Problem Set 5 Solutions, AST 205, Fall 2003

1. The angular size of the Einstien ring surrounding an object (called the lens) of mass M that is a distance D_{OL} from the observer is given by

$$\theta_E = \sqrt{\frac{4GM}{D_{OL}c^2}}.$$

The above equation uses the "contact lens" approximation described in the problem. To make this "plug and chug" type problem easier, we introduce scaling factors so that

$$M = m\, M_{\mathrm{Jup}} \text{ and } D_{OL} = d\, 10 \text{ ly}.$$

The non-dimensional factors m and d give an answer which we can re-scale to avoid repeating all the algebra:

$$\theta_E = 7.9 \times 10^{-9}\sqrt{\frac{m}{d}} \text{ radians}\frac{180 \times 3600 \text{ arcsec}}{\pi \text{ radians}} = 1.6 \times 10^{-3}\sqrt{\frac{m}{d}} \text{ arcsec}.$$

To get answers for Earth, Jupiter, and Solar mass objects we use $m = 1/318$, $m = 1$, and $m = 10^3$, respectively. Similarly we use $d = 1$ and 10^3 for the distances to the observer. It's now easy to present the answers in tabular form.

Table 1: Einstien ring radius in arc seconds for lenses with the given masses and distances from the observer.

	M_{Earth}	M_{Jup}	$M_{\odot}$
10 ly	8.9×10^{-5}	1.6×10^{-3}	5.1×10^{-2}
10^4 ly	2.8×10^{-6}	5.1×10^{-5}	1.6×10^{-3}

2. The Einstien ring must be larger than the (angular) size of the lens for lensing effects to be observed, i.e. $\theta_E > \theta_R = R/D_{OL}$, where R is the radius of the lens. Solving this inequality for D_{OL} gives

$$D_{OL} > \frac{c^2R^2}{4GM}.$$

We can use the same scaling techniques as in problem 1, if we define $R = r\ R_{\mathrm{Jup}}$. Taking the size ratios we find $r = 1/11$ for Earth and $r = 9.9$ for the Sun. This gives the distance criterion as

$$D_{OL} > 6000\frac{r^2}{m} \text{ AU}$$

. The minimum distance for strong lensing around

- an Earth-like object is 1.6×10^4 AU
- a Jupiter-like object is 6000 AU
- a solar-type star is 590 AU

3. a. We use Kepler's law that $P^2 \propto D^3$ for the period, P, and semi-major axis D of planets around the same star. Thus we have:

$$D_V = D_E \left(\frac{P_V}{P_E}\right)^{2/3} = 0.723 \text{ AU}.$$

There are several ways to do parts b & c since the problem is using approximations. We begin with the method which follows the way the problem was written.

b. Earth's speed is $v_E = 2\pi D_E/P_E$ since it travels the circumference of the circle once per period. To express Venus's speed in terms of the here "unknown" $D_E = 1$ AU we use our knowledge of P_V and part (a):

$$v_V = 2\pi \frac{D_V}{P_V} = 2\pi \frac{.723 D_E}{.615 P_E} = 2\pi \times 1.18 \frac{D_E}{P_E}.$$

Now we can calculate the velocity difference between the two planets as:

$$\delta v = v_V - v_E = 2\pi \times 0.18 \frac{D_E}{P_E} = 1.13 \text{ AU/yr} = 3.6 \times 10^{-8} \text{ AU/sec}.$$

c. Since the eclipse travels a distance $d = 2R_E = 12800$ km in $t = 1600$ sec, we use $\delta v = d/t$ to calculate

$$\text{AU} = 2.2 \times 10^8 \text{ km},$$

which is not too far from the actual value of 1.5×10^8 km.

Alternate method: A better way to do the problem is to think in terms of the angular velocity of the planets, not the linear velocity, because they travel on curved tracks. The difference in angular speed is just:

$$\Omega_V - \Omega_E = 2\pi \left(\frac{1}{P_V} - \frac{1}{P_E}\right) = 3.9 \ \frac{\text{rad}}{\text{yr}} = 1.2 \times 10^{-7} \ \frac{\text{rad}}{\text{sec}}.$$

The angular distance that the eclipse travels relative to the Sun is just $2R_E/\text{AU}$ in the small angle approximation. We use the angular equivalent of (angular) velocity times time equals distance (here the angle), namely:

$$(\Omega_V - \Omega_E)t = \frac{2R_E}{\text{AU}}.$$

Solving for the earth-sun distance gives AU $= 6.4 \times 10^7$ km. The point here is not which of the two answers is correct, since the numbers in the problem are artificial, but to note that we get a significantly different answer when we take into account that the curvature of planetary orbits!