12 December 2006 by JJG

Stellar Magnetohydrodynamics

Magnetohydrodynamics (MHD) is the generalization of hydrodynamics to electrically conducting fluids. Major astrophysical applications include stellar interiors, accretion disks, and the interstellar medium.

Derivation of the MHD Equations

MHD makes use of Maxwell's Equations to relate the electrical currents to the electromagnetic fields,

$$\begin{aligned} \boldsymbol{\nabla}\times\boldsymbol{B} &= \frac{4\pi}{c}\boldsymbol{J} + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} \qquad & \boldsymbol{\nabla}\cdot\boldsymbol{B} &= 0 \\ \boldsymbol{\nabla}\times\boldsymbol{E} &= -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} \qquad & \boldsymbol{\nabla}\cdot\boldsymbol{E} &= 4\pi\rho_q \end{aligned} \tag{1}$$

but with some simplifying assumptions and with the addition of Ohm's Law, as described below. We have just made the simplification of replacing the dialectric constant and the permeability by their vacuum values $\epsilon_0 = \mu_0 = 1$ (in gaussian units). The charge density has been written ρ_q to distinguish it from the mass density, ρ.

The basic physical approximation of MHD is that the conductivity is high enough so that ρ_q is negligible. To make this quantitative, in a partially or fully ionized plasma, separation between the positive ions and the negative electrons leads to oscillations at the plasma frequency

$$\omega_{\rm pe} = \left(\frac{4\pi e^2 n_e}{m_e}\right)^{1/2} \approx 2\pi\times 10^4 \left(\frac{n_e}{1\ {\rm cm}^{-3}}\right)^{1/2}\ {\rm s}^{-1}, \tag{2}$$

where n_e is the *mean* density of electrons, until these oscillations damp. MHD should be used only when the macroscopic timescales of interest are $\gg \omega_{\rm pe}^{-1}$, in which case these oscillations are not excited and the plasma is approximately electrically neutral.

Nevertheless, the plasma can carry a significant current. Suppose there are $N \geq 2$ charge species (*e.g.*, e^-, p, He^+, He^{++}, ...) with particle charges q_i, masses m_i, number densities n_i, and mean[1] velocities $\boldsymbol{v}_i$; then the current density is

$$\boldsymbol{J} = \sum_{i=1}^{N} q_i n_i \boldsymbol{v}_i. \tag{3}$$

Here each species is weighted by charge. The plasma velocity is weighted by mass:

$$\boldsymbol{v} \equiv \rho^{-1}\sum_{i=1}^{N} m_i n_i \boldsymbol{v}_i, \qquad \rho \equiv \sum_{i=1}^{N} m_i n_i. \tag{4}$$

Ohm's Law

$$\boldsymbol{J} = \sigma\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c}\times\boldsymbol{B}\right), \tag{5}$$

$$\sigma = \sum_{i=1}^{N} \frac{q_i^2 n_i}{m_i} t_{{\rm coll},i}\,. \tag{6}$$

[1] each species has an approximately Maxwellian distribution of velocities, of which $\boldsymbol{v}_i$ is the mean

relates the current density to the electric and magnetic fields, where σ is the conductivity. The model for the conductivity is that the charge carriers move ballistically under the influence of the Lorentz force until they "collide" with another particle. The collisions tend to isotropize the charge velocities on the timescale $t_{\text{coll},i}$ (different for different species) in a frame comoving with the mean velocity $\boldsymbol{v}$. The (nonrelativistic) model equation is

$$\frac{d\boldsymbol{v}_i}{dt} + \frac{\boldsymbol{v}_i - \boldsymbol{v}}{t_{\text{coll},i}} = \frac{q_i}{m_i}\left(\boldsymbol{E} + \frac{\boldsymbol{v}_i}{c}\times\boldsymbol{B}\right). \tag{7}$$

Two approximations are then made in order to obtain (5) from (7). First, macroscopic timescales are presumed $\gg t_{\text{coll},i}$ so that the first term on the lefthand side of (7) can be neglected. Second, the cyclotron frequency $\Omega_i \equiv q_i B/m_i c$ is presumed small compared to $t_{\text{coll},i}^{-1}$: then it can be shown that $\boldsymbol{v}_i$ can be replaced by $\boldsymbol{v}$ in the $\boldsymbol{v}\times\boldsymbol{B}$ term. In very dilute or very strongly magnetized plasmas where the second assumption fails but the first still holds, the conductivity becomes a tensor rather than a scalar. Electrons normally dominate the conductivity because they are so much lighter than all other species, $\sigma \approx \omega_{\text{pe}}^2 t_{\text{coll},e}/4\pi$. In cold and dense media such as molecular clouds or protostellar disks, however, there may be so few free electrons that metal ions or charged dust grains dominate. Notice that the conductivity has units $[\text{time}]^{-1}$.

The free flow of charge tends to neutralize $\boldsymbol{\nabla}\cdot\boldsymbol{E}$ but cannot eliminate $\boldsymbol{E}$ altogether, if only because electric fields transform between two Lorentz frames O and O' with relative velocity $\boldsymbol{\beta}c$ according to

$$\boldsymbol{E}' = (1-\beta^2)^{-1/2}(\boldsymbol{E} + \boldsymbol{\beta}\times\boldsymbol{B}). \tag{8}$$

Thus, if the electric field were to vanish exactly in the local rest frame of the plasma, we would have $\boldsymbol{E} \to -(\boldsymbol{v}/c)\times\boldsymbol{B}$ in the "lab" frame. More generally, in MHD electric fields are $O(v/c) \ll 1$ compared to magnetic ones. Therefore, the "displacement current" $c^{-1}\partial\boldsymbol{E}/\partial t$ in the first of Maxwell's equations is small compared to $\boldsymbol{\nabla}\times\boldsymbol{B}$. In fact, if L is a characteristic lengthscale and T a characteristic timescale for the flow, then $\boldsymbol{\nabla}\times\boldsymbol{E} \sim O(B/L)$ and $c^{-1}\partial E/\partial t \sim O(E/cT)$, so the ratio of the latter to the former is $\sim O(E/B)\cdot O(L/cT)$ Since $L/T \sim O(v)$, it follows that the displacement current is smaller than the other two terms in its equation by $O((v/c)^2)$. Therefore, MHD neglects the displacement current and takes

$$\boldsymbol{J} = \frac{c}{4\pi}\boldsymbol{\nabla}\times\boldsymbol{B}. \tag{9}$$

This implies $\boldsymbol{\nabla}\cdot\boldsymbol{J} = 0$, which is consistent with $\rho_q \approx 0$. Next, using (9) to eliminate $\boldsymbol{J}$ from Ohm's Law (5) leads to

$$c\boldsymbol{E} = \eta\boldsymbol{\nabla}\times\boldsymbol{B} - \boldsymbol{v}\times\boldsymbol{B}, \qquad \eta \equiv \frac{c^2}{4\pi\sigma}. \tag{10}$$

The quantity η, called the "magnetic diffusivity," has units of a diffusion coefficient: $[\text{length}]^2[\text{time}]^{-1}$. Using (10) to eliminate $\boldsymbol{E}$ from Faraday's Law—the second equation in (1) that contains a time derivative—leads to the "induction equation"

$$\frac{\partial\boldsymbol{B}}{\partial t} - \boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{B}) = -\boldsymbol{\nabla}\times(\eta\boldsymbol{\nabla}\times\boldsymbol{B}). \tag{11}$$

Eq. (11) tells us how the magnetic field evolves under the influence of the velocity field. To complete the MHD equations, we have to see how the magnetic field modifies the evolution of $\boldsymbol{v}$ by exerting a force on the plasma. By summing over particle species, it is easy to show that the Lorentz force per unit volume is $\rho_q \boldsymbol{E} + c^{-1}\boldsymbol{J}\times\boldsymbol{B}$. Since $\rho_q \approx 0$, the first term is negligible. Hence the equation of motion for $\boldsymbol{v}$ becomes

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v} = -\frac{\boldsymbol{B}\times(\boldsymbol{\nabla}\times\boldsymbol{B})}{4\pi\rho} - \rho^{-1}\boldsymbol{\nabla}P - \boldsymbol{\nabla}V. \tag{12}$$

Equations (11) and (12), supplemented by an equation of state $P = P(\rho, S)$ for the pressure and Poisson's equation $\nabla^2 V = 4\pi G\rho$ for the gravitational potential, form a complete, closed system: there is normally no need to consider $\boldsymbol{E}$ or $\boldsymbol{J}$ directly, though sometimes this is conceptually useful (and sometimes unavoidable in posing boundary conditions).

In astrophysics, the dimensionless combination $\eta T/L^2$ is usually extremely small—not because plasmas are perfect conductors (a typical conductivity is comparable to that of copper) but because lengthscales are so large. In this case, the righthand side of the induction equation (11) can be neglected, except in "current sheets" where $\boldsymbol{B}$ is almost discontinous and magnetic reconnection occurs. (This is analogous to saying that viscosity is negligible except in shocks). For $\eta \to 0$, (11) implies that magnetic field lines are "frozen" into the plasma, in the following sense: let Σ be a finite two-dimensional surface that moves with the plasma and is bounded by a closed curve Γ. Then the magnetic flux through this surface is independent of time. To see this, let Σ and Σ' be the positions of this surface at two times t and $t' = t + \Delta t$, and let $\boldsymbol{B}$ and $\boldsymbol{B}'$ be used as abbreviations for the vector fields $\boldsymbol{B}(\boldsymbol{r}, t)$ and $\boldsymbol{B}(\boldsymbol{r}, t')$, respectively. Then

$$\iint\limits_{\Sigma'} \boldsymbol{B}'\cdot d\boldsymbol{S} - \iint\limits_{\Sigma} \boldsymbol{B}\cdot d\boldsymbol{S} = \Delta t \iint\limits_{\Sigma} \frac{\partial \boldsymbol{B}}{\partial t}\cdot d\boldsymbol{S} \; - \iint\limits_{\Delta\Sigma} \boldsymbol{B}\cdot d\boldsymbol{S} \; + O(\Delta t^2), \tag{13}$$

where $\Delta\Sigma$ is a ribbon of width $\boldsymbol{v}\Delta t$ that connects Γ to Γ', so that $\Sigma' - \Sigma + \Delta\Sigma$ forms a complete closed surface; the minus sign in front of Σ indicates that its normal must be reversed in order to point "outward." We have made use of the fact that the flux of $\boldsymbol{B}$ through a closed surface vanishes (since $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$) in order to write the second term on the righthand side of (13) as the change in flux due to the motion of the surface alone. If $d\boldsymbol{l}$ is an element of arc along the bounding curve Γ, then the outward-pointing area element along $\Delta\Sigma$ is $d\boldsymbol{S} = \Delta t\,(d\boldsymbol{l}\times\boldsymbol{v}) + O(\Delta t^2)$. Using this in the second integral on the right side of (13), and using (11) in the first integral, the flux difference becomes

$$\Delta t \iint\limits_{\Sigma} \boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{B})\cdot d\boldsymbol{S} - \Delta t \int\limits_{\Gamma} \boldsymbol{B}\cdot(d\boldsymbol{l}\times\boldsymbol{v}) + O(\Delta t^2). \tag{14}$$

Stokes Theorem shows that the first integral in (14) cancels with the second. QED.

Magnetorotational Instability

One of the most important consequences of adding magnetic fields to the equations of motion is that it becomes possible for angular momentum to be transferred between fluid elements in an axisymmetric configuration. Without the field, and without significant true viscosity, the only collective interactions between fluid elements involve the pressure and potential gradients. In axisymmetry, $\partial P/\partial\phi = \partial V/\partial\phi = 0$, so every fluid element conserves its angular momentum.

Imagine a virtual exchange between two axisymmetric rings of fluid, identified by their masses ΔM_1 & ΔM_2, a rotating barytropic disk or star, *i.e.* $P = P(\rho)$ and therefore

$\partial\Omega/\partial z = 0$ in hydrostatic equilibrium[2], where Ω is the angular velocity, and we are using cylindrical coordinates (r, ϕ, z). Let the initial radii of the rings be r_1 & r_2 with $r_1 < r_2$, and suppose that the rings are initially corotating with their original surroundings, so that their specific angular momenta are $j_i = r_i^2\Omega(r_i)$, $i \in \{1, 2\}$. In keplerian rotation, $j \approx \sqrt{GM_* r}$, where M_* is the mass of the central object, so $dj/dr > 0$ even though $d\Omega/dr < 0$. Hence $j_1 < j_2$. Assume that when the two rings switch position ($r_1 \leftrightarrow r_2$), they conserve their specific angular momenta, and that they come to pressure equilibrium with their new surroundings. Since we have assumed a barytropic equation of state, they also come to the same densities as their surroundings. So the pressure and density fields are unperturbed in the eulerian sense. However, the outwardly displaced ring has smaller specific angular momentum than its new surroundings and therefore feels less centrifugal acceleration. Hence the radial forces on the ring are out of balance, and the ring ΔM_1 feels a net inward acceleration until it falls back to its original position. Similarly, the inwardly displaced element feels a net outward force and "wants" to return to its equilibrium position. We conclude that a barytropic star with a radially increasing angular momentum profile $d(r^2\Omega)/dr > 0$ is stable to axisymmetric perturbations in the absence of magnetic field.[3]

Adding magnetic field changes the situation because even in axisymmetry, the $\boldsymbol{J} \times \boldsymbol{B}$ force can have a nonzero ϕ component, since

$$(\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B} = \boldsymbol{B} \cdot \boldsymbol{\nabla} \boldsymbol{B} - \boldsymbol{\nabla}(\tfrac{1}{2}B^2) \tag{15}$$

$$\hat{\boldsymbol{e}}_\phi \cdot (\boldsymbol{B} \cdot \boldsymbol{\nabla} \boldsymbol{B}) = \left(B_r \frac{\partial}{\partial r} - \frac{B_r}{r} + B_z \frac{\partial}{\partial z} + \frac{B_\phi}{r} \frac{\partial}{\partial \phi} \right) B_\phi \,, \tag{16}$$

and the latter needn't vanish when $\partial_\phi \to 0$. Thus, it is possible for fluid elements to exchange angular momentum through the magnetic field.

Before plunging into an MHD linear stability analysis, it is helpful to consider the following thought experiment, due to Alar Toomre: Consider two orbiting spacecraft connected by an elastic tether. (This is a proxy for two fluid elements connected by magnetic field). If the tether is strong, then it holds the spacecraft together and they orbit at the angular velocity determined by the position of their center of mass. In the opposite limit that the tether has zero strength (*e.g.*, if it breaks), then they orbit independently; if they have different altitudes, then the lower one has the larger angular velocity. Consider the intermediate case of a weak, easily stretched tether. The spacecraft at lower altitude moves ahead of its companion; this stretches the tether, so that the lower craft loses angular momentum to the higher. Since the forces are weak, we may suppose that everything happens slowly and that the orbits remain approximately circular. In this case, each craft moves toward the orbital radius appropriate to its angular momentum. So as the lower craft loses angular momentum, it drifts toward even lower altitude since $dj/dr < 0$, and the upper craft, which gains angular momentum, drifts upward. Since, however, $d\Omega/dj < 0$ on circular orbits, this causes the difference in their angular velocities to increase, which further stretches the tether. This is a runaway situation, and it can be shown that the radial and azimuthal separations between the spacecraft increase exponentially at a rate $\approx |rd\Omega/dr|$ provided that the spring constant of the tether $< 3\mu\Omega^2$, where $\mu = M_1 M_2/(M_1 + M_2)$ is the reduced mass of the two spacecraft. The paradoxical conclusion is that a sufficiently weak *attractive* azimuthal force between orbiting bodies (or fluid elements) causes them to separate; it can be shown that a weak repulsive azimuthal force causes them to stay together.

[2] recall that $\partial\Omega/\partial z = 0$ follows from $P = P(\rho)$ by taking the curl of $\rho^{-1}\boldsymbol{\nabla} P + \boldsymbol{\nabla} V = -r\Omega^2\hat{\boldsymbol{e}}_r$

[3] We assume implicitly that $\Omega > 0$. More generally, the stability condition becomes $d(r^2\Omega)^2/dr > 0$.

Now for the MHD analysis. To minimize the algebra, consider an axisymmetric equilibrium state in which the magnetic field has only a vertical component, $\boldsymbol{B}^{(0)} = B_z^{(0)}\hat{\boldsymbol{e}}_z$, and let this component be constant. Then $\boldsymbol{J}^{(0)} = 0$ so the field exerts no force. The equilibrium velocity field $\boldsymbol{v}^{(0)} = r\Omega(r)\hat{\boldsymbol{e}}_\phi$, so $(\boldsymbol{v}\times\boldsymbol{B})^{(0)} = r\Omega B_z^{(0)}\hat{\boldsymbol{e}}_r$ is purely radial and varies only with radius, hence its curl vanishes, whence $\partial_t\boldsymbol{B} = 0$ by (11). We are assuming here that η is negligible.

It happens that the most rapidly growing instabilities of this equilibrium are those in which $v_z^{(1)} = 0$ and $v_r^{(1)}$ and $v_\phi^{(1)}$ are approximately independent of r as well as ϕ. More precisely, $\boldsymbol{\nabla}\cdot(\rho^{(0)}\boldsymbol{v}^{(1)}) \approx 0$ so that the eulerian density perturbation vanishes. Assuming a barytropic disk, the eulerian pressure perturbation also vanishes, so the only perturbed forces that we need to worry about are magnetic and centrifugal:

$$\frac{\partial}{\partial t}\boldsymbol{v}^{(1)} - 2\Omega v_\phi^{(1)}\hat{\boldsymbol{e}}_r + \frac{1}{r}\frac{dj}{dr}v_r^{(1)}\hat{\boldsymbol{e}}_\phi \approx \frac{1}{4\pi\rho}(\boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{B})^{(1)} = \frac{B_z^{(0)}}{4\pi\rho}\frac{\partial}{\partial z}\boldsymbol{B}^{(1)}. \tag{17}$$

The terms involving Ω and $j \equiv r^2\Omega$ come from evaluating $(\boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v})^{(1)}$ in cylindrical coordinates. With the identity

$$-\boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{B}) = \boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{B} - \boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{v} \ + \boldsymbol{B}\boldsymbol{\nabla}\cdot\boldsymbol{v},$$

the perturbed induction equation becomes

$$\frac{\partial}{\partial t}\boldsymbol{B}^{(1)} \approx r\frac{d\Omega}{dr}B_r^{(1)}\hat{\boldsymbol{e}}_\phi + B_z^{(0)}\frac{\partial}{\partial z}\boldsymbol{v}^{(1)} \ - \hat{\boldsymbol{e}}_z B_z^{(0)}\boldsymbol{\nabla}\cdot\boldsymbol{v}^{(1)}. \tag{18}$$

It's useful to develop some intuition for the magnetic terms. In (17), the sign of the right-hand side is such that a sinusoidal undulation in the initially straight and vertical field lines gives rise to horizontal forces that oppose the displacements of the line. In (18), which encodes the fact that the field lines are "frozen" into the flow, the second term on the right says that vertical shear in the perturbed velocities produces horizontal field out of the initially vertical lines, while the first term says that any resulting radial field gets sheared out into the azimuthal direction by the background differential rotation. The last term in (18) is negligible because $v_z^{(1)} \approx 0$ and $v_r^{(1)}$ depends mainly on z rather than r.

To go further, we assume a (t,z) dependence $\exp(-i\omega t + ikz)$ for the perturbations. This leads to the following linear system for the horizontal perturbations:

$$\begin{bmatrix} -i\omega & -2\Omega & -\frac{ikB_z^{(0)}}{4\pi\rho} & 0 \\ \frac{dj}{dr} & -i\omega & 0 & -\frac{ikB_z^{(0)}}{4\pi\rho} \\ -ikB_z^{(0)} & 0 & -i\omega & 0 \\ 0 & -ikB_z^{(0)} & -r\frac{d\Omega}{dr} & -i\omega \end{bmatrix} \begin{bmatrix} v_r \\ v_\phi \\ B_r \\ B_\phi \end{bmatrix}^{(1)} = 0.$$

The determinant must vanish to allow a nontrivial solution for the perturbations in the column vector. Evaluating the determinant (*e.g.* by expanding along the last column) yields, after some algebra,

$$\omega^4 - (\kappa^2 + 2k^2V_A^2)\omega^2 + k^2V_A^2\left(k^2V_A^2 + r\frac{d\Omega^2}{dr}\right) = 0\,, \tag{19}$$

where

$$\kappa^2 \equiv \frac{1}{r^3}\frac{dj^2}{dr}\,, \qquad V_A \equiv \frac{B^{(0)}}{\sqrt{4\pi\rho}}\,. \tag{20}$$

κ is called *epicyclic frequency* because if $j^2 = R^3 dV/dR$ is the orbital angular momentum of free particles on circular orbits in gravitational potential $V(R)$, then near-circular orbits oscillate radially at this frequency. V_A has dimensions of [length][time]$^{-1}$ and is called *Alfvén velocity*. In a nonrotating magnetized fluid, transverse waves propagate along $\boldsymbol{B}^{(0)}$ with group velocity V_A. Equation (19) is a quadratic equation in ω^2. The discriminant is positive if $\kappa^2 \geq 0$ and $d\Omega^2/dr \leq 0$, as is almost always the case in disks, so ω^2 is real. However, if $0 < (kV_A)^2 < rd\Omega^2/dr$, then the roots are of opposite signs, meaning that one of the four roots for ω is positive imaginary. Since we assumed the time dependence $\exp(-i\omega t)$, this corresponds to an exponentially growing mode. By minimizing ω^2 with respect to k^2, one can show that the fastest growing mode is

$$(-i\omega)_{\max} = \frac{4\Omega^2 - \kappa^2}{4|\Omega|}\,, \tag{21}$$

where we have made use of $rd(\Omega^2)/dr = \kappa^2 - 4\Omega^2$. This growth rate is achieved when

$$(kV_A)^2 = \frac{(4\Omega^2)^2 - \kappa^4}{(4\Omega)^2}\,. \tag{22}$$

Notice that the maximum growth rate is independent of the field strength, in principle: for an arbitrarily small but nonzero V_A, we can choose k large enough (*i.e.* a wavelength small enough) so that (22) is satisfied. Thus, very weak fields *always* give instability; the weaker the field, the smaller the wavelength on which instability occurs. Actually, had we included a finite diffusivity η, we would have found that sufficiently weak fields ($V_A^2 \lesssim \eta\Omega$) become stable, assuming $\kappa^2 > 0$. On the other hand, the largest possible wavelength is of order the the thickness of the disk, since the instability operates by having fluid at one z slide inward ($v_r < 0$) while fluid at another altitude on the same field lines slides outward, and this requires a node within the thickness of the disk. This implies a minimum value for k, and hence (22) cannot be satisfied if the field is sufficiently strong. Since hydrostatic equilibrium implies that the thickness of the disk is $z_0 \approx V_s/\Omega$, where V_s is the sound speed, it works out that the disk is stable if $V_A \gtrsim V_s$. For a more careful and general linear stability analysis, see the original paper by Balbus & Hawley[1].

Nonlinear 3D magnetohydrodynamic simulations show that when η is sufficiently small, just about any weak seed field, no matter what its geometry, leads to self-sustaining turbulence, and the effective viscosity is in the range $\nu \approx (10^{-3} - 10^{-1})\Omega z_0^2$: the larger values occur when there is a net vertical flux threading the disk. This is in satisfactory agreement with observations when the disk temperature and accretion rate can be determined.

References

[1] S. A. Balbus and J. F. Hawley. A powerful local shear instability in weakly magnetized disks. I. linear analysis. 376:214–233, July 1991.