Problem Set 7 Solutions, AST 205, Fall 2003

1. (6 points total).
We are given the relationship between the mean speed of atoms of a given mass m at a given temperature, so this is plug-and-chug. Room temperature is about 300 K. Let's first do this for Hydrogen:

$$v = \left(\frac{3kT}{m}\right)^{1/2} = \left(\frac{3 \times 1.4 \times 10^{-16} \times 300}{1.7 \times 10^{-24}}\right)^{1/2}$$

Let's try this without a calculator. $3 \times 300 = 900$; that's easy. I'll write that as 0.9×10^3. $0.9 \times 1.4 = 1.25$, roughly. $1.25/1.7 \approx 0.7$. Gathering up all our factors of 10, we find:

$$v = \left(0.7 \times 10^{-16+3+24}\right)^{1/2} = \left(7 \times 10^{10}\right)^{1/2} = 2.6 \times 10^5$$

Doing it with a calculator gives me 2.7×10^5, essentially the same. Actually, the calculator formally gives me 2.70293951×10^5, but writing down the number to this precision is of course meaningless.
I was a bit lazy on keeping the units straight here, but I was consistently using cgs, so the units are cm/s. Hydrogen is zipping around at almost 3 kilometers per second! That's almost 6000 miles per hour! Of course, the gas in the room isn't travelling in bulk at those speeds; each atom and molecule is bumping into all the others, all the time, and constantly changing direction. It is the force of all these molecules bumping into a surface that is responsible for atmospheric pressure.
The other two atoms to be considered, oxygen and neon, can be calculated directly from these numbers. In particular, we can write for any atom of mass m:

$$v = \left(\frac{3kT}{m}\right)^{1/2} = v_H \times \left(\frac{m_H}{m}\right)^{1/2},$$

scaling directly from the result from hydrogen. Of course, the ratio of the mass of a given atom to hydrogen is just its atomic weight (12 for the case of carbon[1], and 20 in the case of neon. So the speeds of carbon and neon atoms are down from that of hydrogen by factors of 3.5 and 4.5 respectively, or 75,000 cm/s and 58,000 cm/s, respectively (and here I admit I used a calculator).
Let's now calculate the escape speed from the surface of the Earth. Again we plug and chug:

$$v = \left(\frac{2GM}{R}\right)^{1/2} = \left(\frac{2 \times 2/3 \times 10^{-7} \times 6 \times 10^{27}}{6.4 \times 10^8}\right)^{1/2}$$

Again, let's do this without a calculator. $6/6.4 \approx 1$, and gathering 10's, this gives us:

$$v = \left(1.3 \times 10^{12}\right)^{1/2} \approx 1.1 \times 10^6 \text{ cm/sec.}$$

The escape speed from the surface of the Earth is about 11 km/s, roughly 4 times the average speed of hydrogen atoms in the Earth's atmosphere, and quite a bit more than any

[1] Embarrassingly, the statement of the problem is incorrect; oxygen has **8** protons and **8** neutrons. Even professors make elementary mistakes sometimes!

heavier species. So it seems that nothing can escape the Earth's atmosphere. However, the atomic speed that we've calculated is an average, while in fact different atoms are moving at different speeds (the so-called Maxwell distribution). There is still a non-negligible fraction of the atoms moving at 4 times the mean speed, although the number moving 10 times faster is essentially negligible. So over a fairly short span of time, the high-speed piece of the hydrogen gas will be lost to Earth (at least those atoms in the upper atmosphere). Collisions will restore the Maxwell distribution, to give a new population of high-speed hydrogen atoms, which get lost themselves, and so on. In not very much time at all, all the hydrogen gas is lost. This is why the Earth's atmosphere has essentially no hydrogen atoms or hydrogen molecules in it, even though hydrogen is the most abundant element in the cosmos.

2. (3 points)
This is a straightforward calculation, but first let me take a moment to derive a simpler version of the relationship for the equilibrium temperature of a planet. What you are familiar with is the equation:

$$T_{planet} = \left(\frac{(1-a)L_{star}}{16\pi D^2 \sigma}\right)^{1/4}$$

See the notes to Lecture 5. However, if the star is a black body of temperature T_{star} and radius R_{star}, we can write:

$$L_{star} = 4\,\pi R_{star}^2 \sigma T_{star}^4$$

If we substitute this into the equation above, *all* sorts of stuff cancels, and we get the much simpler:

$$T_{planet} = T_{star}(1-a)^{1/4}\left(\frac{R_{star}}{2\,D}\right)^{1/2}$$

(You should check my algebra yourself). That's the form I'll use for the solution set. Here the star in question is the Sun, $T_{star} = 6000$ K, $R_{star} = 7 \times 10^{10}$ cm, and $D = 1.5 \times 10^{13}$ cm, or one astronomical unit.
OK, let's calculate:

$$T_{planet} = (1-a)^{1/4} \times 6000 \times \left(\frac{7 \times 10^{10}}{3 \times 10^{13}}\right)^{1/2}$$

We can simplify the thing in the parenthesis to $\sqrt{1/400} = 1/20$, or

$$T_{planet} = (1-a)^{1/4} \times 300\,\mathrm{K}.$$

Now let's put in our two albedos. For the ocean-covered Earth, $1-a = 0.8$, the fourth root of which is about 0.95[2]. So for the ocean planet, the temperature is about 285 K, close to the freezing point of water (give it a bit of greenhouse effect, and we'll safely get above that threshold).
For the snow-covered Earth, $1-a = 0.2$. Here I'll use a calculator; the temperature of the Earth is now roughly 2/3 of 300 K, or 200K, *well* below the freezing point of water. So a snowball Earth will stay cold and frozen, unless somehow there is a massive greenhouse effect.

[2] A cool mathematical trick to remember: if $x \ll 1$, then $(1+x)^n \approx 1 + nx$. In this case, $x = -0.2$, and $n = 1/4$.

3. (6 points)

a. This is just a matter of scaling; we can write:

$$R_\star = R_\odot \times \left(\frac{M_\star}{M_\odot}\right)^{0.6} = R_\odot \times 1.4^{0.6}.$$

I can use the trick described on the footnote to write $1.4^{0.6} \approx 1 + 0.4 * 0.6 \approx 5/4$. Multiplying this by the solar radius of 7×10^{10} cm gives 9×10^{10} cm.

b. If the quantity $J = 2/5MR^2 \times 2\pi/P$ is conserved, and if M remains unchanged as the star collapses, then the quantity R^2/P must remain the same. This means that:

$$P_{ns} = P_\star \times \left(\frac{R_{ns}}{R_\star}\right)^2$$

c. So let's plug this in. One month (the original period of the star) is roughly 3 million seconds, so:

$$P_{ns} = 3 \times 10^6 \, \text{sec} \times \left(\frac{15}{900,000}\right)^2$$

where I've put both radii in km. $15/900,000 \approx 1.6 \times 10^{-5}$; the square of this number is 2.5×10^{-10}. So the period of the neutron star is 7.5×10^{-4} seconds, or three quarters of a millisecond.

d. In fact, the fastest spinning neutron star known is about 1.6 milliseconds. This is somewhat faster than that, but it is in the right ballpark.

However, 1.4 solar mass stars are *not* the progenitors of neutron stars; 1.4 solar mass stars end their lives as white dwarfs. Neutron stars are formed in the supernova explosions of much more massive stars, $> 8M_\odot$; most of the material of the star is thrown outward in the explosion, leaving behind the neutron star of typically 1.4 solar masses.