ch} = 3.1\frac{(\hbar c/G)^{3/2}}{(\mu_e m_p)^2} \approx 1.4(2/\mu_e)^2 M_\odot. \tag{1}$$

It is significant that the electron mass does not appear in this formula. One might therefore guess that an object supported by neutron degeneracy could have a similar maximum mass. This would be a shrewd guess, although the full story is more complicated. The electron mass does however appear in the mass-radius relation of white dwarfs; when the electrons at the center are marginally relativistic, $x_F(0) \equiv p_F(0)/m_e c = 1$, the mass $M \approx 0.5(2/\mu_e)^2 M_\odot$, the central density $\rho_c \approx 2\times10^6(\mu_e/2)\,\mathrm{g\,cm^{-3}}$, and the radius

$$R_{x_F(0)=1} \approx 3.8\frac{(\hbar^3/cG)^{1/2}}{\mu_e m_p m_e} \approx 0.021(2/\mu_e)R_\odot.$$

For $x_F(0) \gg 1$, the radius decreases as $x_F(0)^{-1}$, and the mass approaches (1).

While electrons supply the pressure, the mass is dominated by nuclei of atomic weight A and atomic number $Z \approx A/2$. When the Fermi energy $\varepsilon_F = \sqrt{(p_F c)^2 + (m_e c^2)^2}$ of the electrons is large enough, it becomes energetically favorable for nuclei to undergo inverse beta decay,

$$(A, Z) \;+\; e^- \rightarrow (A, Z-1) \;+\; \nu_e, \tag{2}$$

thereby lowering the energy of the electron gas. The threshold for this reaction is

$$\epsilon_F \geq M(A, Z-1) - M(A, Z).$$

For example, at normal densities, the stablest of all nuclei is ^{56}Fe: $(A, Z) = (56, 26)$. The threshold for (2) is then 3.695 MeV, which is reached at $x_F(0) = 7.174$, $M \approx 1.33 M_\odot$. (Observed white dwarfs are probably made of lighter elements—He, C, O, Mg—for which the threshold is higher. But the progenitors of neutron stars are the degenerate iron cores of evolved massive stars.) With increasing density, nuclei of higher and higher A/Z are favored. Eventually, the nuclei become so neutron rich that the reaction

$$M(A', Z') \rightarrow M(A'-1, Z') \;+\; n$$

is favored. This is "neutron drip," and in low-temperature equilibrium conditions, it begins at $\rho_{\rm nd} \approx 4\times10^{11}\,\mathrm{g\,cm^{-3}}$. Nuclei dissolve completely at $\rho_{\rm nuc}/2$, where

$$\rho_{\rm nuc} \approx 2.8\times10^{14}\,\mathrm{g\,cm^{-3}} \tag{3}$$

is the density of nuclear matter: $\rho_{\rm nuc}$ corresponds to a volume $\approx (4\pi/3)\times(1.1\,\mathrm{fm})^3$ per nucleon. Ordinary nuclei have radii $\approx 1.1A^{1/3}$ fm, where $1\,\mathrm{fm} \equiv 1\,\mathrm{Fermi} \equiv 10^{-13}$ cm is a convenient unit of length.

Nuclear matter is somewhat analogous to a liquid (such as water) in that there is an attraction between neighboring constituent particles, but also a strong resistance to compression; the density increases more slowly with pressure than for an ideal gas. Nevertheless, for the moment, suppose that the star *were* an ideal gas of neutrons. Then under nonrelativistic conditions, the equation of state (henceforth EOS) would be

$$P = \frac{1}{5}\left(\frac{3}{8\pi}\right)^{2/3} h^2 m_n^{-8/3}\rho^{5/3} \equiv K\rho^{5/3},$$

where m_n is the neutron mass, and the corresponding stellar parameters would be

$$\begin{aligned}
M &\approx 3.03\left(\frac{K}{G}\right)^{3/2}\rho(0)^{1/2} && \approx 0.6\left(\frac{\rho(0)}{\rho_{\rm nuc}}\right)^{1/2} M_\odot \\
R &\approx 1.63\left(\frac{K}{G}\right)^{1/2}\rho(0)^{-1/6} && \approx 18\left(\frac{\rho(0)}{\rho_{\rm nuc}}\right)^{-1/6} \text{km} \\
\frac{2GM}{Rc^2} &\approx 3.72\frac{K}{c^2}\rho(0)^{2/3} && \approx 0.0965\left(\frac{\rho(0)}{\rho_{\rm nuc}}\right)^{2/3}.
\end{aligned}$$

The last of these quantities is $(v_{\rm esc}/c)^2$, $v_{\rm esc}$ being the escape velocity from the stellar surface. It is clear that relativistic effects are important for these objects, whereas the formulae above are based on a nonrelativistic approximation to the EOS *and* newtonian gravity. Nevertheless, these formulae give the correct orders of magnitude for neutron-star properties. Let us boldly estimate the maximum neutron-star mass by setting $v_{\rm esc} = c$ above: this implies $\rho(0) \approx 33.4\rho_{\rm nuc}$ and $M_{\rm max} \approx 3.4M_\odot$.

The exact general-relativistic equations of hydrostatic equilbrium were derived by Oppenheimer and Volkov (1939):

$$\begin{aligned}
\frac{dP}{dr} &= -\left(\rho + \frac{P}{c^2}\right)\left(\frac{GM_r}{r^2} + \frac{4\pi G P}{c^2} r\right)\left(1 - \frac{2GM_r}{c^2 r}\right)^{-1} \\
\frac{dM_r}{dr} &= 4\pi \int_0^r \rho(\bar{r})\bar{r}^2 d\bar{r}.
\end{aligned} \tag{4}$$

Here $\rho \neq m_n n_n$, which would be the density of rest-mass, but rather $\rho \equiv U/c^2$, where U is the internal energy per unit volume *including* rest mass. For a relativistic ideal gas of neutrons,

$$\begin{aligned}
U &\to \frac{8\pi}{h^3}\int_0^{p_F} p^2\sqrt{(pc)^2 + (m_n c^2)^2}\, dp = \frac{8\pi}{3}\lambda_n^{-3} m_n c^2 \int_0^{x_F} x^2\sqrt{x^2+1}\, dx \\
P &\to \frac{8\pi}{3}\lambda_n^{-3} m_n c^2 \int_0^{x_F} \frac{x^4\, dx}{\sqrt{x^2+1}} \\
n_n &\to \frac{8\pi}{3}\lambda_n^{-3} x_F^3,
\end{aligned} \tag{5}$$

in which $\lambda_n \equiv h/m_n c$ is the neutron Compton wavelength. Notice that as $x_F \to \infty$, $P \to U/3 = \rho/3c^2$ rather than $P \to K\rho^{4/3}$: thus a relativistic ideal gas of nucleons is *not* an $n = 3$ polytrope, and there is no direct counterpart to the Chandrasekhar mass (1) in newtonian gravity for such a star. This is because the particles supplying the pressure are the same as those supplying the gravitating mass. On the other hand, general-relativistic gravity as embodied in eqs. (4) *does* imply a maximum mass for the EOS (5). By integrating the above equations, Oppenheimer and Volkov found $M_{\rm max} \approx 0.7M_\odot$, and a corresponding radius 9.6 km.

Observed neutron-star masses are clearly larger than Oppenheimer & Volkov's value. Well-determined masses come from binary systems, especially those containing a pulsar; less accurate mass estimates are sometimes possible for X-ray binaries, which involve a neutron star accreting from a less compact companion. In favorable cases, very precise pulsar timing allows one to detect subtle general-relativistic effects in the binary orbit and thereby constrain more of the system parameters than would be possible in a strictly newtonian world. Remarkably, all well-determined neutron-star

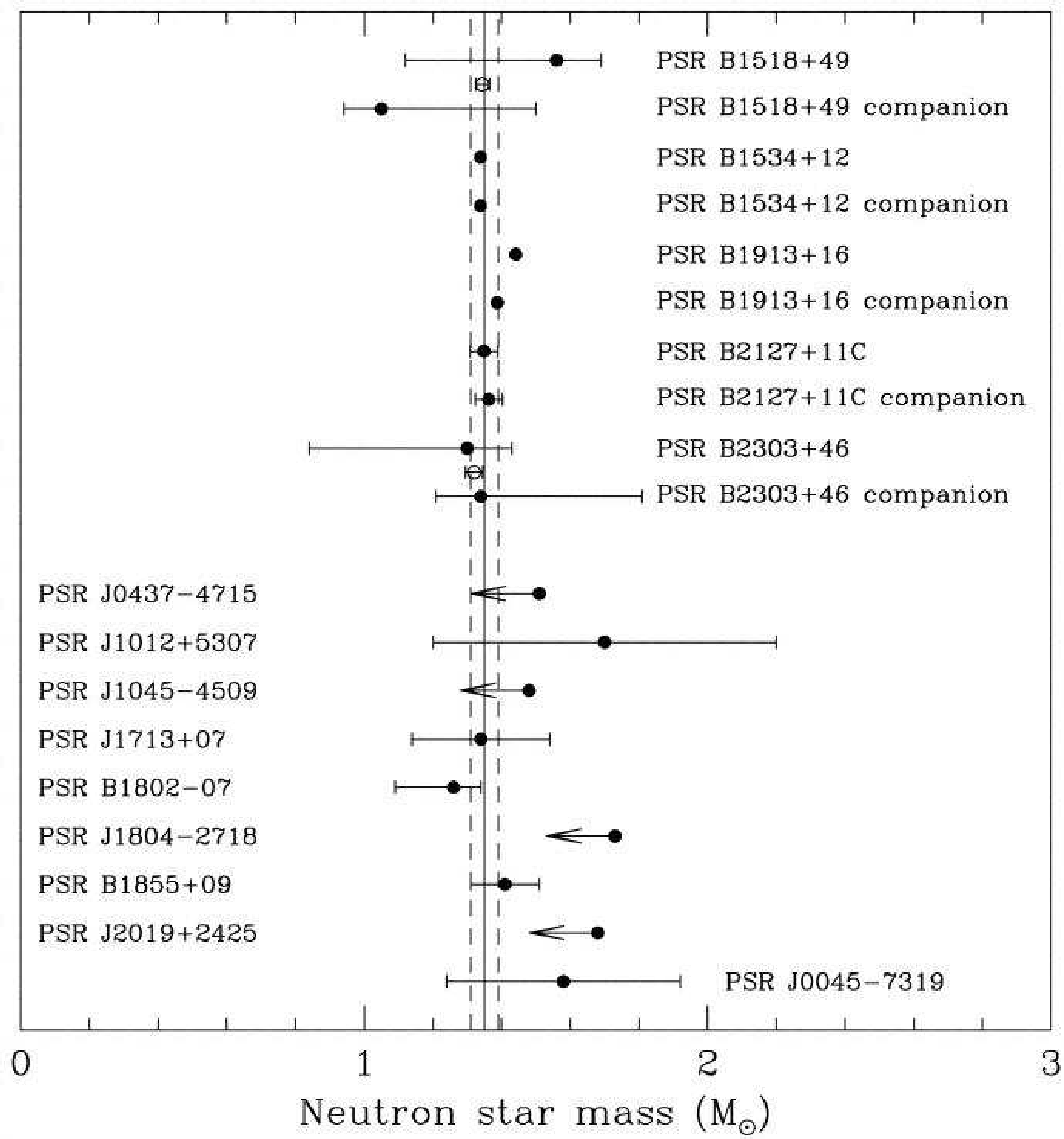

Figure 1: Measured masses of pulsars in binary systems, and of their companions when those are also believed to be neutron stars. From Thorsett & Chakrabarty (1999, ApJ 512, 288).

masses are consistent with a very narrow range: $M_{\rm ns} = 1.35 \pm 0.04 M_\odot$ (Fig. 1). This presents a puzzle, since it is believed that the shortest-period pulsars, which have pulse/rotation periods $< 10\,\text{ms}$, have been "recycled" after spinning down to a long period by accreting $\sim 0.1 M_\odot$ of high-angular-momentum material from an accretion disk fed by the companion.

In any case, the observed masses indicate that the EOS must be stiffer than an ideal gas at $\rho \gtrsim \rho_{\rm nuc}$. This is indeed what nuclear theorists expect, but a consensus as to the correct EOS in this regime has not been reached. Currently plausible choices, when substituted into eqs. (4), predict maximum masses between $1.4M_\odot$ and $2.5M_\odot$. Further constraints on the EOS may be available from neutron-star cooling (Yakovlev & Pethick, 2004, ARA&A, 42, 169): although the discussion here supposes negligible entropy, neutron stars are born at high temperatures in supernova explosions, and their subsequent cooling depends upon their heat capacity, which is a function of the EOS. Because of hydrostatic support by degeneracy and nuclear forces, and efficient cooling by neutrino emission when $kT \gg 1\,\text{MeV}$, cooling has little influence on the mass-radius relation after the first few seconds of the neutron-star's birth, but the surface temperature is measurable by X-ray satellites in favorable cases. Although the data have been improving rapidly, as of this writing their interpretation has been hampered by a poor understanding of neutron-star atmospheres, which may be very exotic because of strong magnetic fields, *etc.*

Before leaving the subject, let us briefly consider the limit of an equation of state so stiff that $\rho =$constant. Now, this is not physically realistic: since the entropy has negligible influence on the pressure in any realistic EOS for neutron-stars,

$$\frac{d\rho/dr}{dP/dr} \approx \left(\frac{\partial\rho}{\partial P}\right)_S = \frac{1}{c_s^2}$$

where c_s is the sound speed. One requires $c_s^2 > 0$ (else sound waves would be unstable), so $d\rho/dP > 0$. Furthermore, $c_s^2 < c^2$ else sound waves would move faster than light, so $d\rho/dP > c^{-2}$. Since (4) ensures $dP/dr < 0$, it follows that $d\rho/dr < 0$. Nevertheless, equations (4) are easily solved for $\rho =$constant and lead to an interesting upper bound on GM/R: For $\rho =$constant we have $M_r = (4\pi\rho/3)r^3$, and the first of eqs. (4) can be rearranged as

$$\frac{dP}{(\rho + c^{-2}P)(\rho + 3c^{-2}P)} = -\frac{4\pi G}{3}\frac{r dr}{1 - (4\pi G\rho/3c^2)r^2} \tag{6}$$

The integrals are elementary, and with the boundary condition $P = 0$ at $r = R$, one has

$$\frac{2GM}{c^2R} = 1 - \left(\frac{P(0) + \rho c^2}{3P(0) + \rho c^2}\right)^2$$

Thus, for any finite value of the central pressure $P(0)$,

$$\frac{2GM}{c^2R} < \frac{8}{9}\,, \tag{7}$$

which means that the star is at least 9/8 times larger than its Schwarzschild radius

$$R_s(M) \equiv \frac{2GM}{c^2} \approx 3.0\left(\frac{M}{M_\odot}\right)\ \text{km}. \tag{8}$$

It is not difficult to prove that the limit (7) applies to any star for which $d\rho/dP > 0$.[1] It follows that

$$M_{\rm max} < 6.8\left(\frac{\bar\rho}{\rho_{\rm nuc}}\right)^{-1/2} M_\odot. \tag{9}$$

[1] Replace ρ with $\bar\rho_r \equiv 3M_r/4\pi r^3$ in (6) and use the fact that $\bar\rho_r \ge \bar\rho \equiv \bar\rho(R)$ to set bounds on the two integrals.

Black holes

A black hole is a mass so concentrated that nothing escapes from it, even particles moving at the speed of light[2]. The theoretical possibility of such objects was raised as far back as the eighteenth century, when Laplace and others postulated the existence of an object whose escape velocity $2GM/R > c^2$. However, these early scientists knew only newtonian gravity, which is not applicable at such large velocities and strong fields. The first exact black-hole solution was found by Karl Schwarzschild in 1916, only months after Einstein published his theory of General Relativity; in fact, the Schwarzschild solution was the first exact solution of Einstein's field equations other than Minkowski space. Any detailed discussion of black hole physics requires general relativity, which is beyond the scope of this course, so we content ourselves with a brief discussion of the observational evidence for their existence.

The properties of black holes are so extreme that astrophysicists struggled for years to establish that such things really exist. This argument has now subsided, due in part to well-established lower bounds to the masses of some X-ray binaries (Fig. 2). Black holes themselves are dark, of course, but gas in orbit around them can dissipate its orbital energy and release the resulting heat as luminous radiation. The sources listed in Fig. 2 are bright—with peak luminosities comparable to the Eddington limit for a solar mass—and emit most of this energy in X-rays (the characteristic photon energy is listed in the penultimate column), indicating a deep potential well, and therefore an object too compact to be a normal (nondegenerate) star. The gas in this disk is supplied by the companion, which usually is a normal star. This star is often observable, and its orbital period (P) and radial velocity semi-amplitude (K) measurable. Using standard newtonian equations for the two-body problem, these observed quantities can be combined to yield a constraint on the masses,

$$\frac{(M_1 \sin i)^3}{(M_1 + M_2)^2} = \frac{PK^3}{2\pi G}\,, \qquad (10)$$

in which M_1 is the mass of the accreting (usually the heavier) star, M_2 is that of the visible companion, and i is the inclination of the orbital angular momentum to the line of sight ($i = 90°$ if the orbit is seen edge-on, which maximizes K for a given intrinsic orbital velocity). The lefthand side of (10) is called the <u>mass function</u>. It is an absolute lower bound on M_1 since $M_2 > 0$ and $\sin i \le 1$. This quantity is given in the second column of Fig. 2. Sometimes other information is available, such as a light curve showing tidal distortion of the secondary star, that allows one to put some constraints on $\sin i$ and thereby obtain a more accurate estimate for M_1. Clearly, many of these X-ray sources are well above the maximum mass of a neutron star or white dwarf, so a black hole is implicated.

Additional evidence for the black-hole nature of these sources comes from their extraordinary variations in X-ray luminosity (L_X). At the peak, as already noted, $L_X \sim L_{\rm Edd}(M)$ in many cases. But these sources also go through intervals of quiescence when L_X is extremely small—for example, $L_{X,\rm min}/L_{X,\rm max} \sim 10^{-8}$ in 0620-003. This ratio is much smaller than in X-ray binaries where the primary is believed to be a neutron star, and it has been argued that faintness of the quiescent state is a signature of the existence of an event horizon (*e.g.* Garcia, McClintock, & Narayan 2001, ApJ 553, L47). The argument goes as follows: (i) the accretion rate of the disk is unlikely to vary by so large a factor; (ii) an accretion rate $\dot{M}$ onto the surface of a neutron star inevitably produces a luminosity $L = GM_{\rm ns}\dot{M}/R_{\rm ns}$; (iii) gas may in principle, however, fall into a black hole without radiating much of its energy, since there is no solid surface to settle on; (iv) and in fact plausible mechanisms exist by which a modest reduction of $\dot{M}$ greatly reduces the density and therefore the radiative efficiency of the accreting flow, consonant with point (iii).

Finally, after four decades of theoretical effort, no physically plausible model has been advanced for quasars and QSOs that does not involve black holes. But in contrast to the black holes in X-ray binaries, where $M \sim 10M_\odot$, the Eddington limit (as well as more indirect evidence) points to $M \gtrsim 10^9 M_\odot$ in the brightest quasars.

[2] barring quantum-mechanical effects, *i.e.* Hawking radiation

Table 4.2. *Confirmed black hole binaries: X-ray and optical data*

Source	$f(M)^a$ ($M_\odot$)	M_1^a ($M_\odot$)	f(HFQPO) (Hz)	f(LFQPO) (Hz)	Radio[b]	E_{max}^c (MeV)	References
0422+32	1.19±0.02	3.2–13.2		0.035–32	P	0.8,1–2:	1,2,3,4,5
0538–641	2.3±0.3	5.9–9.2	–	0.46	–	0.05	6,7
0540–697	0.14±0.05	4.0–10.0:		0.075		0.02	8,7
0620–003	2.72±0.06	3.3–12.9	–	–	P,J?	0.03:	9,10,11,11a
1009–45	3.17±0.12	6.3–8.0		0.04–0.3	[d]	0.40, 1:	12,4,13
1118+480	6.1±0.3	6.5–7.2	–	0.07–0.15	P	0.15	14,15,16,17
1124–684	3.01±0.15	6.5–8.2	–	3.0–8.4	P	0.50	18,19,20,21
1543–475	0.25±0.01	7.4–11.4[e]	–	7	–[f]	0.20	22,4
1550–564	6.86±0.71	8.4–10.8	92,184,276	0.1–10	P,J	0.20	23,24,25,26,27
1655–40	2.73±0.09	6.0–6.6	300,450	0.1–28	P,J	0.80	28,29,30,31,54
1659–487	> 2.0[g]	–	–	0.09–7.4	P	0.45, 1:	32,33,4,13
1705–250	4.86±0.13	5.6–8.3	–	–	–[d]	0.1	34,35
1819.3–2525	3.13±0.13	6.8–7.4	–	–	P,J	0.02	36,37
1859+226	7.4±1.1	7.6–12:	190	0.5–10	P,J?	0.2	38,39,40,41
1915+105	9.5±3.0	10.0–18.0	41,67,113,168	0.001–10	P,J	0.5, 1:	42,43,44,4,13
1956+350	0.244±0.005	6.9–13.2	–	0.035–12	P,J	2–5	45,46,47,48,49
2000+251	5.01±0.12	7.1–7.8	–	2.4–2.6	P	0.3	18,50,51
2023+338	6.08±0.06	10.1–13.4	–	–	P	0.4	52,53

Figure 2: Mass measurements for black-hole candidates in X-ray binary systems. From McClintock & Remillard (2004, "Black Hole Binaries," in *Compact X-ray Sources*, W.H.G. Lewin & M. van der Klis, Cambridge; `astro-ph/0306213`.