Astro 205. Lecture 4 September 24, 2003
Stellar Structure: how do stars "work"?

Stars shine. Why? Because they're hot. Why? Isn't this odd? Instead of expanding and cooling like the rest of the Universe, stars are extremely hot (up to tens of thousands of degrees) while the Universe is cold. Isn't this a bit contrary to thermodynamics?

We'll now begin looking at how stars work, one of the really great triumphs of 20th century physics. And later we'll learn more, the origin of the elements heavier than hydrogen/ helium/ lithium/ beryllium/ boron.

A star's *luminosity* $L_\star$ is:

$$L_\star = 4\pi R_\star^2 \sigma T_\star^4$$

This is the equation for the radiation from a spherical black body of radius $R_\star$ and temperature $T_\star$. σ is the Stefan-Boltzmann constant, 5.7×10^{-5} erg cm^{-2}K^{-4} s^{-1}. Stars are pretty good, though not perfect, black bodies.

Luminosities are measured in *solar luminosities* in astronomy, i.e. in multiples of the luminosity of the Sun, about 4×10^{33} ergs/second. This is a prodigious amount of energy. Even more amazing is that the Sun's energy output has been almost constant over billions of years, as has been the energy output of most stars (we'll return to the "almost constant" a bit later). Most of the stars we see in the sky, including the Sun, are so-called *main sequence* stars; their temperatures and luminosities lie along a sequence between high temperature/high luminosity and low temperature/ low luminosity on the luminosity-temperature or Herzsprung-Russell diagram.

The luminosity of main sequence stars is roughly constant over very long times. How do we know this? There are several lines of evidence. First, we observe that most stars have constant brightness over the astronomical record, which for some stars goes back thousands of years (not much compared with the Universe's age of 13.7 billion years to be sure). The stars that don't have constant luminosity, the *variable stars*, are very interesting in their own right but beyond the scope of the present discussion. Second, we know from the fossil record that the Sun's luminosity has been almost constant over the lifetime of the Solar System; we will discuss this a little next week. Third, we know the distances to many clusters of stars, The stars within a given cluster are the same age, but the cluster ages are different one from the other, yet stars of the same color are found to have the same luminosity from cluster to cluster. Since stars are radiating energy at prodigious rates, they must also be *producing* it at these rates, and this rate must be constant. There is therefore some mechanism controlling the energy output of a main sequence star, and both its temperature and radius are essentially constant with time. The existence of a regulatory mechanism is unusual - think of forest fires as a counter example.

Stellar Structure

A major piece of information to explain the distribution of stars along the main sequence comes from measuring the masses of some stars which are in binary systems, a pair of stars orbiting each other under the force of gravity. These measurements show that the high-luminosity, high-temperature stars have high masses, and the low-temperature, low-luminosity stars have low masses. The main sequence is thus a sequence of *mass*.

Now, let's examine the structure of a star. First, note that large, self-gravitating bodies, e.g. the Sun, are spherical. A rigorous proof of this important result is tricky, and beyond the scope of this course; but just note that, if you are at a distance r from mass M, the force per unit mass is

$$\mathrm{F} = \frac{\mathrm{GM}}{\mathrm{r}^2}$$

and surfaces of constant force are therefore surfaces of constant r, i.e. spheres.

Since main sequence stars neither grow nor shrink, yet are fluid, they must be in what's called *hydrostatic equilibrium* or pressure equilibrium - that is, the gravitational force pulling them together is balanced by pressure "holding them up" - actually, by the pressure gradient from the center to the outside, since the pressure at the photosphere of a star is zero. Consider a cylindrical fluid element at distance r from the center of a star, where the density is ρ(r). The fluid element has area A and thickness dr; it is sketched on P 316 of CO. The mass of this fluid element is

$$\mathrm{dM} = \mathrm{A}\rho(\mathrm{r})\mathrm{dr}$$

and the gravitational force pulling it towards the center of the star is

$$\mathrm{F} = -\frac{\mathrm{GM(r)dM}}{\mathrm{r}^2}$$

where M(r) is the stellar mass within radius r. This is balanced (in equilibrium) by an upwards buoyancy force F = AdP, where dP is the pressure gradient over distance dr. Then in equilibrium

$$\mathrm{AdP} = -\frac{\mathrm{GM(r)}}{\mathrm{r}^2}\rho(\mathrm{r})\mathrm{Adr}$$

or

$$\frac{dP}{dr} = -\rho(r)\frac{GM(r)}{r^2} \tag{1}$$

This is the *equation of hydrostatic equilibrium.*

To get a rough idea of the pressure $\mathrm{P_c}$ in the center of the Sun, let $\mathrm{dP/dr} = \mathrm{P_c/R_\odot}$, $\mathrm{M(r)} = \mathrm{M_\odot}$ and $\rho(\mathrm{r}) = \bar{\rho}$. This gives $\mathrm{P_c} = 3\times10^{15}$ dynes cm^{-2}. (Note: the Earth's atmospheric pressure at sea level is only 10^6 dynes cm^{-2}.) A more rigorous solution of the equations of stellar structure gives as you might expect an increasing density towards the center of the Sun - the central density is about 150 gm cm^{-3} - and a corresponding pressure of 2.5×10^{17} dynes cm^{-2}. The gravitational pressure in the center of a star is balanced by a pressure due to the high temperature. Thus a *main sequence star* (like the Sun) is in two kinds of balance:

The rate at which energy is radiated from the stellar photosphere = the rate at which energy is generated in the center

The gravitational force pulling the star's mass towards its center is balanced by the outward pressure gradient caused by the high temperatures in the interior

Thus the luminosity of a star (the rate at which it generates energy) is closely regulated by the balance between pressure and gravity. Too little pressure and the star contracts (and heats up and produces more energy); too much and the star expands and the central energy source produces less energy, hence less pressure, so the star contracts. This exquisite balance keeps a star shining at essentially constant luminosity for billions of years. And this answers why, this far along in the evolution of the Universe, we still have a situation of hot stars and cold space. Unlike the situation in your kitchen, where pretty well everything is the same temperature, some parts of the Universe *gravitate* and thereby produce energy.

But what *is* the source of energy? Let's look at some possibilities. Take the Sun - its luminosity is 4×10^{33} erg s^{-1} and its main sequence lifetime has been 4.5 billion years (the Earth is at least 3.5 billion years old in any case). The Sun must therefore have produced at least 6×10^{50} ergs over its life. (Note something else very important - the Sun loses energy because it is hot and space is cold, pretty close to absolute zero in fact. If space were very hot a star's balance would be a very different thing).

OK. How about chemistry, i.e. burning stuff? Typical atomic energy levels (and molecular binding energies) are about 1 eV (1.6 $\times 10^{-12}$ ergs). How many H atoms in the Sun? About

$$\mathrm{N} = \frac{\mathrm{M}_\odot}{\mathrm{m_H}}$$

$$= \frac{2 \times 10^{33}}{1.6 \times 10^{-24}} = 1.2 \times 10^{57}\mathrm{H\ atoms}$$

If you burn all this hydrogen (never mind how) you get about 1 eV per H atom, or 1.6×10^{45} ergs, far less than you need.

OK. How about gravity? The *gravitational binding energy* of a massive spherical body is

$$\mathrm{E} = -\frac{3}{5}\frac{\mathrm{GM}^2}{\mathrm{R}}$$

$$= \frac{1.6 \times 10^{59}}{\mathrm{R}} \quad \mathrm{ergs}$$

The solar radius is 7×10^{10} cm, giving 2.3 $\times 10^{48}$ ergs, also not enough. What's going on here is conservation of energy: as a body shrinks under its own internal gravity, the *potential energy* is converted to *kinetic energy* of the atoms and molecules, i.e. to heat. While gravitational contraction can't supply enough energy to "fuel" the Sun or stars, it is the origin of much of the high inner temperatures of the planets, on which more later.

Nuclear energy? When you *fuse* four hydrogen nuclei to make one helium nucleus, .7% of the mass is converted into energy by E $= \mathrm{mc}^2$. Thus we get $0.007\mathrm{m_H c^2}$ per hydrogen nucleus per fusion reaction. (Note: you can fuse to heavier and heavier nuclei as we shall see, but the great bulk of the energy is produced in the first fusion, H to He. *It matters that the Sun is mostly made of hydrogen).*

As we saw above, the energy from the Sun must be nuclear energy, which "converts" mass to energy via the famous equation

$$\mathrm{E} = \mathrm{mc}^2$$

where c is the speed of light. How much mass does the Sun convert to energy over its lifetime?

$$\mathrm{m} = \frac{5.7 \times 10^{50}}{(3 \times 10^{10})^2} = 6.3 \times 10^{29}\ \mathrm{gm} = 1.6 \times 10^{-4}\ \mathrm{M}_\odot$$

What's the rate per second?

$$\frac{4 \times 10^{33}\ \mathrm{erg\ s^{-1}}}{9 \times 10^{20}} = 4 \times 10^{12}\ \mathrm{gm\ s^{-1}}$$

(or the equivalent of 10^8 human beings per second).

How do we get the conversion of mass to energy? This happens in *nuclear fusion reactions* in the core of the Sun. Four hydrogen nuclei fuse to one helium nucleus and the final nucleus has less mass that the sum of the masses of the four H nuclei, by 0.7% per hydrogen nucleus. So we get $0.007\mathrm{m_H c^2}$ of energy per hydrogen atom, where $\mathrm{m_H}$ is the proton mass, the mass of the hydrogen nucleus (or proton). For all of the hydrogen in the Sun, this is $10^{57} \times 0.007 \times 1.67 \times 10^{-24} \times (3 \times 10^{10})^2$ or 10^{52} ergs, which is more than sufficient to power the Sun over its lifetime.

OK. How does this work? For a start, as we have already calculated, the pressure at the center of the Sun is very high, 2.5×10^{17} dyne $\mathrm{cm^{-2}}$. Let's make an order of magnitude estimate of the pressure that holds together normal solid bodies. This can be understood as electrostatic repulsion (really, van der Waals' forces) between neighboring electron clouds in an atom:

$$\mathrm{F} = \frac{\mathrm{e^2}}{\mathrm{r^2}}$$

What is r? Let's take it as the mean "size" of an atom - the Bohr radius is 0.5 Å, so let's take r = 1 Å or 10^{-8} cm. The electron charge e is 4.8×10^{-10} esu. The pressure is the force per unit area; take the area to be $\pi \mathrm{r^2}$. This gives a pressure of 10^{13} dynes $\mathrm{cm^{-2}}$. Look at this as the maximum pressure normal matter can exert to keep its structure. **It is far less than the pressure in the center of the Sun.** The matter in the center of the Sun is thus a very dense soup of electrons and nucleons and the bare nuclei. are free to fuse.

However, since the nuclei are positively charged, there is a force of electrostatic repulsion between the nuclei. This *potential barrier* can be overcome by sufficient kinetic energy, and if you can get to a distance of about 10^{-13} cm (the "size" of the nucleus), the strong nuclear force (an attractive force between nucleons) will fuse the protons and release energy.

To overcome the barrier, the interacting particles need more kinetic energy than the potential energy of electrostatic repulsion. Is the temperature at the center of the Sun sufficient to produce enough kinetic energy? The pressure at the center of the Sun is $\mathrm{P_c} = 2.5 \times 10^{17}$ dynes $\mathrm{cm^{-2}}$. This pressure, from the force of gravity, is balanced by *gas pressure*:

$$\mathrm{P} = \mathrm{nkT}$$

where P is the pressure, T the temperature, where n is the number density (number per cubic centimeter). This is related to the mass density ρ by

$$\mathrm{n} = \frac{\rho}{<\mathrm{m}>}$$

where $<\mathrm{m}>$ is the mean mass per particle (this allows us to incorporate particles of different masses into the calculation). Thus

$$\mathrm{P} = \frac{\rho \mathrm{kT}}{<\mathrm{m}>}$$

Define the *mean molecular weight* as

$$\mu = \frac{< m >}{m_H}$$

where m_H is the mass of the hydrogen nucleus. Then

$$P_g = \frac{\rho k T}{\mu m_H}$$

μ depends on the ionization and composition of the gas.

The central density of the Sun is 150 gm cm^{-3} and the central pressure is $P = 2.5 \times 10^{17}$ dynes cm^{-2}. Also, $\mu = 0.5$ and $m_H = 1.6 \times 10^{-24}$ gm. This gives $T = 10^7$ K, which, together with quantum mechanical *tunneling*, is enough to allow the nuclei to fuse. Each particle has an associated *wavelength*, which you can estimate from the *Uncertainty Principle*:

$$\Delta x \Delta p \geq \hbar$$

If you can get the momentum (mass x velocity, with high temperature $\rightarrow$ high velocity) high enough to ensure that Δx is small (of order 10^{-13} cm), then the nuclei can fuse. Obviously, the rate of nuclear energy generation is very strongly dependent on temperature. The rate for the fusion of two hydrogen atoms is 5×10^{-24} s^{-1} cm^6 at $T = 10^4$ K, 5×10^{-3} at $T = 10^6$ K, and 0.2 at $T = 10^7$ K. So we need temperatures of about 10^7 K to get significant amounts of energy from nuclear reactions, and this is the temperature at the center of the Sun.

The Sun is thus in balance:

Gravity is balanced by gas pressure

The energy radiated per second is balanced by the energy generated per second

And the energy is generated by nuclear reactions, whose rate is controlled by the gravitational pressure on the interior of the Sun.

(Some years ago, in Holland, I was greatly amused by a Green Party poster in favor of solar energy - it showed a Sun smiley face with the slogan "Nuclear Power? No thank you!".)

The Sun is a really superb balancing act: but it's a very *stable* balance. To see this, let's work out two time scales - the *thermal* and *dynamical* time scales. First let's estimate the total energy content of the Sun:

$$E = NkT$$

where N is the total number of particles in the Sun (about 10^{57}), k is Boltzmann's constant and T the temperature; we'll take this to be 10^7 K for the whole Sun. The total energy content is then 1.4×10^{48} ergs. The rate at which the Sun is radiating energy is the solar luminosity, 4×10^{33} erg s^{-1}. Dividing the total energy content by the rate at which the Sun radiates energy gives us the thermal ("cooling") **time scale of 10^7 years**. This is also called the Kelvin-Helmholtz timescale.

There are two ways of working out the dynamical timescale. First, let's look at the *sound crossing time*, i.e. how long it would take a sound wave to cross the Sun's radius. Why is this relevant? Because sound waves carry information, and the sound crossing time is the time in which the whole Sun would respond to a dynamical perturbation. This is $R_\odot/c_S$, where $R_\odot$ is the solar radius and c_S the speed of sound, given by

$$c_s = \sqrt{\frac{kT}{\mu m_H}}$$

For a pure hydrogen plasma (fully ionized gas) and $T = 10^7$ K, the sound speed is 4×10^7 cm s^{-1} and the sound crossing time is then 1.7×10^3 seconds (this is an approximation because we haven't taken into account the temperature gradient in the Sun). This is **about 30 minutes**.

The second way to estimate this is to calculate the *gravitational collapse time*. We won't discuss this, but just for your interest:

$$\begin{aligned}
\frac{d^2r}{dt^2} &= -\frac{GM}{r^2} \\
\frac{vdv}{dr} &= -\frac{GM}{r^2} \\
\frac{1}{2}v^2 &= \frac{GM}{r} \\
v &= \sqrt{\frac{2GM}{r}}
\end{aligned}$$

giving $v = 6 \times 10^7$ cm s^{-1}. This is also an approximation, but note that it is the same as the sound speed; **the Sun's structure is set up this way**.

Thus the Sun responds to any dynamical disturbance extremely rapidly, and it must do so in order to be stable. It does, however, contain enough thermal energy for 10 million years.

The main sequence is a sequence of mass; mor massive stars are hotter and more luminous. But more massive stars also radiate far more energy per unit mass than do low-mass stars; thus, although they have more "fuel" they use it up faster. The most massive stars last only a few million years, far less than the age of the Universe (13.7 billion years). The fact that these stars actually can be seen today (the blue stars in the Orion nebula in Orion's sword) are examples means that star formation must be going on to the present day (a million years is "present day" as far as astronomy is concerned!). So if we are considering the evolution of life on a planet, it needs a steady source of radiation for a very long time. Thus long-lived main sequence stars are required!

This is one of the reasons there are so few high mass/luminosity stars relative to the low-luminosity/mass stars (the other reason is that it's easier to make a low mass star than a high mass star, because you have to get more material together - although it's not quite this simple!). Low mass stars last for a long time, and stars with masses less than about 0.8 $M_\odot$ have main sequence lifetimes longer than the present age of the Universe.

Next time we'll look at the temperatures of planets, the lowest mass star, brown dwarfs, and a "physical" definition of planet.