19 HELIOSEISMOLOGY I: The Wave Mechanics of Solar Sound

Geophysicists have learned much of what they know about the deep interior of the Earth by studying waves that travel in rock. These waves are usually produced by earthquakes, and the branch of geophysics that deals with the waves and what can be learned from them is called *seismology*. *Helioseismology* is the study of the oscillations of the Sun, and *asteroseismology* the corresponding study of other stars. For the Sun at least, the data are now of such quality as to permit quantitative measurements of the radial sound-speed profile, the depth of the convection zone, and the internal differential rotation; the gradient of molecular weight in the core; and interesting constraints on the internal magnetic field. An extensive set of lecture notes on the theory of solar and stellar oscillations by one of the leading experts in the field is available on the Web at [1].

Solar oscillations are detected as motions of the solar photosphere. They were discovered in the 1960's [2, 3], and the growth of the field since then can be seen in recent conference proceedings [4, 5, 6]. Observations are now carried out by a world-wide groundbased network (GONG) and from space (SOHO) [7, 8]. The motions are measured by very small Doppler shifts in selected photospheric absorption lines. Most of the observed periods are between 3 and 6 minutes, so these motions are often called *five-minute oscillations*. Typical root-mean-square velocities averaged within a single resolution element are ≈ 0.4 km sec^{-1}. With measurements of sufficiently high precision and long time span, the motions can be decomposed into a superposition of periodic oscillations, which are interpreted as normal modes of the Sun. The velocity amplitude in a single mode is typically ~ 15 mm sec^{-1}, so the total r.m.s. velocity cited above represents the incoherent sum of $\sim (4\times 10^4 \text{mm sec}^{-1}/15 \text{ mm sec}^{-1})^2 \sim 10^7$ modes!

The modes observed in the Sun are of two basic types. The p-modes are essentially sound waves; pressure is the main restoring force that makes them possible. The f-modes are more akin to the waves seen on the surface of the ocean. At least when their wavelengths are substantially shorter than the solar radius, the f-modes involve very little compression of the fluid. Their main restoring force is the gravitational field, which resists wrinkling of the solar surface. The "f" stands for "fundamental" and will be explained below.

In addition to the p and f modes, stars can support a third type of wave called a g-mode. Like the short-wavelength f-modes, the g-modes are essentially incompressible disturbances relying upon gravity as their restoring force. ("g" stands for "gravity.") Analogous waves occur in the ocean below the surface and are called internal waves. g-modes are possible when displaced fluid elements feel buoyant forces that oppose the displacement. In the ocean, for example, water becomes saltier, colder, and therefore denser with increasing depth. An element of seawater displaced upwards is denser than its environment and therefore tends to sink; one displaced downwards tends to rise. In such a situation the fluid is said to be *stably stratified*. Radiative zones of stars are stably stratified by entropy and sometimes composition gradients; convective zones are always unstably stratified. Unlike the f-modes, the g-modes produce very little vertical motion of the surface and are largely excluded from convection zones. Thus in the Sun and main-sequence stars of later spectral type, g-modes are confined to the radiative core, which makes them difficult to observe. There are no generally-accepted detections of g-modes in the Sun. However, g modes are seen in a class of white dwarfs, the ZZ Ceti stars.

19.1 The wave equation for p and f modes

You have already been introduced to the fluid-dynamical form of $\vec{F} = m\vec{a}$,

$$\frac{d^2\vec{r}}{dt^2} = -\rho^{-1}\vec{\nabla}P - \vec{\nabla}V. \tag{1}$$

Here $\vec{r}(t)$ is the position of some fluid element. In this course, we normally assume that the righthand side of (1) vanishes, i.e., we assume hydrostatic equilibrium. Stellar oscillations involve small departures from hydrostatic equilibrium.

It is convenient to introduce the velocity field $\vec{v}(\vec{r},t)$. Its value is the velocity of the fluid element that happens to be passing through position $\vec{r}$ at time t:

$$\frac{d\vec{r}}{dt} \equiv \vec{v}(\vec{r},t). \tag{2}$$

The acceleration of the fluid element at $\vec{r}$ is

$$\frac{d^2\vec{r}}{dt^2} = \frac{d}{dt}\vec{v}(\vec{r},t) = \frac{\partial \vec{v}}{\partial t} + \frac{d\vec{r}}{dt}\cdot\vec{\nabla}\vec{v} = \frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\vec{\nabla}\vec{v}. \tag{3}$$

The first term on the rightmost side is the time derivative of $\vec{v}$ at a fixed position. But even if the velocity field is steady, fluid elements can be accelerated as they travel with the flow, and the second term accounts for that.

Equation (1) becomes

$$\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\vec{\nabla}\vec{v} = -\rho^{-1}\vec{\nabla}P - \vec{\nabla}V. \tag{4}$$

Equations (1) and (4) are called the "Lagrangian" and "Eulerian" forms of the equations of motion, after the different approaches to fluid mechanics pioneered by these two 18^{th}-century mathematicians. The two forms are equivalent, but the Lagrangian form deals with conditions experienced by a moving fluid element, whereas the Eulerian form describes changes in the fluid properties at fixed positions.

In the Eulerian approach, it is necessary to constrain ρ and $\vec{v}$ by an equation that guarantees conservation of mass:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0. \tag{5}$$

Using Gauss's Theorem, one can show that this equation is equivalent to the statement that the rate of change of the mass within an arbitrary volume V (not to be confused with the gravitational potential) is balanced by the flux of mass through its surface S:

$$\frac{d}{dt}\int_V \rho dV = -\oint_S \rho\vec{v}\cdot d\vec{S}. \tag{6}$$

Small oscillations can be described by linear partial differential equations. The first step in deriving these equations is to write every fluid variable as the sum of a time-independent zeroth-order part that satisfies hydrostatic equilibrium and a time-dependent first-order perturbation that is small:

$$\begin{aligned}
\vec{v} &= \vec{v}_0(\vec{r}) + \vec{v}_1(\vec{r},t), \\
V &= V_0(\vec{r}) + V_1(\vec{r},t), \\
\rho &= \rho_0(\vec{r}) + \rho_1(\vec{r},t), \\
P &= P_0(\vec{r}) + P_1(\vec{r},t). \qquad (7)
\end{aligned}$$

We assume $\vec{v}_0 = 0$ (thus ignoring rotation) and take the equilibrium star to be spherical. When we substitute these expansions into (4) and (5) and expand, we get three types of terms. First, there are terms that involve only the zeroth-order quantities. These terms must cancel one another because the the zeroth-order state is itself a solution of equations (4) and (5). Next, there are terms that contain one of the perturbations $\{\vec{v}_1,\ V_1,\ldots\}$, possibly differentiated, raised to the first power. These terms we keep. Finally, there are terms involving cross products and higher powers of the perturbations. These we discard because they are much smaller than the first-order terms if the perturbations are small. The resulting equations are linear in the perturbations:

$$\frac{\partial \vec{v}_1}{\partial t} = -\frac{\vec{\nabla} P_1}{\rho_0} + \frac{\rho_1}{\rho_0^2}\vec{\nabla} P_0 - \vec{\nabla} V_1, \tag{8}$$

$$\frac{\partial \rho_1}{\partial t} + \vec{\nabla}\cdot(\rho_0 \vec{v}_1) = 0, \tag{9}$$

$$\nabla^2 V_1 = 4\pi G \rho_1. \tag{10}$$

Formula (10) is the first-order form of Poisson's equation.

The discussion becomes a great deal easier with the following two approximations, both of which are very accurate for the observed modes of the Sun.

The first approximation is to neglect V_1, on the grounds that the modes of interest have wavelengths $\lambda \ll R_\odot$. Since ρ_1 averages to zero over the star (mass is conserved), the local perturbation in the gravitational potential depends on the fluctuation in the mass within one wavelength: $V_1 \sim G(\rho_1\lambda^3)/\lambda = G\rho_1\lambda^2$.[1] Hence

$$\frac{V_1}{V_0} \sim \frac{G\rho_1\lambda^2}{GM_\odot/R_\odot} \sim \left(\frac{\lambda}{R_\odot}\right)^2 \left(\frac{\rho_0}{\bar{\rho}}\right)\left(\frac{\rho_1}{\rho_0}\right), \tag{11}$$

in which $\bar{\rho} \equiv 3M_\odot/4\pi R_\odot^3$ is the mean density of the sun. One sees from (11) that $V_1/V_0 \ll \rho_1/\rho_0$ not only because $\lambda \ll R_\odot$ but also because the observed modes are concentrated in the outer part of the Sun where $\rho_0 \ll \bar{\rho}$. However, V_1 should not be neglected when one studies low-order radial pulsations of variable stars because these are very long-wavelength modes that involve the entire star.

The second approximation is that the entropy and chemical composition of the fluid are uniform in the equilibrium star. This is very nearly true in convection zones ($r \gtrsim 0.7R_\odot$ in the Sun) because the convection keeps the fluid well mixed, and the entropy gradient required to sustain the convection is normally extremely small. These things being uniform, the density becomes a function of pressure only: $\rho = \rho(P)$ instead of $\rho = \rho(P, S, \{X_i\})$. Since the perturbations are small and rapid, we can assume that the functional relationship between density and pressure is unchanged by the perturbations, i.e. each fluid element preserves its entropy and composition when it is displaced. To the extent that this is true, the perturbations are said to be *adiabatic*. Nonadiabatic effects are very important in exciting the large-amplitude pulsations of many variable stars, but they are of secondary importance for solar modes, which seem to be excited by convection.

The *enthalpy* of a chemically homogeneous gas is a thermodynamic function of state defined in general by $H \equiv U + (P/\rho)$, where U is the internal energy per unit mass, whence from the First Law,

$$dH = TdS + \frac{dP}{\rho}\,. \tag{12}$$

When the entropy is uniform, as it very nearly is throughout convection zones (as for $r \geq 0.71\mathrm{R}_\odot$ in the Sun), we may write (neglecting a constant of integration)

$$H \equiv \int_0^P \frac{dP'}{\rho(P')} \quad \text{and} \quad dH = v_s^2 \frac{d\rho}{\rho}\,, \tag{13}$$

where

$$v_s^2 \equiv \left(\frac{\partial P}{\partial \rho}\right)_S, \tag{14}$$

[1] The same result follows more formally in the WKBJ approximation where all perturbed quantities $\propto \exp(i\vec{k}\cdot\vec{r})$. It then follows from (10) that $-k^2 V_1 \approx 4\pi G\rho_1$, whence $V_1 \approx -G\rho_1\lambda^2/\pi$ since $k = 2\pi/\lambda$.

is the square of the sound speed. For an isentropic ideal gas, $P \propto \rho^{\Gamma_1}$ and the enthalpy is a constant multiple of the temperature:

$$H = \frac{\Gamma_1}{\Gamma_1 - 1}\frac{P}{\rho} = \frac{\Gamma_1}{\Gamma_1 - 1}\frac{k_B}{\mu m_H}T \; = \frac{v_s^2}{\Gamma_1 - 1}\,,$$

where μ is the molecular weight, and in this lecture the mass of the hydrogen atom is denoted m_H to avoid confusion with the enthalpy. It follows directly from (13) that

$$\rho^{-1}\vec{\nabla}P = \vec{\nabla}H, \tag{15}$$

and that

$$H_1 = v_{s,0}^2\frac{\rho_1}{\rho_0}. \tag{16}$$

Because of (15), the terms involving ρ_1 and P_1 in (8) collapse into $-\vec{\nabla}H_1$, and with V_1 neglected the equation becomes

$$\frac{\partial\vec{v}_1}{\partial t} = -\vec{\nabla}H_1. \tag{17}$$

Using (16) to eliminate ρ_1 from (9) in favor of H_1, we have

$$\frac{\partial H_1}{\partial t} = -\frac{v_s^2}{\rho}\vec{\nabla}\cdot(\rho\vec{v}_1), \quad \text{or equivalently,}$$

$$\frac{\partial H_1}{\partial t} + \vec{v}_1\cdot\vec{\nabla}H = -v_s^2\vec{\nabla}\cdot\vec{v}_1. \tag{18}$$

Here and henceforth, the subscript "0" has been omitted from equilibrium quantities such as ρ, H $(\equiv \rho_0, H_0)$ above. We differentiate (18) with respect to t and eliminate $\partial\vec{v}_1/\partial t$ using (17):

$$\frac{\partial^2 H_1}{\partial t^2} - \frac{v_s^2}{\rho}\vec{\nabla}\cdot(\rho\vec{\nabla}H_1) = 0. \tag{19}$$

This is very similar to the standard wave equation and would reduce to it if ρ and v_s^2 were independent of position. To bring the equation closer to the standard form, we introduce the "wave function"

$$\Psi \equiv H_1\sqrt{\rho}. \tag{20}$$

One can (and should) check that this satisfies

$$\frac{\partial^2\Psi}{\partial t^2} - v_s^2\nabla^2\Psi = -\left(\frac{v_s^2\nabla^2\sqrt{\rho}}{\sqrt{\rho}}\right)\Psi. \tag{21}$$

The righthand side of (21) is normally neglegible, being smaller than the second term on the left by $\sim (\lambda/R_\odot)^2$, except near the surface of the star where the density scale height becomes much smaller than $R_\odot$. If the density were to vanish at the surface, then the coefficient of Ψ on the righthand side would become infinite there.[2] Hence the only hope for a well-behaved solution is to demand $\Psi \propto \sqrt{\rho} \to 0$ as $r \to R$. While this is true, a better representation of the boundary condition can be obtained in terms of the enthalpy. Insofar as the "surface" of the star can be approximated as having zero pressure, temperature, and enthalpy, evaluation of the second form of (18) at the position of the unperturbed surface yields

$$\frac{\partial H_1}{\partial t} + \vec{v}_1\cdot\vec{\nabla}H_0 = 0 \quad \text{at } r = R. \tag{22}$$

[2] This isn't obvious because $v_s^2 \to 0$. In a hydrostatic equilibrium with $P \propto \rho^{\Gamma_1}$ and $\rho = 0$ at $r = R$, one can show that $v_s^2(\nabla^2\sqrt{\rho})/\sqrt{\rho} \propto (R - r)^{-1}$ as $r \to R$.

Note that although H_0 vanishes at the unperturbed surface, its radial gradient does not, so the second term on the lefthand side is nonzero. The lefthand side of (22) is the first-order form of the lagragian time derivative $dH/dt = 0$, *i.e.* the derivative following the fluid motion. The interpretation is that the enthalpy remains constant on the perturbed boundary, but the boundary is displaced from its unperturbed position by the wave. Applying $\partial/\partial t$ to (22) and using (17) to eliminate $\partial\vec{v}_1/\partial t$, we have

$$\frac{\partial^2 H_1}{\partial t^2} - \vec{\nabla} H_1 \cdot \vec{\nabla} H_0 = 0.$$

In a spherical star, hydrostatic equilibrium of the unperturbed state implies

$$\nabla H_0 = \frac{1}{\rho_0}\vec{\nabla} P_0 \to -\frac{GM_r}{r^2}\vec{e}_r.$$

Since (22) is already linearized, we may evaluate the final term above at $r = R$. Thus we arrive at an alternate form of the boundary condition in terms of H_1 alone,

$$\frac{\partial^2 H_1}{\partial t^2} + g\frac{\partial H_1}{\partial r} = 0 \quad \text{at r=R,} \tag{23}$$

where $g \equiv GM/R^2$ is the surface gravity.

In fact the zero-pressure free boundary condition (22) or (23) is only approximate, and its breakdown is responsible for the fact that we do not see modes with periods shorter than about 3 minutes. Modes of shorter period (higher frequency) "leak" through the photosphere into the corona.

19.2 Quantum numbers

Equation (21) is similar to the Schrödinger equation for a particle of mass M moving in a potential U:

$$\frac{\hbar}{i}\frac{\partial \Psi}{\partial t} - \frac{\hbar^2}{2M}\nabla^2\Psi + U\Psi = 0. \tag{24}$$

The main difference between the two is that (24) is first-order in time whereas (21) is second order. However, the same mathematical approach that is used to solve for the wavefunctions of the hydrogen atom can be applied here.

First, since eqs. (21) & (24) are linear and all functions appearing in them except Ψ itself are time-independent, there can be *stationary states:*

$$\Psi(\vec{r}, t) = e^{-i\omega t}\psi(\vec{r}). \tag{25}$$

Here ω is the angular frequency of solution; the period of oscillation will be $2\pi/\omega$. Of course, we are not really doing quantum mechanics, so we should remember that the physical Ψ is really the real part of (25). As long as we deal only with formulae that are linear in Ψ, however, we can postpone taking the real part until the end of our calculations. Thus, $\psi(\vec{r})$ is a complex function that encodes not only the spatial dependence of the mode but also its initial phase. Substituting (25) into (21), we find that ψ must satisfy

$$\nabla^2\psi + \frac{\omega^2 - U}{v_s^2}\psi = 0, \tag{26}$$

where

$$U(\vec{r}) \equiv \frac{v_s^2}{\sqrt{\rho}}\nabla^2\sqrt{\rho}. \tag{27}$$

Next, since U and v_s^2 depend only on the radial part of $\vec{r}$, we write out (26) in spherical polar coordinates (r, θ, ϕ):

$$
\begin{aligned}
0 = &\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{\omega^2 - U(r)}{v_s^2(r)}\psi \\
&+\frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi.
\end{aligned}
$$

The operator in square brackets should be familiar to you. It is the angular part of ∇^2, and its eigenfunctions are the spherical harmonics $Y_{\ell m}(\theta, \phi)$:

$$
\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] Y_{\ell m} = -\ell(\ell+1)Y_{\ell m}, \tag{28}
$$

where the *degree* ℓ is a nonnegative integer. The *order* m is also an integer, and $-\ell \leq m \leq +\ell$. We put

$$
\psi(r, \theta, \phi) = \hat{\psi}(r)Y_{\ell m}(\theta, \phi), \tag{29}
$$

so that $\hat{\psi}$ is analogous to the radial wavefunction of the hydrogen atom. $\hat{\psi}$ satisfies the radial wave equation,

$$
\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\hat{\psi}}{dr}\right) + \left[\frac{\omega^2 - U(r)}{v_s^2(r)} - \frac{\ell(\ell+1)}{r^2}\right]\hat{\psi} = 0. \tag{30}
$$

The solutions of the ordinary differential equation (30) have to satisfy two boundary conditions. At $r = R_\odot$, $\psi = 0$ [eqn. (23)], hence $\hat{\psi} = 0$ too. As $r \to 0$, the only potentially singular terms in (30) are the ones proportional to r^{-2}. By the Method of Frobenius, one can show that the two possible behaviors for $\hat{\psi}(r)$ as $r \to 0$ are $\sim r^{-(\ell+1)}$ and $\sim r^{\ell}$. Only the second is physically acceptable, since $\hat{\psi} \propto r^{-\ell-1}$ would imply an infinite enthalpy perturbation and therefore an infinite acceleration as $r \to 0$. The boundary condition at the surface is found by expressing (23) in terms of the radial part $\hat{h} = \hat{\psi}/\sqrt{\rho}$ of the enthalpy perturbation. Hence

$$
\begin{aligned}
\hat{\psi}(r) \equiv \sqrt{\rho}\,\hat{h}(r) \quad &\propto \quad r^{\ell} \quad \text{as } r \to 0; \\
g\frac{\partial\hat{h}}{\partial r} - \omega^2\hat{h} \quad &\to \quad 0 \quad \text{as } r \to R.
\end{aligned} \tag{31}
$$

With these boundary conditions, an equation such as (30) has nonzero solutions only for discrete values of ω^2. These solutions can be distinguished by their number of radial nodes, i.e. radii at which $\hat{\psi} = 0$. Since we don't count the nodes at $r = 0$ and $R_\odot$, $n \in \{0, 1, 2, 3, \ldots\}$. Thus, every elementary solution of (21) or (26) is uniquely identified by three integer-valued "quantum numbers" (ℓ, m, n). We label the corresponding frequencies and radial wavefunctions accordingly: $\omega_{\ell n}^2$, $\hat{\psi}_{\ell n}(r)$. Arbitrary nonsingular solutions of (21) can be expanded as sums of these elementary solutions. All this is analogous to our experience with the hydrogen atom.

Notice that we have omitted m from the subscripts above. Because m doesn't enter the radial equation (30) or the boundary conditions (31), it has no influence on ω^2 or $\hat{\psi}$. As you know, this is a consequence of spherical symmetry. If we rotate our coordinate axes, $Y_{\ell m}$'s of different m but the same ℓ are transformed into one another (unless we rotate around the z axis only), but since this has no affect on the background spherical star, it can't affect the frequencies or the number and placement of the radial nodes.

We close this section with some miscellaneous remarks.

At a given ℓ, the frequency increases with n. The $n = 0$ mode therefore has the lowest frequency and is called the *fundamental* by analogy with musical instruments. Thus there is one f-mode for each choice of ℓ and m, but many p modes.

In the hydrogen atom, $\hbar\ell$ and $\hbar m$ are interpreted physically as the total orbital angular momentum and its z component. In the helioseismological case, the interpretation of ℓ is less direct. Still, we see from the boundary condition (31) that modes with large ℓ must have very small amplitudes near $r = 0$, and we see that ℓ enters the radial equation (30) through a term that looks like a centrifugal potential. So, like particles with large angular momenta, stellar oscillations of large ℓ avoid the center of symmetry.

One qualitative difference between the hydrogen atom and helioseismology is that in the former case, the frequencies (or energies $E = \hbar\omega$) are independent of ℓ as well as m. This is not a consequence of spherical symmetry but an accident of keplerian/coulomb potentials, *i.e.* $U(r) \propto r^{-1}$.

Finally, in quantum mechanics we interpret $|\Psi(\vec{r}, t)|^2$ as the probability per unit volume of finding the particle near $\vec{r}$. How should we interpret this quantity in helioseismology (or acoustic problems generally)? The answer is that $|\Psi|^2/2v_s^2$ is the energy per unit volume in the mode.

References

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