ACCRETION DISKS

A review article on accretion disks was published by J. E. Pringle (1981, *Annual Review of Astronomy and Astrophysics,* **19** 137). The classical papers on the structure of thin accretion disks are: J. E. Pringle and M. J. Rees (1972, *Astronomy and Astrophysics,* **21** , 1), N. I. Shakura and R. A. Sunyayev (1973, *Astronomy and Astrophysics,* **24** , 337), and D. Lynden-Bell and J. E. Pringle (1974, *Monthly Notices of the Royal Astronomical Society,* **168** , 603).

Consider cylindrical coordinates, (r, z), with r being a distance from the rotation axis, and z being a distance from the equatorial plane. Let there be a massive object at the center of the coordinate system, and let its gravitational field be spherically symmetric. We shall require the gravitational potential Φ to be a function of the distance from the center of the coordinate system, $R = (r^2 + z^2)^{1/2}$:

$$\Phi(R) < 0, \qquad \frac{d\Phi}{dR} > 0, \qquad \Phi(R)_{R \longrightarrow \infty} \longrightarrow 0. \tag{d1.1}$$

Later on we shall consider in some detail two cases, a Newtonian potential $\Phi(R) = -GM/R$, and pseudo-Newtonian potential $\Phi(R) = -GM/(R - R_g)$, where gravitational radius $R_g \equiv 2GM/c^2$. From now on we shall consider motion in the $z = 0$ plane, so we shall have $r = R$.

A test particle on a circular orbit in a plane $z = 0$ has a rotational velocity v that must satisfy a relation

$$\frac{v^2}{r} = \frac{d\Phi}{dr}, \qquad\qquad z = 0, \tag{d1.2}$$

where $d\Phi/dr$ is gravitational acceleration. Therefore, rotational velocity is

$$v = \left(r \frac{d\Phi}{dr} \right)^{1/2}. \tag{d1.3}$$

The angular velocity is given as

$$\Omega = \frac{v}{r} = \left(\frac{1}{r} \frac{d\Phi}{dr} \right)^{1/2}, \tag{d1.4}$$

rotational period is

$$P_{rot} = \frac{2\pi}{\Omega}, \tag{d1.5}$$

angular momentum per unit mass is

$$j = vr = \left(r^3 \frac{d\Phi}{dr} \right)^{1/2}, \tag{d1.6}$$

and total mechanical energy per unit mass is

$$e = \Phi + \frac{v^2}{2}. \tag{d1.7}$$

Consider now a very thin gaseous disk, with a half − thickness $z_0 \ll r$. Within a thin disk rotational velocity is a function of r only, and it is practically constant on cylinders with constant radius r. The surface mass density is defined as

$$\Sigma = \int_{-z_0}^{z_0} \rho dz. \tag{d1.8}$$

If there is some dynamical viscosity η, and there is a shear flow, i.e. $d\Omega/dr \neq 0$, then there is a torque (a couple) acting between two adjacent cylinders

$$g = r \times 2\pi r \times \int_{-z_0}^{z_0} \left(-\frac{d\Omega}{dr} r \right) \eta dz = -2\pi r^3 \frac{d\Omega}{dr} \int_{-z_0}^{z_0} \eta dz, \tag{d1.9}$$

and there is also thermal energy released as a result of viscous interaction between the cylinders

$$\epsilon = \left(r \frac{d\Omega}{dr} \right)^2 \eta \qquad \left[\mathrm{erg\ s^{-1}\ cm^{-3}} \right]. \tag{d1.10}$$

It is very important that the same viscosity that is responsible for the torque acting between two cylinders is also responsible for heat generation. The torque and heat generation are there because there is viscosity and because there is a shear flow, i.e. the adjacent cylinders rotate with respect to each other,

We shell consider now a flow of mass, momentum and energy between cylinders located at a radii r and $r + dr$. The rate of mass flow is called the rate of accretion, and it may be expressed as

$$\dot{M} = 2\pi r \int_{-z_0}^{z_0} \rho v_r dr = 2\pi r v_r \Sigma, \tag{d1.11}$$

where $|v_r/v| \ll 1$ is a very small radial velocity. The rate of angular momentum flow $\dot{J}$ is given as

$$\dot{J} = \dot{M} j + g. \tag{d1.12}$$

The term $\dot{M} j$ gives angular momentum carried with mass flow, while g gives angular momentum transmitted by viscous forces. The rate at which energy flows across a cylinder with a radius r is given as

$$\dot{E} = \dot{M} e + g\Omega, \tag{d1.13}$$

where the first term gives the energy flowing with matter, while the second term gives the energy transmitted by viscous forces. In addition, viscosity dissipates some energy into heat. As the disk is very thin, we shall assume that this energy is radiated locally from the disk surface at a rate $F[\mathrm{erg\ cm^{-2}\ s^{-1}}]$. As the disk has two surfaces, the amount of energy (luminosity) radiated away between radii r and $r + dr$ is given as

$$\frac{dL_d}{dr} = 2\pi r \times 2F = 4\pi r F. \tag{d1.14}$$

The amount of mass, angular momentum, and energy contained between radii r and $r + dr$ is $2\pi r \Sigma$, $2\pi r \Sigma j$, and $2\pi r \Sigma e$, respectively. The equations of mass, angular momentum and energy balance may be written as

$$\frac{\partial}{\partial t} (2\pi r \Sigma) + \frac{\partial \dot{M}}{\partial r} = 0, \tag{d1.15}$$

$$\frac{\partial}{\partial t} (2\pi r \Sigma j) + \frac{\partial \dot{J}}{\partial r} = 0, \tag{d1.16}$$

$$\frac{\partial}{\partial t} (2\pi r \Sigma e) + \frac{\partial \dot{E}}{\partial r} + 4\pi r F = 0. \tag{d1.17}$$

All three equations have very similar form, except there is a term with energy carried with radiation in the last equation, and there is no equivalent term in the first two equations. This is so, because in our approximation radiation carries energy, but no momentum and no mass. The three conservation laws may be expressed in relativistic form, and then there are terms with momentum and mass carried away with radiation. However, those terms are very small, unless the amount of energy carried away is comparable to $\dot{M}c^2$.

Within a thin disk approximation some quantities are functions of radius only: $j(r)$, $\Omega(r)$, $e(r)$, while others are function of radius and time: $\Sigma(r,t)$, $\dot{M}(r,t)$, $\dot{J}(r,t)$, $\dot{E}(r,t)$, $F(r,t)$, $g(r,t)$. Taking this into account the equations (d1.15), (d1.16), (d1.17) may be transformed into

$$2\pi r \frac{\partial \Sigma}{\partial t} + \frac{\partial \dot{M}}{\partial r} = 0, \tag{d1.18}$$

$$\dot{M} \frac{dj}{dr} + \frac{\partial g}{\partial r} = 0, \tag{d1.19}$$

$$g \frac{d\Omega}{dr} + 4\pi r F = 0. \tag{d1.20}$$

Notice, that only some of the derivatives are partial. The last equation gives

$$F = \frac{g}{4\pi r}\left(-\frac{d\Omega}{dr}\right) = \frac{1}{2}\left(-r\frac{d\Omega}{dr}\right)^2 \int_{z_0}^{z_0} \eta dz, \tag{d1.21}$$

where equation (d1.9) has been used to replace the torque g with an integral of viscosity over the disk thickness. The equation (d1.21) may be written as

$$2 \times F = \int_{-z_0}^{z_0} \epsilon dr, \tag{d1.22}$$

where ϵ is defined with equation (d1.10).

These very general equations may be used to demonstrate that if the disk extends between two radii, r_1 and r_2, and there is vacuum for $r < r_1$ and $r > r_2$, then viscosity within the disk will have a tendency to spread the disk over a larger range of radii. We have to assume that angular velocity Ω decreases monotonically with radius, i.e. $d\Omega/dr < 0$, and that specific angular momentum increases monotonically with radius, i.e. $dj/dr > 0$. It can be shown that these are very general requirements, satisfied by all dynamically stable, thin disks. As the surface brightness cannot be negative, the equation (d1.20) requires that torque g cannot be negative. At the inner and outer edges of the disk, i.e. at r_1 and r_2, the density of matter falls off to zero, and hence the torque must fall off to zero as well. Therefore, at some intermediate radius $r_1 < r_m < r_2$ the torque has a maximum, and we have $dg/dr > 0$ for $r_1 < r < r_m$, and $dg/dr < 0$ for $r_m < r < r_2$. Now, the equation (d1.19) implies that the rate of mass flow $\dot{M}$ must vanish at $r = r_m$, and that $\dot{M} < 0$ for $r_1 < r < r_m$, and $\dot{M} > 0$ for $r_m < r < r_2$, i.e. mass flows away from $r = r_m$. This means that the disk will spread out in radius. This phenomenon may be looked at in another way. Viscosity may redistribute angular momentum over the matter within the disk, but it cannot change the total value of angular momentum within an isolated disk which has free boundaries at r_1 and r_2. The same viscosity generates some heat at the expense of total energy of the disk, and this heat is radiated away. Therefore, while the total mass, and the total angular momentum of an isolated disk are conserved, the total energy is decreasing with time. This may be accomplished by spreading the matter over a larger range of radii.

A very important special case is that of a steady – state, time independent accretion, with most quantities remaining functions of radius only, while one, the accretion rate $\dot{M}$, remains constant in time and radius. With the torque g being a function of radius only, the equation (d1.19) may be integrated to obtain

$$g = g_0 + \left(-\dot{M}\right)(j - j_0), \tag{d1.23}$$

where g_0 is a torque at the inner disk radius r_0, and j_0 is the specific angular momentum at r_0. The matter accretes when $v_r < 0$, and according to equation (d1.11) $\dot{M} < 0$. If there is no torque at the inner disk radius, a very common situation, then $g_0 = 0$, and

$$g = \left(-\dot{M}\right)(j - j_0), \qquad F = \left(-\dot{M}\right)\frac{j - j_0}{4\pi r}\left(-\frac{d\Omega}{dr}\right), \tag{d1.24}$$

(cf. equation d1.21) We obtained a very important result: in a steady state accretion the surface brightness of accretion disk does not depend on its viscosity, but it follows from the conservation laws of mass, angular momentum, and energy. Of course, the surface brightness is proportional to the accretion rate. Notice, that the surface brightness approaches zero at very large radii, and also at the inner disk radius r_0, because $j = j_0$ there. The maximum surface brightness is reached at some intermediate radius.

We shall calculate now the total luminosity radiated by a steady – state accretion disk, which extends from r_0 to infinity, and has a no torque condition at r_0. Of course, we have to allow for the luminosity coming out from both sides of the disk. Using the equation (d1.24), changing the variable of integration, and integrating by parts we obtain:

$$L_d = 2\int_{r_0}^{\infty} 2\pi r F dr = \left(-\dot{M}\right)\int_{r_0}^{\infty}\left(-\frac{d\Omega}{dr}\right)(j - j_0)\, dr = \tag{d1.25}$$

$$= \left(-\dot{M}\right)\int_{0}^{\Omega_0}(j - j_0)\, d\Omega = \left(-\dot{M}\right)\int_{0}^{\Omega_0} r^2\Omega d\Omega - \left(-\dot{M}\right) j_0\Omega_0 =$$

$$= \left(-\dot{M}\right)\left[\frac{1}{2}r^2\Omega^2\right]_0^{\Omega_0} + \left(-\dot{M}\right)\int_{r_0}^{\infty}\Omega^2 r dr - \left(-\dot{M}\right) v_0^2 =$$

$$= \left(-\dot{M}\right)\frac{1}{2}\left[v_0^2 - v_\infty^2\right] + \left(-\dot{M}\right)\int_{r_0}^{\infty}\frac{d\Phi}{dr}dr - \left(-\dot{M}\right) v_0^2 =$$

$$= \left(-\dot{M}\right)\left[-\frac{v_0^2}{2} - \Phi_0\right] = \left(-\dot{M}\right)(-e_0),$$

where we used the relations: $j = \Omega r^2$, $v = \Omega r$, $v_\infty = 0$, and $\Phi_\infty = 0$, and where v_0 is the rotational velocity at r_0.

The interpretation of equation (d1.25) is very simple: the total amount of energy released within accretion disk, and radiated away, is equal to the mass accretion rate $(-\dot{M})$, multiplied by the total energy per unit mass at the inner disk orbit, $(-e_0)$. e_0 is the specific binding energy at r_0. The origin of accretion energy is gravitational. However, the amount of radiation emitted between radii r and $r + dr$ is not equal to the difference in binding energies between these two radii, as a large fraction of energy is redistributed throughout the disk by viscous torques.

We shall consider now two special cases: disk accretion onto a non – relativistic, non – rotating star, and accretion onto a black hole. In the first case there is a boundary layer between the stellar surface and the inner disk radius r_0. Across this boundary layer angular velocity increases from $\Omega = 0$ within the non – rotating star, up to $\Omega = \Omega_0$ at the inner radius of the disk. It is believed that the radial extent of the boundary layer is very small. For most stars we may use **Newtonian gravitational potential** , $\Phi = -GM/r$, where M is the stellar mass. In this case the rotational velocity, angular velocity, specific angular momentum, and specific energy are given as

$$v = \left(\frac{GM}{r}\right)^{1/2}, \qquad \Omega = \left(\frac{GM}{r^3}\right)^{1/2}, \qquad j = (GMr)^{1/2}, \qquad e = -\frac{GM}{2r}. \tag{d1.26}$$

These are called Keplerian values. The surface brightness of the disk is given as

$$F = \left(-\dot{M}\right)\frac{3}{8\pi}\frac{GM}{r^3}\left[1 - \left(\frac{r_0}{r}\right)^{1/2}\right]. \tag{d1.27}$$

If the disk is optically thick in the z direction then it radiates as a black body with the effective temperature given by the standard relation: $F = \sigma T_{eff}^4$. Of course, the spectrum of a whole disk is not a black body, because the effective temperature varies with radius.

The total amount of energy radiated by the disk is given as

$$L_d = \left(-\dot{M}\right)(-e_0) = \left(-\dot{M}\right)\frac{GM}{2r_0}. \qquad \text{(d1.28)}$$

While accreting across the boundary layer the rotational velocity of matter must be reduced from $v_0 = (GM/r_0)^{1/2}$ down to zero, while the radial distance hardly changes at all. This kinetic energy, $v_0^2/2 = GM/2r_0$, must be radiated away. Therefore, the luminosity of the boundary layer:

$$L_{bl} = \left(-\dot{M}\right)\frac{GM}{2r_0}, \qquad \text{(d1.29)}$$

is equal to the luminosity of the entire accretion disk! However, as the area of the boundary layer is so much smaller than the area of the disk, the boundary layer must be much hotter than the disk.

A somewhat different situation arises when the accreting star has a very strong magnetic field which can disrupt the accretion disk at the so called *magnetospheric radius* r_m. In this case, there may be some torque present at r_m. Also, the fraction of total accretion energy released within the disk is smaller, while a larger fraction of accretion energy is released between the magnetospheric radius and the stellar radius.

A very different situation arises when accretion disk surrounds a black hole. Even though we do not have a full proof that black holes were detected, there is a number of very good candidates among binary stars emitting X–rays: Cygnus X–1 (cf. J. N. Bahcall, 1978, *Annual Review of Astronomy and Astrophysics,* **16** , 241 for a discussion and references), LMC X–3 (A. P. Cowley, D. Crampton, J. B. Hutchings, R. Remillard, and J. Penfold, 1983, *Astrophysical Journal,* **272** , 118, B. Paczynski, 1983, *Astrophysical Journal (Letters),* **273** , L81), and A0620–00, also known as Nova Monocerotis 1917, 1975 (J. E. McClintock and R. A. Remillard, 1986, *Astrophysical Journal,* **308** , 110). A recent review of stellar mass black holes in binary systems is by A. P. Cowley, (1992, *Annual Review of Astronomy and Astrophysics,* **30** , 287).

A black hole does not have a surface that could be touched. Rather, it has a property that anything, including radiation, that gets below the so called *horizon* , cannot escape. The black holes may have mass, angular momentum, and electric charge. In practice, black holes expected in to be in binary stars are electrically neutral, and their gravitational field is characterized by two parameters only: their mass M, and their angular momentum J. If $J = 0$ then gravitational field is spherically symmetric, and the geometry of space near such a black hole is described by Schwarzschild metric. Its most profound characteristic is the existence of the Schwarzschild radius, also called gravitational radius, $r_g \equiv 2GM/c^2$, which has a property that nothing can escape from a smaller distance. The value of this radius may be estimated with Newtonian gravity, setting the escape velocity equal to the speed of light. It is a coincidence, that dimensionless numerical factor "2" turns out to be the same in Newtonian gravity and in general relativity.

The full general relativistic treatment of disk accretion onto a black hole is fairly complicated (K. S. Thorne, 1974, *Astrophysical Journal,* **191** , 507; C. T. Cunningham, 1975, *Astrophysical Journal,* **202** , 788; 1976, *Astrophysical Journal,* **208** , 534). A reasonably good model is provided by a pseudo − Newtonian potential (P. Wiita and B. Paczynski, 1980, *Astronomy and Astrophysics,* **88** , 23). A very important difference between gravitational field of a Newtonian object and a field of a black hole is the following: in the Newtonian case gravitational acceleration due to a point mass M becomes infinite at $r = 0$, while gravitational acceleration due to a black hole becomes infinite at $r = r_g = 2GM/c^2$. The easiest, though entirely artificial way to model this, is by replacing a Newtonian gravitational potential $\Phi = -GM/r$, with a **pseudo − Newtonian gravitational potential** $\Phi = -GM/(r - r_g)$. At very large radii, $r \gg r_g$, the two potential are almost the same, and they differ strongly only for $r \approx r_g$. We may write down all the expressions like those given with equations (d1.3), (d1.4), (d1.6), (d1.7), and (d1.26), in the following way:

$$\Phi = -\frac{GM}{r - r_g}, \qquad \frac{d\Phi}{dr} = \frac{GM}{(r - r_g)^2} = \frac{GM}{r^2}\left[\left(\frac{r}{r-r}\right)^2\right], \qquad r_g \equiv \frac{2GM}{c^2}, \qquad \text{(d1.30)}$$

$$v = \left(\frac{GM}{r}\right)^{1/2} \left[\frac{r}{r - r_g}\right], \tag{d1.31}$$

$$\Omega = \left(\frac{GM}{r^3}\right)^{1/2} \left[\frac{r}{r - r_g}\right], \qquad \frac{d\Omega}{dr} = -\frac{3}{2}\left(\frac{GM}{r^5}\right)^{1/2} \left[\frac{\left(r - \frac{1}{3}r_g\right) r}{\left(r - r_g\right)^2}\right], \tag{d1.32}$$

$$j = (GMr)^{1/2} \left[\frac{r}{r - r_g}\right], \tag{d1.33}$$

$$e = \left(-\frac{GM}{2r}\right) \left[\frac{(r - 2r_g)\, r}{\left(r - r_g\right)^2}\right], \tag{d1.34}$$

where all formulae are written down like their Newtonian equivalents multiplied by the correcting factors placed in the square brackets. The surface brightness distribution of a steady – state accretion disk may be written as

$$F = \left(-\dot{M}\right) \frac{3}{8\pi} \frac{GM}{r^3} \left[1 - \left(\frac{r_0}{r}\right)^{1/2} \left[\frac{3\,(r - r_g)}{2r}\right]\right] \left[\left(\frac{r}{r - r_g}\right)^3 \left(1 - \frac{r_g}{3r}\right)\right], \tag{d1.35}$$

$$r_0 = 3r_g,$$

which was also written in a form as similar as possible to its Newtonian equivalent. The choice of the inner radius, $r_0 = 3r_g$, will be explained shortly.

There is a very striking difference between the Newtonian and pseudo – Newtonian expressions for the total specific energy, as shown with equations (d1.7) and (d1.34) : the Newtonian expression varies monotonically with radius, and it is always negative; pseudo – Newtonian expression is negative for $r > 2r_g$, and positive for $r < 2r_g$. This is exactly what is found in full general relativistic treatment of the dynamics of a test particle moving around a black hole. Even the numerical factor "2" is the same!

Let us analyze the variation of specific angular momentum with radius. Differentiating equation (d1.33) we obtain

$$\frac{dj}{dr} = \frac{1}{2}\left(\frac{GM}{r}\right)^{1/2} \left[\frac{(r - 3r_g)\, r}{\left(r - r_g\right)^2}\right]. \tag{d1.36}$$

This equation shows that $dj/dr > 0$ for $r > 3r_g$, and $dj/dr < 0$ for $r < 3r_g$, i.e. the specific angular momentum has a minimum at $r = 3r_g$. This is exactly the effect found in a full general relativistic treatment, and even the dimensionless factor "3" is the same! The existence of the minimum angular momentum a test particle may have on a circular orbit has a very profound effect on the dynamics of accretion disks. The shear has always the same sign in Newtonian and in pseudo – Newtonian case, as $d\Omega/dr < 0$ at all radii (cf. d1.32). This means that viscosity always transports angular momentum outwards, because the inner parts of the disk rotate more rapidly than the outer parts. Therefore, any specific element of matter, while accreting, gradually loses its angular momentum. However, when it gets to the orbit with a radius $r = 3r_g$, and looses still more angular momentum, then there is no other circular orbit available. Therefore, the accretion disk has a natural inner radius: $r_0 = 3r_g$. From that point matter falls freely into the black hole.

The total luminosity radiated away by the accretion disk may be calculated integrating the surface brightness as given with equation (d1.35) over all radii from r_0 to infinity. The result is

$$L_d = \left(-\dot{M}\right) e_0 = \left(-\dot{M}\right) \frac{c^2}{16}, \tag{d1.37}$$

where the equation (d1.34) was used to evaluate e_0. We found that while accreting onto a Schwarzschild black hole matter radiates away 1/16 of its rest mass energy, a result fairly close to the correct general relativistic value. This is much more than can be released in any nuclear reaction, and it does not matter what the chemical composition of the accreting matter is, or what the black hole mass is. For this reason accretion onto black holes is suggested whenever there seems to be energy crisis in astrophysics.

Let us consider now stability of motion of a test particle on a circular orbit in arbitrary, spherically symmetric potential $\Phi(r)$. Various quantities for a circular orbit are given with equations (d1.3) – (d1.7). In general, the particle trajectory may be non – circular, and there may be two components to its velocity: $v_r = dr/dt$ and $v_\phi = rd\phi/dt$, where ϕ is the azimuthal angle in the cylindrical coordinate system. There are two constants of motion in this problem: angular momentum and total energy:

$$j_0 = rv_\phi, \qquad e_0 = \frac{1}{2}\left(v_\phi^2 + v_r^2\right) + \Phi. \tag{d1.38}$$

For a particle on exactly circular orbit we have $v_r = 0$. Let us consider now a small perturbation of the particle motion, with the angular momentum and the total energy conserved. The two equations (d1.38) may be combined to obtain

$$v_r^2 = 2\left(\epsilon_0 - \Phi\right) - \frac{j_0^2}{r^2}. \tag{d1.39}$$

Let us find the dependence of radial velocity on the variation of radius Δr, with ϵ_0 and j_0 kept constant. We may expand the relation in a power series

$$v_r^2 = \left(v_r^2\right)_0 + \left(\frac{dv_r^2}{dr}\right)_0 \Delta r + \frac{1}{2}\left(\frac{d^2 v_r^2}{dr^2}\right)_0 \left(\Delta r\right)^2 + ... \tag{d1.40}$$

The subscript "0" refers to the values calculated from equation (d1.39) at $\Delta r = 0$, i.e. at the position of a circular orbit corresponding to ϵ_0 and j_0.

The first term in the power series vanishes, because $v_r = 0$ at the circular orbit. The second term can be calculated from equation (d1.39) as

$$\frac{dv_r^2}{dr} = -2\frac{d\Phi}{dr} + 2\frac{j_0^2}{r^3} = -2\frac{j^2 - j_0^2}{r^3}, \tag{d1.41}$$

where the equation (d1.6) was used to replace the derivative of Φ with specific angular momentum j. Of course, at the circular orbit $j = j_0$, and the second term in the power series (d1.40) vanishes. The third term can be calculated differentiating equation (d1.41) :

$$\frac{d^2 v_r^2}{dr^2} = 6\frac{j^2 - j_0^2}{r^4} - \frac{4j}{r^3}\frac{dj}{dr}. \tag{d1.42}$$

At the circular orbit the first term on the right hand side of equation (d1.42) vanishes, but the second does not. We may now write the equation (d1.40) as

$$v_r^2 = -\frac{2j_0}{r^3}\frac{dj}{dr}\left(\Delta r\right)^2. \tag{d1.43}$$

This equation has solution only if $dj/dr < 0$. Otherwise, there are no solution. This means that for $dj/dr > 0$ there are no particle trajectories for a given ϵ_0 and j_0 except the original circular orbit. Therefore, the orbit is stable if angular momentum increases with radius. However, if angular momentum decreases with radius, then we may take a square root of both sides of equation (d1.44) to obtain

$$v_r \equiv \frac{dr}{dt} = \frac{d\Delta r}{dt} = \pm\left(-\frac{2j_0}{r^3}\frac{dj}{dr}\right)^{1/2} \Delta r, \qquad \frac{dj}{dr} < 0. \tag{d1.44}$$

The last equation may be integrated to obtain

$$\Delta r = \left(\Delta r\right)_0 e^{\pm\left(-\frac{2j_0}{r^3}\frac{dj}{dr}\right)^{1/2}(t-t_0)}, \qquad \frac{dj}{dr} < 0. \tag{d2.45}$$

We find that if $dj/dr < 0$ then there are trajectories that depart exponentially from a circular orbits, keeping the same values of total energy ϵ_0, and angular momentum j_0 on the trajectory, as they were on the circular orbit. Therefore, circular orbits are unstable when angular momentum decreases with radius, and all orbits around black holes with radii smaller than $3r_g$ are unstable. The orbit with a radius $r_{ms} = 3r_g$ is marginally stable, and larger orbits are stable. This is the reason that disks cannot extend inwards of r_{ms}.

Vertical structure of thin disks

We shall consider now very thin disks with small, but final extent in the vertical, i.e. z , direction. Any disk with a finite surface mass density must have a finite pressure, which makes its thickness final as well. Let us consider disk with negligible mass, and negligible self − gravity. The vertical pressure gradient has to be balanced by the vertical gradient of the gravitational potential of the central massive object. The equation of hydrostatic equilibrium in the z direction may be written as

$$\frac{1}{\rho}\left(\frac{\partial P}{\partial z}\right)_r = -\left(\frac{\partial \Phi}{\partial z}\right)_r = -\frac{d\Phi}{dR}\left(\frac{\partial R}{\partial z}\right)_r = -\frac{d\Phi}{dR}\frac{z}{R}, \tag{d1.46}$$

where we noticed that $R = (r^2 + z^2)^{1/2}$. As $R \approx r$ for $|z/r| \ll 1$, we may combine equations (d1.4) and (d1.46) to obtain

$$\frac{1}{\rho}\left(\frac{\partial P}{\partial z}\right)_r = -\Omega^2 z, \tag{d1.47}$$

where angular velocity Ω practically does not vary with z. Therefore, gravitational, acceleration in the vertical direction is proportional to z.

Let us consider now a simple, polytropic relation for the disk, with

$$P = K\rho^{1+\frac{1}{n}}, \qquad K = \text{const}, \qquad n = \text{const}. \tag{d1.48}$$

Inserting (d1.48) into (d1.47) we may integrate the equation of hydrostatic equilibrium in the z direction to obtain

$$K\,(n+1)\,\rho^{\frac{1}{n}} = \frac{1}{2}\Omega^2\left(z_0^2 - z^2\right), \tag{d1.49}$$

where z_0 is the distance from the equatorial plane to the disk surface. A polytropic speed of sound is given as

$$v_s^2 = \frac{dP}{d\rho} = K\frac{n+1}{n}\rho^{\frac{1}{n}}. \tag{d1.50}$$

Combining equations (d1.49) and (d1.50) we have

$$v_s^2 = \frac{1}{2n}\Omega^2\left(z_0^2 - z^2\right). \tag{d1.51}$$

The speed of sound vanishes at the surface, while at the equator we have

$$v_{s,e} = (2n)^{-1/2}\,\Omega z_0 = (2n)^{-1/2}\,v\frac{z_0}{r} = (2n)^{-1/2}\,\Omega z_0. \tag{d1.52}$$

We found that the ratio $v_{s,e}/v$ is about equal to the ratio z_0/r.

The observations indicate that there are bright accretion disks in many binary systems. Therefore, there is a high viscosity in those disks. For a long time, the nature of this viscosity was unknown, and the viscosity was represented by the simple parametrization described below in eqs. (d1.54-55). It is now believed that the "viscosity" is actually the result of magnetohydrodynamic turbulence excited by the so-called "magnetorotational" instability [MRI]. In its simplest form, MRI is axisymmetric in a differentially-rotating, electrically conducting disk with a background magnetic field parallel to the the rotation axis (z axis). Imagine two fluid elements at different heights (z and $z + \Delta z$) threaded by the same bundle of field lines. In perturbation, one of these elements moves slightly inward ($\Delta r < 0$) and the other outward. Since the field is "frozen" in the fluid, the field lines develop an S-shaped bend. Magnetic tension tries to resist this and to straighten the lines. However, if the field is weak, it does not succeed. The ingoing element, to the extent that it conserves its angular momentum, wants to increase its angular velocity; the reverse is true for the outgoing element. This stretches the lines in the azimuthal as well as the radial direction. Consequently, the weak magnetic tension removes angular momentum from the ingoing element and transfers it to the outgoing one. The loss of centrifugal support causes the ingoing element to fall farther inward and increase its angular velocity further, and the reverse for the outgoing element. This is a runaway instability until nonlinear effects (reconnection of the field lines, or Kelvin-Helmholtz instabilites between ingoing and outgoing streams, or buoyant escape of the field vertically) causes it to saturate in turbulence.

For the moment, however, let us return to the traditional "viscous" picture, which proceeds by analogy with the kinetic theory of gases. Imagine, that disk is made of particles which have the volume averaged mass density ρ, random velocities v_p, superimposed on Keplerian rotation of the disk, and the mean distance they travel between collisions is λ_p. The viscosity in such a system may be written as

$$\eta = \frac{1}{3}\rho v_p \lambda_p. \tag{d1.53}$$

The highest velocity a particle (or a blob of gas) may have is the speed of sound, and the largest mean distance it can travel in radial direction with such velocity is approximately equal to the disk thickness $2z_0$. Therefore, the maximum viscosity a disk matter may have is

$$\eta_{max} = \frac{1}{3}\rho v_s 2z_0 \approx \rho z_0^2 \Omega \approx \frac{P}{\Omega} \approx \Sigma v_s, \tag{d1.54}$$

and the actual viscosity may be parametrized as

$$\eta = \alpha \eta_{max}, \qquad 0 < \alpha < 1. \tag{d1.55}$$

This is the basis of the so called **alpha disk model** . In practice, 3D MHD simulations find that nonlinearly saturated MRI produces a roughly constant $\alpha \approx 10^{-3} - 10^{-1}$ (the range reflects in part the geometry of the background field): see Balbus & Hawley 1998, Rev. Mod. Phys. **70**, 1.

We may use the alpha disk model to estimate radial velocity of matter within accretion disk. Throughout this estimate we shall be replacing derivatives by the ratio of corresponding quantities, like $-d\Omega/dr \approx \Omega/r$. We may combine equations (d1.9), (d1.54) and (d1.55) to obtain

$$g \approx 2\pi r^2 \Omega \int_{-z_0}^{z_0} \eta dz \approx 2\pi j \alpha \Sigma v_s z_0. \tag{d1.56}$$

Combining equations (d1.11) and (d1.19) we have

$$\dot{M} = 2\pi r v_r \Sigma \approx \frac{g}{j}. \tag{d1.57}$$

Finally, combining equations (d1.56) and (d1.57) we find

$$v_r \approx \alpha v_s \frac{z_0}{r} \ll v_s \ll v, \tag{d1.58}$$

i.e. the radial velocity is very much smaller than the speed of sound in a thin disk, even for $\alpha = 1$.