Astro 205. Problem set 1. Solutions

1. The Moon's distance from the Earth is 3.84×10^{10} cm and its revolution period around the Earth is 27.3 days. (i) assuming that the Moon's mass is very much smaller than the Earth's, calculate the Earth's mass from this information (you also need the constant of gravity G: 6.7×10^{-8} gm^{-1} cm^3 s^{-2}). (ii) The Moon's actual mass is 7.4×10^{25} gm. How far from the center of the Earth is the center of mass of the Earth-Moon system? The Earth's radius is 6.4×10^8 cm. Is the center of mass of the Earth-Moon system inside or outside the Earth?

Solution

For the first part, let's assume that the Moon's orbit around the Earth is circular and that M(Earth) >> M(Moon), i.e. the center of mass of the Earth-Moon system can be taken to be the center of mass of the Earth. Then

$$\frac{\mathrm{mV}^2}{\mathrm{r}} = \frac{\mathrm{GmM}}{\mathrm{r}^2}$$

where V = orbital velocity of Moon, m = mass of Moon, M = mass of Earth, G = constant of gravity and r = distance between Earth and Moon. Now

$$\mathrm{V} = \frac{2\pi \mathrm{r}}{\mathrm{P}}$$

where P = period of Moon's orbit. So

$$\mathrm{M} = \frac{\mathrm{rV}^2}{\mathrm{G}} = \frac{4\pi^2 \mathrm{r}^3}{\mathrm{GP}^2}$$
$$= 6 \times 10^{27} \text{ gm}$$

The second part calculates the location of the center of mass:

$$\mathrm{r} = \frac{\mathrm{Mr}_1 + \mathrm{mr}_2}{\mathrm{M} + \mathrm{m}}$$

Let $\mathrm{r}_1 = 0$ = position of Earth, so r_2 = Earth-Moon distance. Then

$$\mathrm{r} = \frac{\mathrm{mr}_2}{\mathrm{M} + \mathrm{m}}$$

$$= \frac{7.4 \times 10^{25}}{7.4 \times 10^{25} + 6 \times 10^{27}} \times 3.84 \times 10^{10} \text{ cm}$$
$$= 4.7 \times 10^8 \text{ cm}$$

This is the distance of the center of mass of the Earth-Moon system from the center of the Earth. It is less than the Earth's radius, 6.4×10^8 cm. Thus the center of mass of the Earth-Moon system is "inside" the Earth.

2. Here is a list of the periods (P), eccentricities (e), and semi-major axes (a) of several well-known comets. Comets orbit the Sun. (i) Verify that their orbits obey Kepler's third law (note: Kepler's 3rd law holds for the period and semi-major axis). (ii) Calculate the perihelion and aphelion distances for each comet.

Table 1. Comets

Comet	P(yr)	a(AU)	e
Halley	76.1	17.8	0.97
Encke	3.3	2.2	0.85
Wolf	8.42	4.15	0.40
1943.1	512	64	0.999914

Solution

The perihelion distance is

$$\mathrm{r_p} = \mathrm{a}(1 - \mathrm{e})$$

where a is the semi-major axis and e the orbital eccentricity. The major axis is 2a, so that the aphelion distance is

$$\begin{aligned} \mathrm{r_a} &= 2\mathrm{a} - \mathrm{r_p} \\ &= 2a - a(1-e) \\ &= a(1+e) \end{aligned}$$

So the values of $\mathrm{P^2/a^3}$ and of the perihelion and aphelion distances are:

(where the units of $\mathrm{P^2/a^3}$ are $\mathrm{years^2/AU^3}$).

Note some interesting things: Halley and Encke (and 1943.1) have perihelion distances less than 1 AU. This means that their orbits can cross that of Earth, and there is the possibility of collision.

Halley's aphelion distance is about at the distance of Pluto, so it may originate in the *Kuiper Belt.* Wolf's orbit takes it from just beyond Jupiter to inside Jupiter's orbit.

Table 2

Comet	P^2/a^3	Perihelion(AU)	Aphelion(AU)
Halley	1.03	0.534	35.1
Encke	1.02	0.33	4.07
Wolf	0.99	2.49	5.81
1943.1	1.00	0.0055	128

The perihelion distance of 1943.1 is 0.0055 AU, or 8.2×10^{10} cm. This is not much larger than the Sun's radius! 1943.1 is an example of a "sun-grazing" comet.

3. The Sun's present rotation period is 27 days. If all the angular momentum in the Solar System were returned to the Sun, what would its rotation period be? Do this problem approximately: just take Jupiter and ignore the other planets. Use the masses, distances etc. from Table 1, lecture 1.

Solution

Angular momentum. I messed this up in class by mixing units: the angular frequency ω should have been in radians/second rather than in seconds^{-1}.

Orbital angular momentum of Jupiter $=$ $m_J V_J D_J$ Now

$$V = \frac{2\pi r}{P} = \omega r$$

i.e.

$$\omega = \frac{2\pi}{P}$$

where P is the orbital period of Jupiter. So Jupiter's orbital angular momentum is

$$m_J D_J \omega_J D_J = m_J D_J^2 \omega_J$$

where D_J is the distance between Jupiter and the Sun, $\omega_J = 2\pi/P_J$ and M_J is Jupiter's mass. Let's approximate it as 10^{-3} $M_\odot$.

The Sun's rotational angular momentum is

$$\frac{2}{5} M_\odot R_\odot^2 \omega_\odot = 1.06 \times 10^{49}$$

where

$$\omega_\odot = \frac{2\pi}{P_\odot}$$

So the total angular momentum (neglecting all other angular momentum in the system, such as Jupiter's rotational angular momentum) is 2.14×10^{50}.

If all of this angular momentum is transfereed to rotational angular momentum of the Sun, the ratio of the "new" to "old" periods is just the ratio of the current solar angular momentum to the total angular momentum:

$$\frac{\mathrm{P(new)}}{27\mathrm{days}} = \frac{1.06 \times 10^{49}}{2.14 \times 10^{50}}$$

giving P(new) = 1.3 days.