DYNAMICAL STABILITY OF SPHERICAL STARS

We consider first a spherical star in a hydrostatic equilibrium. r_0, ρ_0, and P_0 are the radius, density, and pressure, assumed to be known function of the mass fraction, M_r, with $0 < M_r < M$, where M is the total mass of the star. Hydrostatic equilibrium means that

$$\frac{dP_0}{dM_r} = -\frac{GM_r}{4\pi r_0^4}, \tag{d.1a}$$

$$\frac{dr_0}{dM_r} = \frac{1}{4\pi r_0^2 \rho_0}. \tag{d.1b}$$

We shall make now a homologous, adiabatic perturbation, with radius of every mass shell changed by the same factor: $r = r_0(1+x)$, where x is a very small number that may vary with time, but not in space. Mass conservation expressed with eq. (d.1b) demands that $\rho = \rho_0(1-3x)$. We shall assume that the change is adiabatic, with a constant adiabatic exponent γ :

$$P = K\rho^\gamma, \qquad \frac{P-P_0}{P_0} = \gamma \frac{\rho - \rho_0}{\rho_0}. \tag{d.2}$$

It follows that $P = P_0(1-3\gamma x)$. Naturally, we assume that density and pressure perturbations are very small, i.e. $|(\rho-\rho_0)/\rho_0| \ll 1$, $|(P-P_0)/P_0| \ll 1$.

The perturbed quantities no longer satisfy the equation of hydrostatic equilibrium, but rather the equation of motion:

$$\frac{\partial^2 r}{\partial t^2} = -\frac{GM_r}{r^2} - 4\pi r^2 \frac{\partial P}{\partial M_r}. \tag{d.3}$$

As the only time variable quantity is x we may write the equation of motion in the form

$$\frac{\partial^2 r}{\partial t^2} = r_0 \frac{d^2 x}{dt^2} = -\frac{GM_r}{r_0^2}(1-2x) - 4\pi r_0^2 \frac{dP_0}{dM_r}(1+2x-3\gamma x). \tag{d.4}$$

Notice, that all quantities with subscript "0" satisfy the equation of hydrostatic equilibrium. Therefore, combining equations (d.1a) and (d.4) we obtain

$$\frac{d^2 x}{dt^2} = \frac{GM_r}{r_0^3}(4-3\gamma)x. \tag{d.5}$$

Now we make another crude approximation, $M_r/r_0^3 \approx \rho_{av} = \text{const}$, and we obtain

$$\frac{d^2 x}{dt^2} = G\rho_{av}(4-3\gamma)x = \sigma^2 x; \qquad \sigma^2 \equiv G\rho_{av}(4-3\gamma). \tag{d.6}$$

This equation has a solution

$$x = x_1 e^{\sigma t} + x_2 e^{-\sigma t}. \tag{d.7}$$

If $\sigma^2 < 0$ then the motion is oscillatory, i.e. the star is dynamically stable. If $\sigma^2 > 0$ than there is a solution that increases exponentially, i.e. the star is dynamically unstable. Therefore, the star is dynamically unstable if $\gamma < 4/3$. We obtained this result in a crude way. The proper analysis leads to the requirement that **average** value of γ within the star has to be less than 4/3 for the star to be dynamically unstable. The instability or the oscillations develop on a **dynamical time scale**, which is defined as

$$\tau_d \approx \sigma^{-1} \approx (G\rho_{av})^{-1/2}. \tag{d.8}$$

The critical value for the adiabatic exponent, $\gamma_{cr} = 4/3$, is a consequence of the gravitational force varying as r^{-2}. If the gravitational acceleration was given as $-GM_r/r^{2+\alpha}$, then the critical value would be $\gamma_{cr} = (4+\alpha)/3$. One of the effects of general relativity is to make gravity stronger than Newtonian near a black hole, but the effect is there even when the field is weak. For $r \gg r_{Sch} \equiv 2GM/c^2$ we have roughly $\alpha \approx r_{Sch}/r$. This means that dynamical instability may set in when γ is just very close to, but somewhat larger than 4/3.